4.1. Introduction

It was mentioned earlier that interesting and elegant results have been obtained for time delay systems as regards stabilisation, observation, pole placement and other related control problems, inspite of inherent associated difficulties. However realisation and implementation of the theoretically obtained control laws are still not at a satisfactory level. Application of complicated control laws, like a 'dynamic feedback', can achieve many desired system properties. It is felt that achievement of only a few such properties (like a simple stabilisation with a degree of stability in lieu of complete spectrum control; or obtaining a stable observer having same structure as the system instead of one with arbitrary dynamics etc.), by means of simple but implementable control laws, should be a satisfactory design strategy in many applications. In this respect the polynomial state feedback or memoryless state feedback controls are still of interest due to their relative simplicity. However, the conditions for their existence (which can be obtained by resorting to ring theoretic approach) are more stringent compared to those for the complicated 'true state feedback' obtained through functional analytic method. This chapter is devoted to the study of a few sufficient conditions for
existence of such simple laws in the form 'partial spectrum assignment'. Only simple matrix manipulations are involved in achieving stabilising laws, stable observers, and observer based compensators. To begin, with, a brief survey of various design strategies recently available (which are too detailed) on stabilisation, observer designs and spectrum control, is felt to be appropriate.

4.1.1 Stabilisation of DDE System

It is well known now that stability and stabilisation of a DDE system pose more problems compared to systems without delays. In Sec. 1.3 few well known verifiable tests for stability and methods of stabilisation for LTI retarded DDE system were described. Broadly speaking for the system

\[ \dot{x}(t) = A_0 x(t) + A_1 x(t - h) + B_0 u(t), \quad t \geq 0, \]

\[ x \in \mathbb{R}^n, \ u \in \mathbb{R}^m, \ h > 0 \quad (4.1) \]

several different stabilisation problems can be formulated depending on the information made available to the feedback controller.

(I) State feedback stabilisation (e.g., [17], [18], [34], [35]): where control

\[ u(t) = K_0 x(t) + \int_{-h}^{0} K(\theta) x(t + \theta) \, d\theta \quad (4.2) \]

depends on the true state \( x_t \) at time \( t \). One interesting way of deriving such control strategy is LQOC problem (e.g., [21], [94], [97], [99], [129]-[132]).
(II) Discrete-time feedback stabilisation (e.g., \([75],[84],[85]\)): where
\[
u(t) = \sum_{i=0}^{N} K_i x(t - \Delta_i), \quad \Delta N = h
\] (4.3)
and is an approximate discrete time implementation of (4.2). Here each value of \(x(.)\) is to be stored and updated as time continuously progresses. The Repin-type modelling also gives rise to such feedback law (Secs. 1.5 and 2.2).

(III) Sampled-data feedback stabilisation (e.g., \([133],[134]\); (A) and (B)): where,
\[
u(t) = \sum_{i=0}^{N} K_i x((r-i)\Delta), \text{ for all } t \in [r \Delta, (r+1)\Delta],
\] (4.4)
where control signal \(u\) remains constant until further sampled information regarding the state of the system becomes available, and so its updating is required only at each sampling instant. This makes the strategy very attractive from implementational (e.g., computer-controlled) viewpoint.

When systems are represented over rings \(\mathbb{R}[d]\) (Sec. 1.1.3)
\[
\dot{x}(t) = A(d) \ x(t) + B(d) \ u(t)
\] (4.5)
two types of strategies have been used.

(IV) Constant(nondynamic or polynomial)-state feedback (e.g., \([24],[25],[40],[41],[55],[57],[107],[132],[135]\): where
\[
u(t) = K(d) \ x(t)
\] (4.6)
where the polynomial matrix $K(d)$ is over $\mathbb{R}[d]$; i.e., of the form
\[ u(t) = \sum_{i=0}^{N} K_i x(t - ih) \] (4.6a)
with real coefficient matrices $K_i$ are over $\mathbb{R}$. Implementation wise this scheme is very attractive although of quite restrictive nature.

(V) Dynamic state feedback (e.g., [24],[90],[91]): when
\[ u(t) = K_g(\sigma) x(t) \] (4.7)
where $K_g(\sigma)$ is a matrix containing rational functions $p(\sigma)/q(\sigma)$ in $d$ (i.e., $K_g(\sigma)$ may be viewed as a transfer function of continuous-time linear systems built from delays rather than integrators); or this may correspond to adding a number of integrators to the system $(A(d), B(d))$. The motivation for its use is that the constant feedback is in fact not so simple since $K(d)$ may contain memory. Moreover when the state feedback problem is seen just one step in the complete (I/O) regulator construction, a dynamic feedback will be implemented in any case, and so may be viewed as a true-state feedback.

(VI) Memoryless state feedback (e.g., [62],[68],[70]-[72]): when
\[ u(t) = K x(t), K \text{ over } \mathbb{R}. \] (4.8)
Implementation wise this popular feedback is the simplest but most restrictive. Stabilising conditions for this were discussed in Sec. 1.3.2 (Methods 9, 10 and 11).
This work is primarily concerned with the control of DDE systems by means of non-dynamic (or constant) feedback. Therefore, the discussions are mainly in the light of the Polynomial State Feedback (PSF).

Stability and stabilisation independent of delay (IOD) is an important design criterion, which was discussed in Sec. 1.3 (Methods 4, 5, 7 and 8). These conditions are comparatively easier to test than those for fixed delays.

Definition of $\gamma$-stable IOD was given in Defn. 1.10. For the stabilisation problem (Sec 1.3.2) let the system (4.5) over $\mathbb{R}[d]$ be represented as $(A, B)(z)$ for a fixed $z = \exp(-hs) \in \mathbb{C}$, matrices $A(z), B(z)$ over $\mathbb{C}$. This defines a 'finite-dimensional' (delay free) system over $\mathbb{C}$, called a 'Local System' (Kamen [57]). Now consider the PSF

$$u(t) = \sum_{j=0}^{N} L_j x(t - jh), L_j \in \mathbb{R}^{n \times m}.$$  (4.9)

Let $L(z) = \sum_{j=0}^{N} L_j z^j$, and the polynomial matrix

$$U(z) = [B(z), A(z) B(z), \ldots, A^{n-1}(z) B(z)].$$  (4.10)

Definition 4.1:

System $(A, B)(z)$ is $\gamma$-stabilisable IOD iff there exists $L(z)$ in (4.10) such that the closed loop system $[A(z) + B(z) L(z)]$ is $\gamma$-stable IOD.
Definition 4.2:

Given a fixed $\gamma > 0$, a system is locally $\gamma$-stabilisable iff for every $z \in \mathbb{F}$, $|z| < e^{\gamma}$ there is an $(n,m)$ matrix $L_z$ over $\mathbb{F}$ or $\mathbb{R}$ such that

$$\Re \lambda_j[A(z) + B(z)L_z] < -\gamma, \quad j = 1, 2, \ldots, n.$$ 

Some useful results [24], [40], [57] are:

(R1) Suppose that rank $\mathcal{J}(z) = n \quad \forall z \in \mathbb{F}$. Then system $(A, B)(z)$ is $\gamma$-stabilisable IOD for any $\gamma \geq 0$.

(R2) A weaker version is: for a fixed $\gamma \geq 0$ suppose that

$$\text{rank } [sI - A(z), B(z)] = n, \quad s \in \mathbb{F}, \quad |z| < e^{\gamma}.$$  \hspace{1cm} (4.11a)

Then system is $\gamma$-stabilisable IOD.

(R3) (a) Local $\gamma$-stabilisability is necessary but not sufficient for $\gamma$-stabilisability IOD.

(b) System $(A, B)(z)$ is locally $\gamma$-stabilisable iff

$$\text{rank } [sI - A(z), B(z)] = n, \quad \Re s \geq -\gamma, \quad |z| < e^{\gamma}.$$  \hspace{1cm} (4.12)

(c) Any necessary and sufficient condition for $\gamma$-stabilisability IOD must 'sit' between the two rank conditions (4.11) and (4.12). However presently there is no such exclusive rank condition available.

