From chapter 2 we know that in order to get the energy bands, we need to solve the Schrödinger equation for periodic potential as illustrated by Kronig-Penny Model (Kronig and Penney, 1931). Kronig and Penney suggested a potential in the form of an array of square well potential. In this chapter, how Schrödinger equation can be arrived at by applying Lie algebraic method is discussed. More precisely, in this chapter we explain different states of Pöschl-Teller potential which is a periodic potential with the help of Lie algebraic method.

3.1 The Methodology for Treating Schrödinger Equation Problems:

The generalized methodology for treating the problems of interest relating to Schrödinger equation is as follows:

i. Choice of relevant group/ algebra:

The general rule for the correct choice of group or the algebra is that for any $n$-dimensional problems, the Lie algebra is $u(n+1)$ and the respective Lie group is $U(n+1)$ (Oss, 1996). In our present work we concentrate on the algebra rather than the group.
ii. **Lie algebraic condition:**

Once the group is correctly chosen, one needs to find the generators of that group. The correct generators will satisfy the defining conditions of Lie algebra. On the other hand if for treating particular problem, algebra is chosen first, then one needs to find the various operators which obey the said algebra.

iii. **Finding the appropriate Casimir operator:**

For finding the Casimir operator, one desires to find out a single operator which commutes with all the generators of the chosen group or the operators which obey the respective Lie algebra. The advantage of finding the Casimir operator is that the eigen values of Casimir operators can be easily obtained.

iv. Setting up of eigen value equation for the Casimir operator and further mathematical simplifications.

### 3.2 Bound States of Pöschl-Teller Potential:

To begin with we consider the group SU(2) generated by three operators $\hat{J}_x$, $\hat{J}_y$, $\hat{J}_z$ which have their usual form obtained from $\hat{J}_r = r \times (-i\nabla)$.

Now one interesting fact about the generators of SU(2) is that they obey Lie algebra as they satisfy the following two equations which are the defining equations for Lie algebra.

$$
\begin{align*}
\left[ \hat{J}_x, \hat{J}_y \right] &= i\hat{J}_z, \\
\left[ \hat{J}_y, \hat{J}_z \right] &= i\hat{J}_x, \\
\left[ \hat{J}_z, \hat{J}_x \right] &= i\hat{J}_y
\end{align*}
$$

(3.01)

and the condition of Jacobi identity as

$$
\left[ \left[ \hat{J}_x, \hat{J}_y \right], \hat{J}_z \right] + \left[ \left[ \hat{J}_y, \hat{J}_z \right], \hat{J}_x \right] + \left[ \left[ \hat{J}_z, \hat{J}_x \right], \hat{J}_y \right] = 0
$$

(3.02)

The above two equations are the defining equations for Lie algebra.
As the algebra is su(2) Lie algebra, so the corresponding Casimir Operator can be expressed in the form

\[
\hat{C} = \hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2
\]  

(3.03)

Now expressing \( \hat{J}^2 \) and \( \hat{J}_z \) in spherical polar coordinates \((r, \theta, \phi)\), we get,

\[
\hat{J}^2 = -\left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right]
\]  

(3.04)

and

\[
\hat{J}_z = -i \frac{\partial}{\partial \phi}
\]  

(3.05)

Also we can write an eigen value equation of the form

\[
\hat{C} \psi = \hat{J}^2 \psi = E \psi
\]  

(3.06)

Further we know that the eigen value of \( \hat{J}^2 \) is \( j(j+1) \), so combining equation (3.04) and Eq. (3.06) we can write

\[
\hat{C} \psi = \hat{J}^2 \psi = -\left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \psi = j(j+1) \psi
\]  

(3.07)

\[
\Rightarrow \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \psi = Q \psi
\]  

\[
\Rightarrow \left[ \cot \theta \frac{\partial \psi}{\partial \theta} + \frac{\partial^2 \psi}{\partial \theta^2} + \frac{1}{\sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} \right] \psi = Q \psi
\]  

(3.08)

where \( Q = -j(j+1) \)  

(3.09)

writing \( \psi = f(\theta)g(\phi) \), we get

\[
\cot \theta \frac{\partial}{\partial \theta} \left[ f(\theta)g(\phi) \right] + \frac{\partial^2}{\partial \theta^2} \left[ f(\theta)g(\phi) \right] + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \left[ f(\theta)g(\phi) \right] = Q \left[ f(\theta)g(\phi) \right]
\]
\[ \cot \theta \frac{\partial f}{\partial \theta} + \frac{\partial^2 f}{\partial \theta^2} + \frac{f}{\sin^2 \theta} \frac{\partial^2 g}{\partial \phi^2} = Qg \]