(R4) Randolfi's condition [35] for the special case of $B(z) = B$ (delay free) using distributed delays in feedback gives the necessary and sufficient stabilising condition for a fixed delay as
rank \([sI - A(e^{-hs}), B] = n, \text{Re } s > 0\). \quad (4.13)

In general it is not true that condition (4.12) with \(Y = 0\) be necessary and sufficient for \(O\)-stabilisability IOD, particularly under \(2SF\). Kamen et al. [57A; Sec. 5] showed that if rank \([sI - A_o - A_1d, B] = n\) for all \(s, \text{Re } s > 0\) and \(d, |d| < 1\), then there exists a stabilising \(2SF\).

To summarise

(i) Local \(Y\)-stabilisability is necessary but not sufficient for the existence of a non-dynamic (\(2SF\)) stabilising feedback.

(ii) Recent work of Emre [91] indicates that local-stabilisability is sufficient for the existence of a dynamic stabilising feedback.

4.1.2 Observers for DDE Systems

If the entire state vector of a dynamical system \(S\) is not directly accessible, a common procedure is to construct an observer whose output is then feedback. Roughly, an observer \(\hat{S}\) is a dynamical system that accepts as input the inputs \(u(.)\) as well as the outputs \(y(.)\) of \(S\), and produces an output \(\hat{x}(t)\) at time \(t\) that asymptotically approaches the internal state \(x(t)\) of \(S\), whatever \(x(0)\) was. A reasonable design requirement is that \(\hat{S}\) be somehow stable itself, and that \(\hat{S}\) be obtained with given rates of convergence of the estimation error \(e(t) = \hat{x}(t) - x(t)\) to zero; or more generally \(e(t)\) to have specific dynamics. Then \(S\) and \(\hat{S}\) are linear systems; the error convergence rates are independent of system input \(u(.)\). It is also of interest to determine when an observer
The estimator theory for LTI ODE systems both in deterministic (e.g., a Luenberger observer) or in a stochastic setting (e.g., a Kalman filter) is complete and well understood (e.g., [111], [112]). However, for the DDE system the state of this theory is still not at quite a satisfactory level, particularly in regards to its implementation. Therefore, the procedures used for implementing an optimal control law are more ad hoc; and so observers are necessary for estimating $x(t)$ as well as $x_t$, the latter being more complicated. The success of standard (text book) approach to observer designs has initiated studies in obtaining Luenberger-like observers. But this has run into a number of highly technical difficulties ([89],[90]).

Consider a deterministic LTI DDE system which for simplicity is given by

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + B_0 u(t), \quad t > 0,$$
$$y(t) = C_0 x(t),$$
$$x(t) = \phi(t), \quad t \in [-h, 0]$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $m \leq n$, $p < n$, rank $C_0 = p$.

The basic approaches for designing an observer were mentioned earlier (Sec. 1.2.2). Important classes of observers for time delay systems found in literature are:

(i) asymptotic (with $e(t) \to 0$ as $t \to \infty$),
(ii) exact (with $e(t) \equiv 0 \forall t \geq t_1$, $t_1$ is arbitrary),
(iii) optimal stochastic or deterministic.

Few other types are given in Lee and Olbrot [88].
This section is devoted to only asymptotic observers for which the Luenberger idea (e.g., [100], [111], [112]) dualising the concept of stabilising state feedback discussed earlier fully applies [93]. As a natural extension of Luenberger observers for ODE system, consider for the system (4.14) a second \((n-p)\)-dimensional dynamical system driven by the DDE system, and is called its observer

\[
\dot{z}(t) = F_0 z(t) + F_1 z(t-h) + G_0 y(t) + G_1 y(t-h) + H u(t), \quad t > 0,
\]

\[
z(t) = \xi(t), \quad t \in [-h, 0]
\]

(4.15)

where \(z(.) \in \mathbb{R}^{(n-p)}\), \(y(.) \in \mathbb{R}^p\), \(u \in \mathbb{R}^m\), \(\xi \in \mathbb{C}^n\); \(F_0, F_1 \in \mathbb{R}^{(n-p),(n-p)}\); \(G_0, G_1 \in \mathbb{R}^{(n-p),p}\); \(H \in \mathbb{R}^{(n-p),m}\).

Mimicking the procedure for the non-delay case (e.g., [132]) following existence results for the observer (4.15) are obtained.

Lemma 4.1:

Let there exists a \((n-p,n)\)-dimensional transformation matrix \(T\) satisfying

\[
F_0 T + G_0 C_0 = T A_0
\]

(4.16)

\[
F_1 T + G_1 C_0 = T A_1
\]

(4.17)

\[
H = T B
\]

(4.18)

In that case

(i) if \(\xi(t) = T \varnothing(t), t \in [-h, 0]\)
then
\[ z(t) = T x(t), \quad t \geq 0 . \] (4.19)

Otherwise, (ii) if \( \xi(t) \neq T \tilde{\varphi}(t) \) and
\[ n(t) : = \xi(t) - T \tilde{\varphi}(t), \quad t \in [-h, 0] \] (4.20)
then the reconstruction error
\[ e(t) : = z(t) - T x(t) \] (4.21)
satisfies the DDE (i.e., the error-dynamics)
\[ \dot{e}(t) = F_0 e(t) + F_1 e(t - h), \quad t \geq 0 \]
with initial data
\[ e(t) = n(t), \quad t \in [-h, 0] . \] (4.22)

Following remarks analogous to the finite-dimensional case can be made on the observer:

(RO1) System (4.15) is called the reduced or minimal-order dynamic observer for system (4.14). If instead \( T = I_n \), it is an identity or full order observer in which case system matrices are to satisfy
\[
F_0 = A_0 - G_0 C_0 , \\
F_1 = A_1 - G_1 C_0 , \\
H = B . \] (4.23)

(RO2) In practical situations the knowledge of the initial data \( \varphi(.) \) is not precisely available, and therefore, condition
(4.19) is hardly satisfied in real systems. This imperfect initialisation (4.20) of the observer yields an initial estimate bias (4.21) which should asymptotically approach zero. Unlike the delay-free system the matrix $T$ must now satisfy two simultaneous conditions (4.16)-(4.17) instead of one. Obviously, existence of such a matrix for arbitrarily specified $(A_0, A_1, F_0, F_1)$ is not guaranteed; and therefore, design of an observer for time delay system is more limited. It is visualised from remark (R02) that as $t \to \infty$ error $e(t)$ should approach zero for any value of initial data, and preferably at a specified rate. The error evolves according to the DDE (4.22) and the rate of decay is governed by the 'proper' location of error (or observer)-system eigenvalues.

With similar reasoning as in non-delay systems, the observer should be stable and real parts of its eigenvalues should be more negative than those of the observed system. At the same time observer eigenvalues should not be moved towards minus infinity so that it does not act as a differentiator with attendant difficulties. For the DDE the problem of proper pole placement is known to be quite difficult particularly under the restriction that $G_0, G_1$ are constant matrices (later in Sec 4.1.3).

It is essentially this situation in (R04) that has led to various design procedures for asymptotic observers, most of which are summarised in [88] and [96D]. A few of them, essential in the present context are reviewed here. Hewer and Nazaroff [92] developed the formal observer theory for
DDE systems where they attempted a Luenberger-type minimal order one. But construction of $T$ matrix and stabilisation of error equation are not well treated, except for some special situations. Gressang and Lamont [93] used the SDT (Ch. 3) and gave sufficient conditions for both identity and reduced order observers. Spectral separation property for the observer based closed loop system was also proved. Bhat and Koivo [37] extended this result in a very elegant manner by developing an explicit form of the observer equation for system (4.14) which is an integro-differential equation, and so the computational requirement for its implementation is appreciable. Lee and Manitius [132] pointed out that observer design was basically an iterative process arising from the difficulties of verifying the stability of a DDE. They also gave an easily computable sufficient condition for existence of an identity observer.

Lemma 4.2:

Let the pair $(A_0, C_0)$ be observable. Then there exists an asymptotic observer for system (4.14) in case

$$\| R(A_1 - G_1 C_0) \| < 1/2,$$

where $P$ is positive definite solution to

$$R(A_0 - G_0 C_0) + (A_0 - G_0 C_0)' P = -I.$$

Other sufficient conditions include

Lemma 4.3 (Ikeda et al [136]): Let

(i) $(A_0, C_0)$ be observable and be written in a Luenberger canonical form; (ii) $A_1 = A_{11} + A_{12}$, where matrix $A_{11}$
has rows written as linear combinations of rows of \( C_0 \) and \( A_{12} \) is an upper triangular matrix. Then an observer with arbitrary decay rate for \( e(t) \) exists.

Lemma 4.4 (Hashemi and Leondes[137]):

Let conditions (4.16)-(4.17) be satisfied and matrices
\[ F_0 \text{ and } F_1 \text{ satisfy (i) } F_0F_1 = F_1F_0, \text{ (ii) } \lambda_{\text{max}}(F_0) \leq -\|F_1\|. \]
Then the observer (4.15) is uniformly asymptotically stable.

They gave a design procedure for a 3rd order system but did not indicate that the conditions were satisfied only by matrices of special structures. When the DDE systems are expressed over ring \( R[d] \)
\[
\begin{align*}
\dot{x}(t) &= A(d)x(t) + B(d)u(t), \\
y(t) &= C(d)x(t)
\end{align*}
\]
(4.24)

interesting asymptotic observer designs have been obtained (e.g., [25],[26],[57],[88],[89],[91],[138]). Various pertinent definitions on this are given in Sec. 1.2.2.

Few commonly known methods for observer designs are enumerated below

(1) For systems observable over \( R[d] \):
(1) Dualisation of Morse's result (Sec 1.2.2 (C03)) gives rise to an observer construction with only point delays, where for systems of small dimensions it is possible to find a stabilising \( K \) over \( R[d] \). (e.g., the proof of [40; Prop 1], Lee and Zak [107]). For application of this idea the number
of outputs should be \( \geq 2 \) and not all components of output vector be delayed. In fact the reconstruction problem is equivalent to design of state feedback controllers for the dual system \((A'(d), C'(d))\).

(ii) Method of Emre and Khargonekar [90] suggests that for strongly observable systems observers with arbitrary dynamics can be constructed with guaranteed internal stability. Observers have also been realised under the weaker hypothesis of detectability, Emre [91].

(II) Via Spectral Projection Method:

When \( \text{deg } C(d) = 0 \) observer designs are based on the technique suggested by Bhat and Koivo [37] and Randolfi [35], under natural detectability condition. These observers may contain distributed delays; (also in Olbrot [87A]). The main difficulty in this method is the computation of system eigenvalues which is rather a nontrivial numerical problem.

(III) Ogunnaike's Procedure [138]:

By splitting \( x(t), A(d), \) and \( B(d) \) into the usual observed and unobserved (but observable) portions, this observer gives an asymptotic estimate of \( x(t) \) as the combination of outputs of a Luenberger observer and a complementary system which is an ODE.

(IV) Method of Hautus and Sontag [89]:

This is an interesting idea of obtaining an asymptotic non-Luenberger type observer using the concept of detectability, when the system is described over rings of rational functions.
\((\forall)\gamma\)-stable Independent of Delay Observer, Kamen [57]:

Based on the concept of \(\gamma\)-stabilisability IOD (Sec. 4.1.1) an observer for the DDE system \((A,B,C)(z)\) may be given by the dynamical system equation

\[
\frac{d}{dt}z(t) = \sum_{i=0}^{N} A_i z(t - ih) + \sum_{j=0}^{M} F_j [y(t - jh) + \sum_{i=0}^{q} \sum_{p=0}^{r} C_i z(t - ph - ih)] + \sum_{k=0}^{\infty} B_k u(t - kh) \quad (4.25)
\]

where \(z \in \mathbb{R}^n\) is the instantaneous state and \(F_j\) are gain matrices of the observer. The error \(e(t) (\equiv x(t) - z(t))\) satisfies

\[
\frac{d}{dt}e(t) = \sum_{i=0}^{N} A_i e(t - ih) - \sum_{j=0}^{M} \sum_{i=0}^{q} F_j C_i e(t-jh-kh) \quad (4.26)
\]

If the characteristic function of \((4.26), \Phi(s, e^{-hs}) \neq 0\) for \(\Re s > -\gamma\) for a fixed \(\gamma > 0\), the system \((4.25)\) is called \(\gamma\)-stable IOD observer.

4.1.3. Spectrum Control for DDE System

Recently there has been wide interest in the problem of spectrum control (or assignment) for the time-delay systems. This has more challenging features than the finite-dimensional cases, since control has to be exercised for infinite-number of system eigenvalues. Consider for simplicity system \((4.1)\) whose CQP is given by

\[
H = \det [\lambda I_n - A_0 - A_1 e^{-\lambda h}]
= \sum_{i=0}^{n} \sum_{j=0}^{M} \lambda^{n-i} \exp (-j\lambda h) \quad (4.27)
\]
where $M_{ij}$ are obtained from the elements of $A_0$ and $A_1$ matrices. In Sec. 1.2.1 various forms of controllability criteria in $M_2$-state space setting were discussed where the following significant implication holds:

$M_2$-approx. controllability $\Rightarrow$ spectral controllability $\Rightarrow$ exponential stabilisability with an arbitrary decay rate.

Verifiable conditions for $M_2$-approx. controllability are known ([17],[18],[43]). Also it is known that([17],[22],[23],[118]) a property called $F$-controllability (weaker than $M_2$-approx. contr.) implies spectral controllability. Other such related conditions are in ([32],[41]-[43],[139],[140], and references therein).

Many useful results available at present are for systems (4.5), described over the commutative (polynomial) ring $R[d]$. Repeating from Sec. 1.2.1 for the sake of continuity the following controllability matrix is defined:

$U(d) = [B(d), A(d) B(d), \ldots, A^{n-1}(d) B(d)]$

Definition 4.2: The pair $(A(d), B(d))$ is

(i) $R[d]$-controllable (or ring-reachable) in case $\text{Span} <A(d)|B(d)> = R^2[d]$, where $R^2[d]$ is the free finitely-generated module of $(n,1)$ matrices over $R[d]$.

(ii) Weakly controllable in case

$\text{rank} <A(d)|B(d)> = n$ over $R[d]$.
(iii) \( R(d) \) - controllable in case \( \operatorname{rank} \begin{pmatrix} A(d) & | & B(d) \end{pmatrix} = n \) over \( R(d) \), where \( R(d) \) is the quotient field of \( R[d] \) (i.e., the field of rational functions in \( d \) with real coefficients).

For the system (4.5) the controller considered is of the constant feedback (the RBF) form (4.6). Then, the following problems can be stated:

a) The problem of Coefficient Assignment (CA): where the closed-loop system (4.5)-(4.6) characteristic polynomial (Ch. poly.) can be arbitrarily assigned.

b) The problem of Pole Assignment (PA): where the closed-loop Ch. poly. has arbitrary zeros \( q_1, q_2, \ldots, q_n \); \( q_i \in R[d] \).

Definition 4.3:

The pair \((A,B)\) over \( R[d] \) is coefficient assignable (C-A) if for any given arbitrary monic polynomial \( p(z) = (p(d)) \) \((z) \in (R[d])(z)\) of degree \( n \) and real polynomial coefficients \( \{p_0, p_1, \ldots, p_{n-1}\} \) belonging to \( R[d] \), there is some feedback matrix \( K \) over \( R[d] \) such that the Ch. Poly.

\[
p(z) = \det [zI_n - A - BK] = z^n + \sum_{i=0}^{n-1} p_i z^i. \quad (4.29)
\]

Definition 4.4:

The pair \((A,B)\) over \( R[d] \) is pole (or zero)-assignable (P-A) if for any arbitrary real polynomials \( q_1, q_2, \ldots, q_n \) belonging to \( R[d] \) there is some feedback matrix \( K \) over \( R[d] \)
such that
\[ \det [z I_n - A - BK] = \prod_{i=1}^{n} (z - q_i). \quad (4.30) \]

In case \( R[d] \) is a field, these two imply each other and are known to be equivalent to reachability of \((A, B)\). However, the case of \( R[d] \neq \text{field} \) is more difficult and a few results are only available, the main approach being to extend the results of field case. Since in general a polynomial \( p(z) \) does not have all its 'roots' in \( R[d] \), the \( C-A \) is a stronger statement than the \( F-A \) in which existence of \( K \) is required for only some polynomials. In general \( P-A \) implies \( P-A \) but not vice versa. However, although \( P-A \) is a weaker property to \( C-A \) it is enough for many applications.