Dividing both sides by \( fg \), we get

\[ \cot \theta \frac{\partial f}{f} \frac{\partial}{\partial \theta} + \frac{1}{f} \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{\sin^2 \theta} \frac{1}{g} \frac{\partial^2 g}{\partial \phi^2} = Q \]

Multiplying both sides by \( \sin^2 \theta \), we get

\[ \cot \theta \frac{\sin^2 \theta}{f} \frac{\partial f}{\partial \theta} + \frac{\sin^2 \theta}{f} \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{\sin^2 \theta} \frac{\partial^2 g}{\partial \phi^2} = Q \sin^2 \theta \]

\[ \cot \theta \frac{\sin^2 \theta}{f} \frac{\partial f}{\partial \theta} + \frac{\sin^2 \theta}{f} \frac{\partial^2 f}{\partial \theta^2} - Q \sin^2 \theta = -\frac{1}{\sin^2 \theta} \frac{\partial^2 g}{\partial \phi^2} \]

\[ \cos \theta \frac{\sin \theta}{f} \frac{\partial f}{\partial \theta} + \frac{\sin^2 \theta}{f} \frac{\partial^2 f}{\partial \theta^2} - Q \sin^2 \theta = -\frac{1}{\sin^2 \theta} \frac{\partial^2 g}{\partial \phi^2} \] (3.10)

The L.H.S. of Eq. (3.10) is a function of ‘\( \theta \)’ while R.H.S. is a function of ‘\( \phi \)’ alone. This is possible only when both sides equal a constant (say \( m^2 \)) i.e.

\[ -\frac{1}{\sin^2 \theta} \frac{\partial^2 g}{\partial \phi^2} = m^2 \] (3.11)

and

\[ \cos \theta \frac{\sin \theta}{f} \frac{\partial f}{\partial \theta} + \frac{\sin^2 \theta}{f} \frac{\partial^2 f}{\partial \theta^2} - Q \sin^2 \theta = m^2 \] (3.12)

For solving equation (3.11), we put \( \frac{\partial}{\partial \phi} = n \) in Eq. (3.11) and get

\[ n^2 g = -m^2 g \]

\[ \Rightarrow n^2 + m^2 = 0 \]

\[ \Rightarrow n = \pm im \]

So \( g = Ae^{\pm im\phi} \) is the general solution.

For \( g(\phi) \) to be single valued, \( g(\phi) = g(\phi + 2\pi) \). Therefore
\[ Ae^{\pm im\phi} = Ae^{\pm im(\phi+2\pi)} \]
\[ \Rightarrow 1 = e^{\pm im2\pi} \]

This is possible only if \( m = 0, 1, 2 \ldots \ldots \ldots \ldots \)

Further, the normalization condition gives

\[ 1 = \int_{0}^{2\pi} g^* g \, d\phi = |A|^2 \int_{0}^{2\pi} d\phi \]
\[ \Rightarrow |A|^2 2\pi = 1 \]
\[ \Rightarrow A = \frac{1}{\sqrt{2\pi}} \]

except for an arbitrary phase factor which can be taken as zero. The normalized solution is then

\[ g = \frac{1}{\sqrt{2\pi}} e^{im\phi}, \quad m = 0, \pm 1, \pm 2, \ldots \ldots \quad (3.13) \]

Now multiplying both sides of Eq. (3.12) by \( f \) and rearranging the terms we get

\[ \sin^2 \theta \frac{d^2 f}{d\theta^2} + \sin \theta \cos \theta \frac{df}{d\theta} = \left( Q \sin^2 \theta + m^2 \right) f \quad (3.14) \]

Now let us substitute \( z = \cos \theta \)

\[ \therefore \quad dz = -\sin \theta d\theta \]
\[ \Rightarrow \quad \frac{dz}{d\theta} = -\sin \theta \]

Again,

\[ \frac{df}{d\theta} = \frac{dz}{d\theta} \frac{df}{dz} = -\sin \theta \frac{df}{dz} \quad (3.15) \]

and

\[ \frac{d^2 f}{d\theta^2} = \frac{d}{d\theta} \left( \frac{df}{d\theta} \right) = \frac{d}{d\theta} \left( -\sin \theta \frac{df}{dz} \right) \]
\[ \frac{d}{d\theta} (\sin \theta \frac{df}{dz} - \sin \theta \frac{df}{d\theta} \frac{d}{dz}) = -\cos \theta \frac{df}{dz} - \sin \theta \frac{d}{d\theta} \left[ \frac{df}{dz} \right] = -\cos \theta \frac{df}{dz} - \sin \theta \frac{d}{d\theta} \left[ -\sin \theta \frac{df}{dz} \right] \]

\[ \Rightarrow \frac{d^2 f}{d\theta^2} = -\cos \theta \frac{df}{dz} + \sin^2 \theta \frac{d^2 f}{dz^2} \quad (3.16) \]

Putting these values of \( \frac{df}{d\theta} \) and \( \frac{d^2 f}{d\theta^2} \) in Eq. (3.14) and we get

\[ \sin^2 \theta \left( -\cos \theta \frac{df}{dz} + \sin^2 \theta \frac{d^2 f}{dz^2} \right) + \sin \theta \cos \theta \left( -\sin \theta \frac{df}{dz} \right) = \left( Q \sin^2 \theta + m^2 \right) f \quad (3.17) \]

as \( z = \cos \theta \), so Eq. (3.17) can also be expressed as

\[ -z \left( 1 - z^2 \right) \frac{df}{dz} + \left( 1 - z^2 \right)^2 \frac{d^2 f}{dz^2} - (1 - z^2) \frac{df}{dz} = \left\{ Q \left( 1 - z^2 \right) + m^2 \right\} f \]

\[ \Rightarrow \left( 1 - z^2 \right)^2 \frac{d^2 f}{dz^2} - 2z \left( 1 - z^2 \right) \frac{df}{dz} = \left( Q \left( 1 - z^2 \right) f + m^2 f \right) \]

Dividing both sides by \( \left( 1 - z^2 \right) \) we get

\[ \left( 1 - z^2 \right) \frac{d^2 f}{dz^2} - 2z \frac{df}{dz} + \left\{ Q - \frac{m^2}{\left( 1 - z^2 \right)} \right\} f = 0 \quad (3.18) \]

This Eq. (3.18) is nothing but Associated Legendre’s differential equation. Eq. (3.18) has poles at \( z = \pm 1 \).

The solution of Eq. (3.18) is the Legendre polynomial \( P_j(z) \) for \( m = 0 \) and associated Legendre polynomial \( P_j^m(z) \) for \( m \neq 0 \). So the ultimate solution for \( m \neq 0 \) becomes
where $\epsilon = (-1)^n$, $\epsilon = 1$ for $m \leq 0$ and $P_j^m(\cos \theta)$ is the associated Legendre polynomial.

As $\psi = f(\theta)g(\phi)$, so combining Eq. (3.13) and Eq. (3.19), we get

$$\psi = \psi_{jm}(\theta, \phi) = \epsilon \sqrt{\frac{2j+1(1+|m|)!}{4\pi(1+|m|)!}} P_j^m(\cos \theta) \, e^{im\phi}$$  \hspace{1cm} (3.20)$$

where $j = 0, 1, \ldots$; $m = 0, \pm 1, \pm 2, \ldots, \pm j$

Now we assume $Z = \cos \theta = \tanh \rho$ ($-\infty \leq \rho \leq \infty$)

$$\therefore \frac{df}{dz} = \frac{df}{d\rho} \frac{d\rho}{d(\tanh \rho)} = \frac{df}{d\rho} \frac{1}{\sec^2 \rho} = \cos^2 \rho \frac{df}{d\rho}$$  \hspace{1cm} (3.21)$$

$$\frac{d^2f}{dz^2} = \frac{d}{dz} \left( \cos^2 \rho \frac{df}{d\rho} \right) = \frac{d}{dz} \left( \cos^2 \rho \frac{df}{d\rho} \right) + \cos^2 \rho \frac{d^2f}{d\rho^2} \frac{d\rho}{dz} = \frac{d}{d\rho} \left( \cos^2 \rho \frac{df}{d\rho} \right) + \cos^2 \rho \frac{d^2f}{d\rho^2} \frac{1}{\sec^2 \rho}$$

$$= 2 \cos \rho \sin \rho \cos^2 \rho \frac{d\rho}{dz} + \cos^2 \rho \frac{d^2f}{d\rho^2} \frac{1}{\sec^2 \rho}$$

$$\Rightarrow \frac{d^2f}{dz^2} = 2 \cos \rho \sin \rho \frac{df}{d\rho} + \cos^2 \rho \frac{d^2f}{d\rho^2}$$  \hspace{1cm} (3.22)$$