A property called Feedback Cyclization (F-C) is of interest in the control problems (Kamen [25]) because it allows the problems of \( F-A \) and \( C-A \) to be treated as the simple case of single input system \((m = 1)\). Characteristic features of this property are well documented in [25]. Various types of rings characterising \( C-A \), \( P-A \) and \( F-C \) properties are treated in Bumby et al [135]. An important result by Lee and Olbrot [47] concerns the genericity of reachability, summarised as follows:

Let \( A(d) \in R^{n \times n}[d] \), \( B(d) \in R^{n \times m}[d] \), \( \deg A(d) \leq k \), \( \deg B(d) \leq k \) (i.e., maximal degrees of elements of \( A \) and \( B \)), \( m \geq 2 \). Then the property that \((A(d), B(d))\) be \( R[d] \)-controllable is generic. In general, for systems over
reachability is generic as long as the number of input channels is larger than \( r \). For \( r = 1 \), which are good models of many DDE systems, there should be at least two control channels. However, for \( m = 1 \) the property of \( R(d) \)-controllability is not generic, whereas \( R(d) \)-controllability is so.

A few important spectrum control results available are summarised below, mostly for system (4.5) over the commutative ring \( R[d] \).

(SCR1). The \( R-A \) (Morse [40], Lee and Zak [107]):

For any \( R[d] \), \( R-A \) implies reachability. But if \( R[d] \) is a PID (Sontag [24; p 31]) and \((A,B)\) is reachable then \( R-A \) is possible by the non-dynamic feedback control (NFC). In case of \( m = 1 \), reachability (although a quite restrictive condition) is both necessary and sufficient for \( R-A \). This result gives a new insight into the meaning of ESF control for DDE systems, creating the possibility of stabilisation and finite spectrum assignment (e.g., [139], [140]). Morse gave a construction for \( K(d) \), but an interesting design procedure was worked out by Lee and Zak. They also showed that even in case of \( R(d) \)-controllability poles can be shifted, but only to some places. Further, for the class of DDE systems with \( m \geq 2 \) arbitrary pole placement using ESF is a generic property.

Kamen's [57] results state that \( R-A \) implies stability IOD; in general, stabilisability IOD is a much weaker property than pole assignability.
(SCR2). The P-CA :-

(a) Sontag [24] shows that reachability of $(A,B)$ is necessary for both C-A and P-A by NFC, for any ring. In $R[d]$ when $m=1$ or $n=1$ reachability is also sufficient for C-A.

(b) Kamen [25] shows that if a single input system $(A,b)$ is cyclic with generator $b$ then it is both C-A and P-A by SSF. For the multi-input system his results are contained in [25; Prop. 4.11 and Thm. 4.12].

(SCR3). The dynamic feedback :-

(a) Emre and Khargonekar [90] developed C-A results for a reachable system over a commutative ring by dynamic feedback, that also guaranteed internal stability.

(b) Regulator synthesis under detectability conditions are found in [89]-[91].

(c) In general stabilisation of DDE (4.1) by dynamic feedback is equivalent to $[sI - A(z),B]z = e^{-hs}$ being 'stabilisable'; $(A(z),B)$ is stabilisable [91] iff there exists stable rational matrices $V_1, V_2$ such that

\[
[sI - A(z), B] \begin{bmatrix}
V_1 \\
V_2
\end{bmatrix} = I_n . \tag{4.31}
\]

For the case of stabilisation IOD [57], (4.31) amounts to

\[
\text{Rank } [sI - A(z),B] = n ~ \forall ~ \text{Re } s \geq 0, \ |z| \leq 1 . \tag{4.32}
\]

Few other results in this regard are in [141]. Finite-dimensional compensation for time delay systems and other infinite-dimensional systems in the semigroup settings are
in [117], [117A].

Identical solvability criteria for arbitrary spectrum control of LTI state-delayed system using $\mathcal{H}_\infty$ (4.65) were obtained where a sufficient value of $N = n(n + 1)/2 + 1 < \infty$ was suggested. For multi-input system rank $\mathcal{U}(d) = n$ is a necessary condition. For the multiple state and control-delay system (S5) defined in Sec 1.1.1, necessary conditions for modal controllability (the $\mathcal{H}_\infty$-CA) are: (i) rank $\mathcal{U}(0) = n$, (ii) rank $\mathcal{U}(e^{-\lambda}) = n \forall \lambda \in \Lambda = \text{a set of complex numbers}$. Two sufficient conditions (in case the necessary conditions are satisfied) are also given by identifying the maximal sets of linearly independent vectors in $\mathcal{U}(d)$.

They developed the design of feedback controllers for linear systems with discrete, commensurate or distributed delays in the states and/or inputs such that the closed loop system had any arbitrary set of finite ($=n$) number of poles without preliminary knowledge of open loop spectrum. The law in general is complicated, requiring dynamic feedbacks whose parameters are obtained by solving a set of polynomial equations. As a sensitive study it was found out that under small perturbations, stability is preserved but the perturbed system might no longer be finite, its infinity part remaining located far in the LHP. Kamen [27] also attempted such a problem earlier, but did not give a systematic design procedure.
They gave computable necessary and sufficient conditions for spectral controllability for the LTI system (4.1) and its representation over $\mathbb{R}[d]$ as in (4.5) with $B(d) = B$. It was shown that (definition of spectral controllability is in Sec. 1.2.1):

spectral controllability of (4.1) $\implies$ weak $\mathbb{R}[d]$ controllability of (4.5).

Let the set $\Sigma_0 := \{d_0 \in \Phi : \text{rank}[B, A(d_0)B, \ldots, A^{n-1}(d_0)B] < n \}$ and the set

$$\Lambda_0 := \{ \lambda_0 \in \Phi : \exp(-\lambda_0 h) \in \Sigma_0 \}.$$

Then, system (4.1) is spectrally controllable iff

a) system (4.5) is weakly controllable, and

b) $\text{rank} \begin{bmatrix} \lambda_0 \end{bmatrix} = n, \forall \lambda_0 \in \Lambda_0 \cap \sigma(\lambda)$

where $\Delta(\lambda) = [\lambda I - A_0 - A_1 e^{-\lambda h}]$ and $\sigma(\lambda) = \text{spectrum of the system} = \{\lambda \in \Phi : \det \Delta(\lambda) = 0\}$.

As pointed out earlier (e.g., in (SCR1)) it is now well known that for a single input ($m = 1$) system to be $R_A$ by ESF it is necessary and sufficient that it be ring $\mathbb{R}[d]$ - reachable, which for instance for a system without control delay is equivalent to (Lee-Olbro [47], and Sec 1.2.1)

$$\det U(d) = \det[A(d)b, A(d)A(d)b, \ldots, A^{n-1}(d)b]$$

$$= \text{constant} \neq 0 \quad \text{(in } \mathbb{R}) \quad \text{(4.33)}$$

This is a very restrictive condition and is hardly met with in practice. Certain results may however be obtained that
help in stabilisation via the \( B^3F \) control (4.6a).

Assertion 4.1. Let system (4.5) with \( B(\xi) = b \in \mathbb{R}^{n \times 1} \) be weakly controllable and \( \det U(d) = \prod_{i=0}^{q} \alpha_i d^i, q \leq n, \alpha_i \in \mathbb{R} \). Then coefficients of \( \exp(-i \lambda h), i = q, q+1, \ldots, n \) in the CQP (4.27) can be arbitrarily controlled by law (4.6a).

Assertion 4.2. Let for the system (4.1) with \( B_0 = b \in \mathbb{R}^{n \times 1}, (A_0, b) \) be at least stabilisable and let \( q = 1 \). Then the system is stabilisable by the law (4.6a) with \( N = 1 \).

Above results can be verified once it is understood how satisfaction of (4.33) effects the pole assignment.