Using Eq. (3.21) and Eq. (3.22) in (3.18), we get
\[(1 - \tan h^2 \rho) \left[ 2 \cos h^2 \rho \sin h \rho \frac{df}{d\rho} + \cos h^2 \rho \frac{d^2 f}{d\rho^2} \right] - 2 \tan h \rho \frac{df}{d\rho} \cos h^2 \rho + \left[ -Q - m^2 \cos h^2 \rho \right] f = 0 \]

\[\Rightarrow \sec h^2 \rho \cdot 2 \cos h^2 \rho \sin h \rho \frac{df}{d\rho} + \sec h^2 \rho \cos h^2 \rho \frac{d^2 f}{d\rho^2} - 2 \tan h \rho \frac{df}{d\rho} \cos h^2 \rho - Qf = m^2 \cos h^2 \rho f \]

\[\Rightarrow 2 \cos h^2 \rho \sin h \rho \frac{df}{d\rho} + \cos h^2 \rho \frac{d^2 f}{d\rho^2} - 2 \tan h \rho \cos h^2 \rho \frac{df}{d\rho} - Qf = m^2 \cos h^2 \rho f \]

\[\Rightarrow \cos h^2 \rho \frac{d^2 f}{d\rho^2} - Qf = m^2 \cos h^2 \rho f \]

\[\Rightarrow \frac{d^2 f}{d\rho^2} - \frac{Q}{\cos h^2 \rho} f = m^2 f \quad (3.23)\]

Using Eq. (3.09) in Eq. (3.23), we get

\[\left[ \frac{d^2}{d\rho^2} + \frac{j(j+1)}{\cos h^2 \rho} \right] f = m^2 f \quad (3.24)\]

As \(Z = \cos \theta = \tanh \rho(\rho < \infty)\), so Eq. (3.19) can also be expressed as

\[f(\rho) = \varepsilon \sqrt{\frac{2j+1(j+|m|)!}{2(j+|m|)!}} P^m_j(\rho) \quad (3.25)\]

Using Eq. (3.25) in Eq. (3.24), we get,

\[\left[ - \frac{d^2}{d\rho^2} - \frac{j(j+1)}{\cos h^2 \rho} \right] \varepsilon \sqrt{\frac{2j+1(j+|m|)!}{2(j+|m|)!}} P^m_j(\rho) = -m^2 \varepsilon \sqrt{\frac{2j+1(j+|m|)!}{2(j+|m|)!}} P^m_j(\rho)\]

or

\[\left[ - \frac{d^2}{d\rho^2} - \frac{j(j+1)}{\cos h^2 \rho} \right] P^m_j(\rho) = -m^2 P^m_j(\rho) \quad (3.26)\]

This Eq. (3.26) is the Schrödinger equation for the bound states of Pöschl-Teller potential. The strength of this potential is given by \(V_o = j(j+1)\). Here a single representation \('j'\) will account for the spectral properties, given by \(m^2\). So here
starting from the su(2) Lie algebra, the Schrödinger equation for bound states of Pöschl-Teller potential is achieved.

### 3.3 Scattering States of Pöschl-Teller Potential:

In order to explain the scattering states of Pöschl-Teller potential we use su(1,1) Lie algebra and proceed.

As mentioned in chapter 1, the commutation relations for su(1,1) algebra adopt a very suggestive form:

\[
\begin{align*}
[\hat{J}_x, \hat{J}_y] &= -i\hat{J}_z, \\
[\hat{J}_y, \hat{J}_z] &= i\hat{J}_x, \\
[\hat{J}_z, \hat{J}_x] &= i\hat{J}_y \\
\end{align*}
\]

But for the minus sign in the first commutator distinguishes it from that of su(2) Lie algebra. Formally one gets the commutation relations of su(1,1) from those of su(2) by making any one of the replacement

\[
\hat{J}_x \rightarrow i\hat{J}_x, \quad \hat{J}_y \rightarrow i\hat{J}_y, \quad \hat{J}_z \rightarrow i\hat{J}_z
\]

(However, this projection of the real Lie algebra of SU(2) on the real Lie algebra of SU(1,1) is no isomorphism, because imaginary coefficients are used.)

In this chapter we use the following three generators of SU(1,1) obeying su(1,1) Lie algebra

\[
\begin{align*}
\hat{J}_x &= -i \left[ y \frac{\partial}{\partial z} + z \frac{\partial}{\partial y} \right] \\
\hat{J}_y &= i \left[ x \frac{\partial}{\partial z} + z \frac{\partial}{\partial x} \right] \\
\hat{J}_z &= -i \left[ x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right]
\end{align*}
\]

Now as the algebra is su(1,1), so the Casimir operator \( \hat{C} \) can be written as,
\[ \hat{C} = \hat{J}^2 = \hat{J}_z^2 - \hat{J}_x^2 - \hat{J}_y^2 = \hat{J}_z (\hat{J}_z - 1) - \hat{J}_x \hat{J}_y \]  

(3.28)

where the operators \( \hat{J}_x \) and \( \hat{J}_y \) are defined as \( \hat{J}_z = \hat{J}_x \pm i \hat{J}_y \)

Also with the above expressions of \( \hat{J}^2 \) and \( \hat{J}_z \), the simultaneous eigen states can be represented as

\[ \hat{J}_z |j,m\rangle = m |j,m\rangle \]  

(3.29)

Introducing the polar hyperbolic coordinates \( x = r \cos h \rho \cos \phi \), \( y = r \cos h \rho \sin \phi \), \( z = r \sin h \rho \), (where \( 0 < \phi < 2\pi \), \( 0 \leq \rho < \infty \)), we have

\[ \frac{\partial}{\partial x} = \cos h \rho \cos \phi \frac{\partial}{\partial r} - \frac{1}{r} \sin h \rho \cos \phi \frac{\partial}{\partial \rho} - \frac{1}{r} \sin \phi \frac{\partial}{\partial \phi} \]
\[ \frac{\partial}{\partial y} = \cos h \rho \sin \phi \frac{\partial}{\partial r} - \frac{1}{r} \sin h \rho \sin \phi \frac{\partial}{\partial \rho} + \frac{1}{r} \cos \phi \frac{\partial}{\partial \phi} \]
\[ \frac{\partial}{\partial z} = -\sin h \rho \frac{\partial}{\partial r} + \frac{1}{r} \cos h \rho \frac{\partial}{\partial \rho} \]  

(3.30)

Using the above values of \( \frac{\partial}{\partial x} \), \( \frac{\partial}{\partial y} \) and \( \frac{\partial}{\partial z} \) in the expressions of \( \hat{J}_x \), \( \hat{J}_y \), and \( \hat{J}_z \), we get

\[ \hat{J}_x = -i \left[ y \frac{\partial}{\partial z} + z \frac{\partial}{\partial y} \right] \]
\[ = -i \left[ r \cos h \rho \sin \phi \frac{\partial}{\partial r} - \sin h \rho \frac{\partial}{\partial \rho} + \frac{1}{r} \cos h \rho \frac{\partial}{\partial \phi} \right] + r \sin h \rho \left[ \cos h \rho \sin \phi \frac{\partial}{\partial r} + \frac{1}{r} \sin h \rho \frac{\partial}{\partial \rho} - \frac{1}{r} \sin \phi \frac{\partial}{\partial \phi} \right] \]
\[ = -i \left[ \cos h^2 \rho \frac{\partial}{\partial \rho} - \sin h^2 \rho \sin \phi \frac{\partial}{\partial \rho} + \cos \phi \tan h \rho \frac{\partial}{\partial \phi} \right] = -i \left[ \sin \phi \frac{\partial}{\partial \rho} + \cos \phi \tan h \rho \frac{\partial}{\partial \phi} \right] \]

\[ \therefore \hat{J}_x = -i \left[ \sin \phi \frac{\partial}{\partial \rho} + \cos \phi \tan h \rho \frac{\partial}{\partial \phi} \right] \]  

(3.31)
\[ \hat{j}_y = i \left[ x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z} \right] \]
\[ = i \left[ r \cos \rho \cos \phi \left( - \sin \rho \frac{\partial}{\partial r} + \frac{1}{r} \cos \rho \frac{\partial}{\partial \rho} \right) + r \sin \rho \left( \cos \rho \cos \phi \frac{\partial}{\partial r} - \frac{1}{r} \sin \rho \cos \phi \frac{\partial}{\partial \rho} + \frac{1}{r \cos \phi} \frac{\partial}{\partial \phi} \right) \right] \]
\[ = i \left[ \cos h^2 \rho \cos \phi \frac{\partial}{\partial \rho} - \sin h^2 \rho \cos \phi \frac{\partial}{\partial \rho} - \sin \phi \tan h^2 \rho \frac{\partial}{\partial \phi} \right] = -i \left[ \cos \phi \frac{\partial}{\partial \rho} - \sin \phi \tan h^2 \rho \frac{\partial}{\partial \phi} \right] \]
\[ \therefore \hat{j}_y = -i \left[ \cos \phi \frac{\partial}{\partial \rho} - \sin \phi \tan h^2 \rho \frac{\partial}{\partial \phi} \right] \quad (3.32) \]

\[ \hat{j}_z = -i \left[ x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right] \]
\[ = -i \left[ r \cos \rho \cos \phi \left( \cos \rho \sin \phi \frac{\partial}{\partial r} - \frac{1}{r} \sin \rho \sin \phi \frac{\partial}{\partial \rho} + \frac{1}{r \sin \rho} \frac{\partial}{\partial \phi} \right) \right] \]
\[ = -i \left[ \cos^2 \phi \frac{\partial}{\partial \phi} + \sin^2 \phi \frac{\partial}{\partial \phi} \right] = -i \frac{\partial}{\partial \phi} \]
\[ \therefore \hat{j}_z = -i \frac{\partial}{\partial \phi} \quad (3.33) \]