In the next section certain cases of stabilisation and observation problems are treated. It is observed that for the second order single input (or single output) systems exhaustive conditions could be derived for existence of non-dynamic controllers, or observers of simple structure to estimate instantaneous states \( x(.) \). For higher order systems with multiple inputs (or outputs) special structures of system matrices become necessary. In fact, the most important direction of generalisation of the second order cases is the case with scalar input (or output for the dual problem), since as could be noticed from the genericity results of Lee and Olbrot [47] discussed earlier (also in Sec 1.2.2) the case with commensurate delays and number of inputs (outputs) \( \geq 2 \) can be treated generically by algebraic methods initiated by Morse [40].
4.2 Partial-Spectrum Control ([171]–[176])

The theory and progress made on the spectrum control of LTI retarded DDE system were discussed in Sec 4.1.3. This section builds on a simplification of Morse's result [40] given in equation (4.30). Morse's construction as in the proof of [40; Prop 1]) for a stabilising feedback gain $K(d)$ may be used for systems of low dimensions. It is only recently that Lee and Zak [107] derived an algorithm for the construction of $K(d)$ (i.e., the computation of RSF control) which assigns an arbitrary spectrum placement for the closed-loop system when the DDE system is R(d) (or even R(d))-reachable. Kamen [56] also gave a design procedure to compute $K(d)$ for asymptotic stability by RSF. However, the present setup is developed independent of [107] and is more straightforward although along similar lines. The admissible control law is simpler but restrictive: only proportional feedback of $x(t)$ and/or $x(t - h)$ is used, mainly because the primary objective is not the full control of the system spectrum. The basis is again (as in Morse's method) the 'decomposition' or factorisation of the nth order system CQP (4.27) into $n$-first order quasi-polynomials (whenever possible) in $(s, e^{-sh})$, with coefficients fully or partially controlled. Naturally it is evident that the above requires special structures of involved matrices; i.e., $A_0, A_1$ and $B_0$. The motivation behind this formulation is presented below.

Consider the LTI system
\[ \dot{x}(t) = A_0 x(t) + A_1 x(t-h) + B u(t), \quad t > 0 \] (4.1)

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), and its CQP given by

\[ H = H(\lambda, e^{-\lambda h}) = \det \left[ \lambda I_n - A_0 - A_1 e^{-\lambda h} \right] \] (4.27)

It is indicated in Sec 3.2.2 that obtaining zeros of \( H \) is a difficult task except for special cases, and the scalar case (\( n = 1 \)) is one such for which detailed procedure for computation is outlined in the Chapter 3 Appendix. Now consider the linear feedback control law

\[ u(t) = K_0 x(t) + K_1 x(t-h), \quad K_0, K_1 \in \mathbb{R}^{(m,n)}. \] (4.34)

In the sequel following notations will be followed. Since system (4.1) can be normalised to a system with unit delay (e.g., by transformation \( t = \tau h \), \( x(\tau \cdot h) = \tilde{x}(\tau) \), \( u(\tau \cdot h) = \tilde{u}(\tau) \) and then

\[ \frac{\tilde{x}(\tau)}{\tilde{u}(\tau)} = h A_0 \tilde{x}(\tau) + h A_1 \tilde{x}(\tau-1) + h B \tilde{u}(\tau), \]

\( h = 1 \) will be assumed subsequently without loss of generality; and hence matrices \( A_0, A_1, B \) in (4.1) with \( h = 1 \) has implicitly delay time \( (h) \) dependency. Let

\[ h_{oi} = h_{oi}(\lambda) = \lambda - \alpha_{oi} \],

\[ h_{li} = h_{li}(\lambda, e^{-\lambda}) = \lambda - \alpha_{oi} - \alpha_{li} e^{-\lambda} \],

\[ h_{2i} = h_{2i}(\lambda, e^{-\lambda}) = \lambda - \alpha_{li} e^{-\lambda}, \quad \alpha_{ji} \in \mathbb{R} \],

\[ \overline{A}_0 = A_0 + B K_0, \quad \overline{A}_1 = A_1 + B K_1, \quad \overline{a}_{ji} = \alpha_{ji} + b_i/k_{ji} \]
where, $a_{ji}$ (resp $\bar{a}_{ji}$) are obtained from elements of $A_j$
(resp $\bar{A}_j$), $j = 0$, 1. $P_0$ and $P_1$ are respectively the modal
matrices of $\bar{A}_0$ and $\bar{A}_1$. The CQP $H = \det \left[ \lambda I_n - \bar{A}_0 - \bar{A}_1 e^{-\lambda} \right]$, and similarly are defined $h_{pi}$, $p = 0$, 1, 2. $\{a_{1i}\}$ is the
set of eigenvalues of the DDE systems, always meant to be arranged in descending order of their real parts; i.e.,
$\text{Re } \lambda_{j+1} \leq \text{Re } \lambda_j$ for all $j = 0$, 1, 2, ....... The spectrum (or eigenvalues) of a matrix $A$ is denoted by $\sigma(A)$, $\lambda_i(s)$
indicates a specific eigenvalue and $\lambda_0$ is the most dominant or largest one; $\sigma(\bar{A}_0)$ and $\sigma(\bar{A}_1)$ are assumed reals. $A_{ij}$
denotes an element of matrix $A$.

Definition 4.5:

The scalar coefficients $a_o$ and $a_1$ of the CQP $h_1 = \lambda - a_o - a_1 e^{-\lambda}$ will be called compatible for the stabilisation whenever they make $h_1$ stable, as evident from Hayes' result
[1; p 444].

From the descriptions in Sec. 3.2.2 it is known that zeros of the scalar CQP $h_{1i}(\lambda, e^{-\lambda})$ for the first order
DDE system

\[(SS) : \dot{x}_i = a_{0i} x_i(t) + a_{1i} x_i(t-1) + b_{0i} u(t)\]

are comparatively easily computable by any number and to any degree of desired accuracy (Ch. 3 Appendix). Further, the stability zone of scalar system (SS) for any $i$ is well known [1; p 444]. From the properties of $\lambda$'s enumerated in Sec. 3.2.2 it is realised that by any selection of $K_{0i}$
$K_{1i} \in \mathbb{R}$ alone, if not all, at least one eigenvalue (e.g.,
the most dominant one) of (SS) can be assigned arbitrarily. Thus, if the CQP $\bar{H}$ of the controlled system (4.1)-(4.34) can be factorised into $\bar{H} = \prod_{i=1}^{n} \bar{h}_i$, then at least $n$ eigenvalues of it can be placed in an arbitrarily chosen zone in $\phi$; for example in $\gamma_1 \leq \text{Re} \lambda_i \leq \gamma_2$, $\gamma_1$, $\gamma_2 < 0$. This type of assignment (whenever possible), as distinct from (and a specialised form of) the P-A problem, will be called Partial-Spectrum Assignment (PSA).

Definition 4.6:

The system (4.1) is called partial-spectrum assignable if for any given set of monomials $\{q_1, q_2, \ldots, q_n\}$ in $(e^{-\lambda})$ of the form $q_i(e^{-\lambda}) = \alpha_{0i} + \alpha_{1i}e^{-\lambda}$, $i = 1, 2, \ldots, n$ and $\alpha_{0i}, \alpha_{1i} \in \mathbb{R}$, there exist feedback gain matrices $K_0, K_1 \in \mathbb{R}_{m \times n}$ such that the closed loop (4.1)-(4.34) system CQP $\bar{H}$:

(i) is expressible as $H(\lambda, e^{-\lambda}) = \prod_{i=1}^{n} (\lambda - q_i)$ and

(ii) has at least $n$ zeros $\lambda_i$, $i = 1, 2, \ldots, n$ satisfying $\text{Re} \lambda_i \in [\gamma_1, \gamma_2]$ for arbitrary $\gamma_1$, $\gamma_2 < 0$.

With the above decomposition the complete spectrum of the DDE system (4.1) can be easily computed, being the union of the zeros of all the constituent scalar CQPs $(\lambda - q_i)$. Therefore, with properly chosen $\gamma_1$, $\gamma_2 < 0$ an arbitrary degree of stability (DS) can be achieved for the controlled system. However, since characteristically the eigenvalues are
closely spaced (Sec 3.2.2), and most of them will be 'almost' beyond the effect of the assumed control, their influence on other transient behaviours (although not affecting the DS) cannot be ruled out. This aspect has to be tolerated in lieu of the simplicity of the control law. Further, since the results to be stated mainly depend on choosing coefficients \( \alpha_0, \alpha_1 \) of a scalar CQP \( h_1(\lambda, e^{-\lambda}) = \lambda - \alpha_0 - \alpha_1 e^{-\lambda} \), the following deduction may serve as a guide for considering stabilisation. This is derived from Hayes's diagram [1; p 445], (also Ch. 3, Appendix).