Using these values of \( \hat{j}_x, \hat{j}_y, \) and \( \hat{j}_z, \) we get

\[ \hat{j}_+ = \hat{j}_x + i \hat{j}_y \]
\[ = -i \left[ \sin \phi \frac{\partial}{\partial \rho} + \cos \phi \tan h^2 \rho \frac{\partial}{\partial \phi} \right] - \cos \phi \frac{\partial}{\partial \rho} + \sin \phi \tan h^2 \rho \frac{\partial}{\partial \phi} \]
\[ = -\left( \cos \phi + i \sin \phi \right) \frac{\partial}{\partial \rho} - i \tan h^2 \rho \frac{\partial}{\partial \phi} \left( \cos \phi + i \sin \phi \right) = \left( - \frac{\partial}{\partial \rho} - i \tan h^2 \rho \frac{\partial}{\partial \phi} \right) \left( \cos \phi + i \sin \phi \right) \]
\[ \therefore \hat{j}_+ = \left( - \frac{\partial}{\partial \rho} - i \tan h^2 \rho \frac{\partial}{\partial \phi} \right) e^{i \phi} \quad (3.34) \]
\[ \hat{J}_+ = \hat{J}_x - i\hat{J}_y \]
\[ = -i \left[ \sin \phi \frac{\partial}{\partial \rho} + \cos \phi \tan h \rho \frac{\partial}{\partial \phi} \right] + \cos \phi \frac{\partial}{\partial \rho} - \sin \phi \tan h \rho \frac{\partial}{\partial \phi} \]
\[ = (\cos \phi - i \sin \phi) \frac{\partial}{\partial \rho} - i \tan h \rho \frac{\partial}{\partial \phi} (\cos \phi - i \sin \phi) = \left( \frac{\partial}{\partial \rho} - i \tan h \rho \frac{\partial}{\partial \phi} \right) (\cos \phi - i \sin \phi) \]

\[ \therefore \hat{J}_+ = \left( \frac{\partial}{\partial \rho} - i \tan h \rho \frac{\partial}{\partial \phi} \right) e^{-i\phi} \quad (3.35) \]

Now we do a similarity transformation by \( \sqrt{\cosh \rho} \) i.e. we perform the following transformation.

\[ \frac{\partial f}{\partial \rho} = \cos h^{1/2} \rho \frac{\partial}{\partial \rho} \cos h^{-1/2} \rho f = \cos h^{1/2} \rho \left[ -\frac{1}{2} \cos h^{-1/2} \rho \sin h \rho f + \cos h^{-1/2} \rho \frac{\partial f}{\partial \rho} \right] \]
\[ = -\frac{1}{2} \tan h \rho f + \frac{\partial f}{\partial \rho} \]
\[ \Rightarrow \frac{\partial f}{\partial \rho} = -\frac{1}{2} \tan h \rho + \frac{\partial f}{\partial \rho} \]

using this value of \( \frac{\partial f}{\partial \rho} \) in Eq. (3.34) and Eq. (3.35), we get

\[ \hat{J}_+ = \left[ -\frac{\partial}{\partial \rho} + \tan h \rho \left( \frac{1}{2} - i \frac{\partial}{\partial \phi} \right) \right] e^{i\phi} \quad (3.36) \]

and

\[ \hat{J}_- = \left[ \frac{\partial}{\partial \rho} - \tan h \rho \left( \frac{1}{2} + i \frac{\partial}{\partial \phi} \right) \right] e^{-i\phi} \quad (3.37) \]