Lemma 4.5:

Let \( \alpha_0 < 0 \) be chosen for better stability conditions of CQP \( h_1(\lambda, e^{-\lambda}) \). With a fixed value of \( \alpha_1 \) system stability increases with increasing values of \( |\alpha_0| \). On the other hand, with a fixed value of \( \alpha_0 \) and a set of values of \( \alpha_1 \), if the first zero \( \lambda_0 \) of \( h_1 \) is real (resp. complex) stability increases with decreasing (resp. increasing) values of \( \alpha_1 \). In addition, when \( \alpha_0 \in (0, 1) \) the system can still be stable if \( \alpha_1 \in ((\pi/2-1) \alpha_0 - \pi/2, -\alpha_0) \); and when \( \alpha_0 = 0 \) it is stable iff \( \alpha_1 \in (-\pi/2, 0) \), the greatest DS achievable being one.

This section is devoted to exploring possibilities of having such special structures of \( A_0, A_1, B_0 \) matrices effecting the above CQP-decomposition. The strategy is to make \( A_0 \) and \( A_1 \) simultaneously diagonal or identically triangular.
(i.e., both either lower or upper triangular), or one diagonal and other triangular by suitable choice of controls and similarity transformation. A case by case study of the situations is made giving rise to sufficient conditions for the purpose [171]. Any reachability conditions etc. to be satisfied are not explicitly dealt with, only that these are bound to be restrictive.

4.2.1. Stabilisation

CASE -1: Stabilisation of a second order \((n = 2, m = 1)\) DDE system [172],

Consider a second order single input system given by

\[
A_0 = \begin{bmatrix} a_{01} & a_{02} \\ a_{03} & a_{04} \end{bmatrix}, \quad A_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{13} & a_{14} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad h = 1 \tag{4.35}
\]

where \(a_{0i}, a_{1i}, b_j \in \mathbb{R} \). Let the feedback gain matrices \(K_0, K_1 \in \mathbb{R}^{2 \times 2} \) for the law (4.34) be given by

\[
K_0 = [K_{01}, K_{02}], \quad K_1 = [K_{11}, K_{12}], \quad K_{ij} \in \mathbb{R} \, .
\]

Also let the modal matrix \(P_0\) of \( \bar{A}_0 = A_0 + b K_0 \) be given by, without loss of generality

\[
P_0 = \begin{bmatrix} 1 & 1 \\ w_2 & w_4 \end{bmatrix}, \quad w_2 \neq w_4 \, .
\]

Lemma 4.6: With above defined matrices,

\[
P_0^{-1} \bar{A}_0 P_0 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad P_0^{-1} \bar{A}_1 P_0 = \begin{bmatrix} \lambda_3 & 0 \\ \alpha \lambda_4 & \lambda_4 \end{bmatrix} \tag{4.36a,b}
\]
and the closed loop CQP of the transformed (i.e.,
\( z(t) = P_0 x(t) \)) system is

\[
H(\lambda, e^{-\lambda}) = (\lambda - \lambda_1 \cdots \lambda_3 e^{-\lambda}) (\lambda - \lambda_2 \cdots \lambda_4 e^{-\lambda})
\] (4.37)

whenever the following conditions are satisfied:

\[
w_2 = \left( \lambda_1 b_2 - a_{01} b_2 + a_{03} b_1 \right) / \left( \lambda_1 b_2 + a_{02} b_2 - a_{04} b_1 \right),
\] (4.38)

\[
w_4 = \left( \lambda_2 b_2 - a_{01} b_2 + a_{03} b_1 \right) / \left( \lambda_2 b_1 + a_{02} b_2 - a_{04} b_1 \right),
\] (4.39)

\[
w_4 p_1 - p_2 + w_2 (w_4 p_3 - p_4) = (w_4 - w_2) \lambda_3,
\] (4.40)

\[
p_1 + w_4 p_3 = \lambda_4,
\] (4.41)

\[
p_2 + w_4 p_4 = \lambda_4 \cdot w_4
\] (4.42)

where \( \{\lambda\}_n \in \mathbb{R} \) is a desired set of eigenvalues, \( \alpha \in \mathbb{R} \) is a quantity obtained by way of above transformation, and

\[
p_1 := a_{11} + b_1 K_{11}, \quad p_2 := a_{13} + b_2 K_{11},
\]

\[
p_3 := a_{12} + b_1 K_{12}, \quad p_4 := a_{14} + b_2 K_{12}.
\]

Proof: The derivation of above results is lengthy but straightforward. Conditions (4.38)–(4.42) directly follow from satisfaction of identities (4.36a,b) and simultaneous reductions of \( \bar{A}_0 \) and \( \bar{A}_1 \) matrices. Factorisation (4.37) is then obvious from the (similarity) transformed system.
Choice of $\lambda_1$ will naturally dependent on elements of $A_0$, $A_1$ and $b$. Following result describes such situations.

Proposition 4.1:

Let the DDE system be described by (4.35) and let $x(t)$ and $x(t-1)$ be available for feedback through the law (4.34). Then following are the constraints for the system to be ESA.

Also, system spectrum is determined by $\lambda_i$, $i = 1, 2, 3$ as follows from (4.37).

(A) Let $b_1 \neq 0$, $b_2 \neq 0$. System is ESA and arbitrary degree of stability (DS) is achieved if $A_1$ and $b$ are such that (from (4.41) and (4.42))

$$\frac{b_2}{b_1} a_{12} - a_{14} = \left(\frac{b_1}{b_2}\right) a_{13} - a_{11} = D_1 \text{ (say)}$$

and $\lambda_4 = -D_1$, $w_4 \neq b_2/b_1$.

(B) Let $b_1 \neq 0$, $b_2 = 0$. Then $(\lambda_1, \lambda_2)$ can be made arbitrary but $(\lambda_2, \lambda_4)$ cannot be independently chosen, since they are related by

$$\lambda_4 a_{03} - \lambda_2 a_{13} = a_{03} a_{14} - a_{04} a_{13} = D_2 \text{ (say)}.$$ 

However, in case $(a_{04}, a_{14})$ are compatible for stability system is directly stabilisable by choosing $(\overline{a}_{01}, \overline{a}_{11})$ compatible for stability and $\overline{a}_{02} = \overline{a}_{12} = 0$. In general, use of Lemma 4.5 specify relationship between elements $\{a_{0i}\}$ and $\{a_{1i}\}$ for stability as:

(i) if $a_{13} = 0$, $a_{03} \neq 0$ then $\lambda_4 = a_{14}$, $\lambda_2$ arbitrary and so DS is arbitrary;
(ii) if $a_{03} = 0$, $a_{13} \neq 0$ then $\lambda_2 = a_{04}$, $\lambda_4$ arbitrary and system is stable whenever $a_{04} < 1$;

(iii) if $a_{03} = a_{13} = 0$ then system is stable whenever $(a_{04}, a_{14})$ are compatible for stability;

(iv) if $a_{03} = a_{13} \neq 0$ then system is stable whenever $(a_{14} - a_{04}) > 0$, and in case $(a_{14} - a_{04}) < 0$ system is stable if $|a_{14} - a_{04}| < \pi/2$;

(v) if $a_{03} \neq 0$, $a_{13} \neq 0$ and $D_2 \neq 0$ then system is stable whenever $D_2/a_{13} > 0$, or $0 > D_2/a_{03} > -\pi/2$, or $(-a_{03}/a_{13}) > 1$;

(vi) if $a_{03} \neq 0$, $a_{13} \neq 0$ and $D_2 = 0$ then system is stable whenever $(a_{03}/a_{13}) \neq -1$.

(C) Let $b_1 = 0$, $b_2 \neq 0$ then system is stable if conditions of (B) are satisfied with $a_{03}$, $a_{04}$, $a_{13}$ and $a_{14}$ being replaced by $a_{02}$, $a_{01}$, $a_{12}$ and $a_{11}$ respectively.

Above results are derived from drawing the $\lambda_2/\lambda_4$ slope on the Hayes' diagram [1; p 444].

Algorithm 4.1:

As an illustration, a suggested algorithm for conditions in (A) (i.e., when $b_1 \neq 0$, $b_2 \neq 0$) is as follows: (i) Find $\lambda_4$ from (4.43); (ii) choose $\lambda_2$ for desired stabilisation and find $w_4$ from (4.39) satisfying (4.43); (iii) choose a pair $(\lambda_1, \lambda_3)$ arbitrarily; (iv) since $\lambda_1$, $\lambda_2$ and $w_4$ are known find $w_2$ from (4.38) and so $(K_{01}, K_{02})$; (v) from (4.41) find $K_{11}$ in terms of $K_{12}$.

In this connection it is pointed out in Sec. 4.1.3 that ring reachability is a quite restrictive condition for
single input system. However for the second order system when \( n = 2 \) and \( m = 1 \) this condition interestingly reduces to a simple result.

Theorem 4.1:

For the second order, single input \((n = 2, m = 1)\) DDE system given by (4.35), \( \det U(d) = \text{constant} \neq 0 \) (in \( \mathbb{R} \)) if and only if \((A_0, b)\) is controllable and \((A_1, b)\) is not controllable.

Proof: By direct substitution.

It is to observe that expression (4.43) is equivalent to the condition of \((A_1, b)\) being not controllable.

CASE-2 (Trivial cases).

(A) \( n = m = 1 \). The scalar system can be given any desired DS by \( x(t) \) feedback alone.

(B) \( m = n \) and rank \( B = n \). Let \( \bar{A}_0 \) and \( \bar{A}_1 \) represent a set of desirable system matrices compatible for ESA. Then

\[
K_i = B_i^{-1} [\bar{A}_1 - A_1], \quad i = 0, 1.
\]

CASE-3:

(A) Let \( A_0 = \alpha_0 I_n \), \((A_1, B)\) controllable, \( \alpha_0 \in \mathbb{R} \). Then it is possible that

\[
\bar{M} = \frac{n}{\prod_{i=1}^{n} (\lambda - \alpha_0 \bar{a}_{1i} e^{-\lambda})}, \quad \text{where} \quad \{\bar{a}_{1i}\} = \sigma(\bar{A}_1)
\]

are arbitrary by choosing \( K_1 \) such that

\[
E_1^{-1} \bar{A}_1 P_1 = \text{diag} [\bar{a}_{11}, \bar{a}_{12}, \ldots, \bar{a}_{1n}]
\]
and

$P^{-1} A \cdot P = \alpha_0 I_n$. Thus, if

(i) $\alpha_0 < 1$, then stability is guaranteed;

(ii) $\alpha_0 = 0$, then $\max D_s = 1$;

(iii) $\alpha_0 < 0$, then $D_s$ increases with $|\alpha_0|$. 

(B) If instead $(A_0, B)$ is controllable, $A_1 = \alpha_1 I_n$, $\alpha_1 \in \mathbb{R}$, then arbitrary $D_s$ is achievable irrespective of value of $\alpha_1$.

CASE-4: $(A_0, B)$ controllable and columns of $A_1$ are linear combinations of columns of $B$.

(A) If $K_1$ is chosen such that $A_1 = 0$, the system is (at least mathematically) finite-spectrum assignable (FSA).

(B) When an nth order single input scalar DDE system has a state variable representation (the class of system (86) in Sec 1.1.11) the system is known (e.g., [68]) to be stabilizable by $x(t)$ feedback alone. However it can be FSA by $x(t)$ and $x(t-1)$ feedbacks.

(C) When $m > 1$ the stability conditions through $x(t)$ feedback need special structures of $A_1$ (e.g., [70]). Analogously for a multi-input DDE system to be FSA or RSA the conditions may be: $(A_0, B)$ controllable, rank $B = m$, and structures of $A_1$ and $B$ are such that by reordering or by a nonsingular transformation (if possible) these are

$A_1 = \begin{bmatrix} 0 & 0 \\ A_2 & A_3 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ B_1 \end{bmatrix}$, $A_3$ and $B_1 \in \mathbb{R}^m \cdot m$. 

that is the system consists of a set of \((n - m)\) unforced ODEs and a set of \(m\) forced DDEs

(D) Certain sub-cases may be of nature

(i) \(A_1 = a I_n + B F\) for some \(F \in \mathbb{R}^{n \times n}\)

(ii) \(A_0 = p A_1 + q I_n\) for some \(p, q \in \mathbb{R}\).

**CASE-5**: \((A_0, B)\) controllable, rank \(B = m\).

Then the order of \(m\) for the system to be FSA or ESA is determined as follows:

**Proposition 4.2**: Let \((A_0, B)\) be controllable and rank \(B = m\).

Then the system is FSA (and also ESA) if \(m \geq (2n + 1)/3\).

**Derivation**. Let \(K_0\) be such that \(P_0^{-1} A_0 P_0 = \text{diag}[\alpha_0, \alpha_0, \ldots, \alpha_{n}]\), where \(\alpha_0, i \in \mathbb{R}, i = 1, 2, \ldots, n\) is a desired set. Then it is required that a matrix \(K_1 P_0 = \hat{K}_1 \in \mathbb{R}^{m \times n}\) be determined such that \(P_0^{-1} [A_1 P_0 + B \hat{K}_1]\) is at least triangularised with arbitrary diagonal elements. Let

\[
B = \text{Colm }[B_1, B_2], \quad \hat{K}_1 = [K_3, K_2]
\]

where \(B_1, K_3\) are \(((n-m) \times m)\), and \(B_2, K_2\) are \((m \times m)\) real matrices. It is observed that although effectively \(n(n+1)/2\) elements of \(A_1\) are to be controlled for FSA the condition \(n m \geq n(n+1)/2\) may not be sufficient for this purpose. Then expanding and partitioning \(P_0^{-1} [A_1 P_0 + B \hat{K}_1]\) in the form of \(\begin{bmatrix} A_2 & A_3 \\ A_4 & A_5 \end{bmatrix}\) following observation can be made:

(i) \((n-m) (n-m+1)/2\) elements of \(A_2 \in \mathbb{R}^{(n-m) \times (n-m)}\) are to be changed by proper choice of \(m(n-m)\) elements of \(K_3\), giving the estimate \(m \geq (n + 1)/3\);
and (ii) all the elements of $\Lambda_2 \in \mathbb{R}^{(n-m) \times m}$ together with $m(m + 1)/2$ elements of $\Lambda_2 \in \mathbb{R}^{m \times m}$ are to be changed by proper choice of $m^2$ elements of $K_2$, giving the estimate $m \geq (2n + 1)/3$.

Thus for FSA (and so for ESA) $m \geq (2n + 1)/3$. This restriction on the order of matrix $B$ is significant for higher order systems, for example if $n = 2, 3, 4, 10, 50$ and $100$ then $m = 2, 3, 3, 7, 34$ and $67$ respectively. If instead the diagonal elements of $P_0^{-1} \Lambda_1 P_0$ are not to be changed then $m \geq (2n - 1)/3$ is the estimate, and system will only be ESA provided of course these elements are 'small' enough to be compatible for stability with $\{a_{0i}\}$. The gains can then be obtained by an iterative process.

CASE-6: $(A_0, B)$ and $(A_1, B)$ controllable, rank $B = m$. Let there exist gain matrices $K_0$ and $K_1$ such that $\Lambda_0$ and $\Lambda_1$ commute (i.e., $\Lambda_0 \Lambda_1 = \Lambda_1 \Lambda_0$).

In that case, Lemma 4.7 ([113; p 56], [143; pp 215-241]):

If $\Lambda_0$ and $\Lambda_1$ commute then a single nonsingular transformation (e.g., the modal matrix of $\Lambda_0$ or $\Lambda_1$) simultaneously diagonalises them, with eigenvalues of $\Lambda_0$ and $\Lambda_1$ as their respective diagonal elements.

Without loss of generality let the control matrix and the closed loop system matrices be partitioned (e.g., by a nonsingular transformation) as
\[
B = \begin{bmatrix}
1 & \cdots & 0 \\
0 & & \\
\vdots & & \\
0 & & 1
\end{bmatrix}, \quad \bar{A}_0 = \begin{bmatrix}
\bar{A}_{01} & \bar{A}_{02} \\
\bar{A}_{03} & \bar{A}_{04}
\end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix}
\bar{A}_{11} & \bar{A}_{12} \\
\bar{A}_{13} & \bar{A}_{14}
\end{bmatrix}
\]

where the lower blocks \( A_{0j} \) and \( A_{1j} \), \( j = 3, 4 \) are respectively same as those of the identically partitioned \( A_0 \) and \( A_1 \) matrices. Also partitioning \( K_0 \) and \( K_1 \) as

\[
K_0 = \begin{bmatrix}
m & n-m \\
m & n-m
\end{bmatrix}, \quad K_1 = \begin{bmatrix}
m & n-m \\
m & n-m
\end{bmatrix}
\]

obtain

\[
\bar{A}_{01} = A_{01} + B_1 S_0, \quad \bar{A}_{02} = A_{02} + B_1 S_0, \quad \bar{A}_{11} = A_{11} + B_1 S_1
\]

and \( \bar{A}_{12} = A_{12} + B_1 S_1 \).