As for representing the Casimir operator the value of \( \hat{J}_+ \hat{J}_- \) is required, so to calculate \( \hat{J}_+ \hat{J}_- \) let us assume a function \( f(\rho, \phi) \) over which it acts. Then

\[ \left( \hat{J}_+ \hat{J}_- \right) f(\rho, \phi) = \left\{ -\frac{\partial}{\partial \rho} + \frac{1}{2} \tan h \rho - i \tan h \rho \frac{\partial}{\partial \phi} \right\} e^{i\phi} \left\{ e^{-i\phi} \left( \frac{\partial f}{\partial \rho} - \frac{1}{2} \tan h \rho f - i \tan h \rho \frac{\partial f}{\partial \phi} \right) \right\} \]
\[ \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{2} \sec h^2 \rho f + \frac{1}{2} \tan h \rho \frac{\partial f}{\partial \rho} + i \sec h^2 \rho \frac{\partial f}{\partial \phi} + i \tan h \rho \frac{\partial^2 f}{\partial \rho \partial \phi} + \frac{1}{2} \tan h \rho \frac{\partial f}{\partial \phi} - \frac{1}{4} \tan h^2 \rho f \]

\[ = -\frac{i}{2} \tan h^2 \rho \frac{\partial f}{\partial \phi} - e^{i\phi} \tan h \rho f \left( -\frac{1}{2} \sec h^2 \rho f - i \tan h \rho \frac{\partial f}{\partial \phi} \right) - i \tan h \rho \frac{\partial^2 f}{\partial \phi^2} \]

\[ + \frac{i}{2} \tan h^2 \rho \frac{\partial f}{\partial \phi} - \tan h^2 \rho \frac{\partial^2 f}{\partial \phi^2} \]

\[ = -\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{2} f + i \frac{\partial f}{\partial \phi} - \frac{1}{4} \tan h^2 \rho f - \tan h^2 \rho \frac{\partial^2 f}{\partial \phi^2} \]

\[ \therefore \left( \hat{J}_z \right) f(\rho, \phi) = -\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{2} f + i \frac{\partial f}{\partial \phi} - \frac{1}{4} \tan h^2 \rho f - \tan h^2 \rho \frac{\partial^2 f}{\partial \phi^2} \]

Now, we have the expression of Casimir operator as:

\[ \hat{C} = \hat{J}_z \left( \hat{J}_z - 1 \right) - \hat{J}_+ \hat{J}_- = -\hat{J}_+ \hat{J}_- + \hat{J}_z^2 - \hat{J}_z. \]

so,

\[ \hat{C} f(\rho, \phi) = \left( -\hat{J}_+ \hat{J}_- + \hat{J}_z^2 - \hat{J}_z \right) f = \frac{\partial^2 f}{\partial \rho^2} - \frac{1}{2} f - i \frac{\partial f}{\partial \phi} + \frac{1}{4} \tan h^2 \rho f + \tan h^2 \rho \frac{\partial^2 f}{\partial \phi^2} - \frac{\partial^2 f}{\partial \phi^2} + i \frac{\partial f}{\partial \phi} \]

\[ = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{4} \tan h^2 \rho f - \frac{1}{2} \left( \tan h^2 \rho - 1 \right) \frac{\partial^2 f}{\partial \phi^2} \]

\[ = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{4} \left( 1 - \sec h^2 \rho \right) f - \frac{1}{2} \sec h^2 \rho \frac{\partial^2 f}{\partial \phi^2} \]

\[ = \frac{\partial^2 f}{\partial \rho^2} - \sec h^2 \rho \left( \frac{f + \frac{\partial^2 f}{\partial \phi^2}}{4} \right) - \frac{f}{4} \]

\[ \therefore \hat{C} = -\hat{J}_+ \hat{J}_- + \hat{J}_z^2 - \hat{J}_z = \frac{\partial^2 f}{\partial \rho^2} - \sec h^2 \rho \left( \frac{1}{4} + \frac{\partial^2 f}{\partial \phi^2} \right) - \frac{1}{4} \] \hspace{1cm} (3.38)

The wave functions in Eq. (3.29) have the form

\[ |j, m\rangle = \psi_j^m(\rho)e^{im\phi} \] \hspace{1cm} (3.39)

Using Eq. (3.38) and Eq. (3.39) in Eq. (3.29), we get
\[ \hat{C} | j, m \rangle = \left[ \frac{\partial^2}{\partial \rho^2} - \sec h^2 \rho \left( \frac{1}{4} + \frac{\partial^2}{\partial \phi^2} \right) - \frac{1}{4} \right] | j, m \rangle = j(j+1) | j, m \rangle \]

\[ \Rightarrow \quad \left[ \frac{\partial^2}{\partial \rho^2} - \sec h^2 \rho \left( \frac{1}{4} + \frac{\partial^2}{\partial \phi^2} \right) - \frac{1}{4} \right] \psi^j_m (\rho) e^{im\phi} = j(j+1) \psi^j_m (\rho) e^{im\phi} \]

\[ \Rightarrow \quad \left[ -\frac{\partial^2}{\partial \rho^2} - \frac{m^2 - 1}{4} \sec h^2 \rho \right] \psi^j_m (\rho) = -\left( j + \frac{1}{2} \right)^2 \psi^j_m (\rho) \quad (3.40) \]

### 3.4 Results and Discussions:

This Eq. (3.40) is the Schrödinger equation for a particle in one-dimensional Pöschl-Teller potential, the strength of which is proportional to \( \left( m^2 - \frac{1}{4} \right) \). The same equation also shows that, since the energy of the particle is \(-\left( j + \frac{1}{2} \right)^2\), thus in this case the basis states correspond to bound states of the Pöschl-Teller potential. For a given potential (i.e., \( m \) fixed positive integer or half-integer) the bound state spectrum is given by \( E_j = -\left( j + \frac{1}{2} \right)^2 \). Also one may easily write its Hamiltonian in a group theoretic form as \( \hat{H} = -\left( \hat{C} + \frac{1}{4} \right) \).

Now if we consider \( j' \) is a negative integer or half integer (Bargmann, 1947) i.e.

\[ j = -\frac{1}{2}, -1, -\frac{3}{2}, -2, \ldots \]

and \( m \) is unbounded from above as

\[ m = -j, -j+1, -j+2, \ldots \]
Then the basis states correspond to bound states of the Pöschl-Teller potential. For a given potential (i.e., $m$ fixed positive integer or half-integer) the bound state spectrum is given by $E_j = -\left( j + \frac{1}{2} \right)^2$.

On the other hand if we consider $j$ is a negative integer or half integer i.e. 

$$j = -\frac{1}{2}, -1, -\frac{3}{2}, -2, \ldots$$

and $m$ is unbounded as 

$$m = j, j-1, j-2, \ldots$$

Then also it describes the same physical state (bound state) as the potential retains the same symmetry as that before.

Now if we take $j = -\frac{1}{2} + ik$, $k = real \neq 0$, then Eq. (3.29) becomes

$$\left[ -\frac{\partial^2}{\partial \rho^2} - \frac{m^2 - \frac{1}{4}}{\cosh^2 \rho} \right] \psi_m^j(\rho) = k^2 \psi_m^j(\rho)$$

(3.30)

This is the Schrödinger equation for scattering states of Pöschl-Teller potential. In this case the basis states correspond to the scattering states with energy $E_n = k^2$ but at different potential strengths.

### 3.4.1 Periodicity of the Pöschl-Teller Potential:

Periodicity of the Pöschl-Teller potential is checked by following method:

Consider, 

$$f(\rho) = \frac{1}{\cosh^2(\rho)} = \left( \frac{e^{\rho} + e^{-\rho}}{2} \right)^{-2}$$

Now let us put $a = 2\pi i$, then

$$f(\rho + a) = f(\rho + 2\pi i) = \left( \frac{e^{(\rho+2\pi i)} + e^{-(\rho+2\pi i)}}{2} \right)^{-2}$$
The above calculation confirms the periodicity of the Pöschl-Teller potential.

3.4.2 The Graphical Form of the Pöschl-Teller Potential:

![Graphical form of the Pöschl-Teller potential](image)

**Figure 3.1: Graphical form of the Pöschl-Teller potential**

3.5 Conclusion:

In this chapter at first we have shown how Schrödinger equation for the bound states of Pöschl-Teller potential can be explained with the help of su(2) Lie algebra. Then using a different realization of the su(1,1) algebra both the bound and scattering states of Pöschl-Teller potential are described. This Lie algebraic approach has the advantage that it can be generalised to cases where the Hamiltonian is specified in terms of the generators of the group rather than as a differential Schrödinger operator. Here the energy spectrum is discussed using purely group theoretical techniques without solving the Schrödinger equation. Also in most of the earlier works (Arima and Iachello, 1976, 1978; Iachello, 1981; Roosmalen et al., 1982; Iachello and Levine, 1982; Levine and Wulfman, 1979;
Roy et al., 2010, 2011), su(2) algebra was used to describe the bound states while su(1,1) was used to describe the scattering states, but here we have shown how one can explain both the bound as well as the scattering states employing the su(1,1) Lie algebra.

In this chapter we have explained how different states of a single potential can be explained just by varying the algebra.

Further the Pöschl-Teller potential is found to exhibit periodicity. But as the period of the potential is found to be imaginary which is not feasible for an ideal or real crystal, so we need to look for some other potentials. In order to get different potentials we need to change the operators which obey Lie algebra.

In the next Chapter, we use a different realization of su(1,1) Lie algebra and find the Schrödinger equation for a one dimensional periodic potential viz. Scraf potential having real period.