Following result may be stated.

Proposition 4.3: Let structural constraints on \( A_0 \) and \( A_1 \) be

(i) \( n \leq 2m \)

(ii) \( A_{04} A_{14} = A_{14} A_{04} \)

(iii) \( A_{13} = \alpha A_{03}, \alpha \in \mathbb{R} \)

and also let \( K_0 \) and \( K_1 \) be such that

(iv) \( \bar{A}_{01} \bar{A}_{11} = \bar{A}_{11} \bar{A}_{01} \)

(v) \( \alpha \bar{A}_{02} = \bar{A}_{12} \).

Then (a) \( \bar{A}_0 \bar{A}_1 = \bar{A}_1 \bar{A}_0 \);

(b) \( K_0 \) and \( K_1 \) can further be chosen so that pairs \( (\lambda_i(\bar{A}_0), \lambda_i(\bar{A}_1)), i = 1, 2, \ldots, n \) are 'almost' arbitrary, in the
sense that the choice can be varied over a reasonably wide range, if not arbitrary.

(c) Therefore the system is ESA.

Verification: The commutating condition (a) involves \(n^2\) equations for \(2nm\) unknowns in the elements of \(K_0 = \{(K_0)_{ij}\}\) and \(K_1 = \{(K_1)_{ij}\}\) in which the set of \(mn\) nonlinear (in elements of \(K_0\) and \(K_1\)) equations, by blockwise multiplications are

\[
(m^2) : \overline{A}_{01} \overline{A}_{11} + \overline{A}_{02} \overline{A}_{13} = \overline{A}_{11} \overline{A}_{01} + \overline{A}_{12} \overline{A}_{03} , \quad (4.45)
\]

\[
(m(n-m)) : \overline{A}_{01} \overline{A}_{12} + \overline{A}_{02} \overline{A}_{14} = \overline{A}_{11} \overline{A}_{02} + \overline{A}_{12} \overline{A}_{04} . \quad (4.46)
\]

Similarly, the set of \(n(n-m)\) linear equations in the same elements are

\[
(m(n-m)) : \overline{A}_{03} \overline{A}_{11} + \overline{A}_{04} \overline{A}_{13} = \overline{A}_{13} \overline{A}_{01} + \overline{A}_{14} \overline{A}_{03} . \quad (4.47)
\]

\[
(n-m)^2 : \overline{A}_{03} \overline{A}_{12} + \overline{A}_{04} \overline{A}_{14} = \overline{A}_{13} \overline{A}_{02} + \overline{A}_{14} \overline{A}_{04} . \quad (4.48)
\]

At the outset the purpose is to 'avoid' solving nonlinear equations (4.45)-(4.46) so that a set of 'real' solutions may be obtained; i.e., to satisfy (4.45)-(4.46) through satisfaction of (4.47)-(4.48). Assumption (i) is for obtaining more number of linear equations than number of unknowns (= \(2nm\), in the elements of \(K_0\) and \(K_1\)) so that 'infinitely-many' solutions may be available for possible manipulations later. Now under assumptions (i)-(iv), relations (4.45) and (4.46) are automatically satisfied. Relation (4.46) assumes (by using (v) and manipulating) the form
\[ M \overline{A}_{02} = \overline{A}_{02} N \]  

(4.49)

where \((n,m)\) - matrix \(M = (a_{01} - \overline{A}_{11})\) and \((n-m)\) - matrix \(N = (a_{04} - \overline{A}_{14})\). Similarly relation (4.47) assumes

\[ A_{03} M = N A_{03} \]  

(4.50)

Thus, iterative solution of comutative condition (iv), relations (4.49) and (4.50) simultaneously are required, so as to select proper values of \(K_{1}\) to effect the BSA.

It is to observe that matrix \(M\) has its \((i,j)\)th element as a linear combination of those of \(\overline{A}_{01}\) and \(\overline{A}_{11}\). So there are \(m^2\) - pairs (or total \(2m^2\)) of unknowns, in elements of \(K_{0}\) and \(K_{1}\), for \(m(n-m)\) linear equations of (4.50); and the number of linearly independent equations of these are governed by ranks of \(A_{03}\) and \(N\). Assumption (i) assures the required freedom of choice.

Following algorithm is suggested.

**Algorithm 4.2**

1. **Step-1**: Since \((A_{0},B)\) and \((A_{1},B)\) are controllable find constraints on \(K_{0}\) and \(K_{1}\) so that eigenvalues of \(\overline{A}_{0}\) and \(\overline{A}_{1}\) are real, distinct, compatible for stability (Defn. 4.5), and \(\lambda_{1}(\overline{A}_{0})\) are 'sufficiently large' negative (e.g., \(\leq \beta_{0}, \beta_{0} > 0\) arbitrary), \(\lambda_{1}(\overline{A}_{1})\) are 'sufficiently small' (e.g., \(\leq \beta_{1}, |\beta_{1}| < 1\)). Let these be denoted by the sets \(K_{0}\) and \(K_{1}\) respectively.

2. **Step-2**: Find \(N\) and solve (4.50) to find the region of feasible solutions for the \(m^2\) - pairs of unknowns of the form \(\{(a_{0})_{ij} - (K_{1})_{ij}\} = (K_{ij})\). Let this set of solutions be denoted by \(\mathcal{K}_{1}\).
Step-3: Find subsets \( \{ S_0 \} \subseteq \mathcal{K}_0 \) and \( \{ S_1 \} \subseteq \mathcal{K}_1 \) so that \( \overline{A}_0 \) and \( \overline{A}_1 \) commute (e.g., they are diagonal matrices).

Step-4: Go to Step 2 and by iteration with Step 3 find the \( \{(\overline{R})_{ij}\} \subseteq \mathcal{K}_0 \), \( \{S_0 \}_{ij} \subseteq \mathcal{K}_0 \), \( \{S_1 \}_{ij} \subseteq \mathcal{K}_1 \).

Step-5: Solve, as in Step 2, equation (4.49) for \( M \) using \( (\overline{K})_{ij} \) of Step 4. Find sets \( \{(G_0)_{ij}\} \subseteq \mathcal{K}_0 \) and then \( \{(G_1)_{ij}\} \subseteq \mathcal{K}_1 \) using assumption (v).

Step-6: Iterate and find a wide range of suitable \( K_0, K_1 \) and \( \lambda_1 (\overline{A}_0), \lambda_1 (\overline{A}_1) \).

It was observed that although complicated to look at, the procedure worked well for systems of low order. In fact, systems upto 4th order can be worked out using a pocket calculator.

4.2.11 Examples

Example 4.1. Let

\[
A_0 = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad h = 1.
\]

Although on direct triangularisation, the closed loop CQP is factorised the system cannot be stabilised due to the presence of the factor \((\lambda - 2 - 4 e^{-\lambda})\). However since condition CASE-1 B(v) holds the stabilising (and CQP-factorising) gains \( K_0, K_1 \) exist. From the expression for \( D_2 \), \( (\lambda_4 - \lambda_2) = 2 \) and \( \lambda_2/\lambda_4 \) slope on the Hayes' diagram (Lemma 4.5) gives a choice \( \lambda_2 = -2, \lambda_4 = 0 \). Choosing arbitrarily \( \lambda_1 = -4, \lambda_3 = 1 \) one gets \( K_0 = [-9, -12], w_2 = -1/2 \).
\[ w_4 = -\frac{3}{4}. \] From \( P_0^{-1} \bar{A}_1 P_0 \) to make \( \lambda_4 = 0, \bar{a}_{12} = 0 \) one gets \( K_1 = [-5, -5] \), and so \( \bar{H} = (\lambda + 2) (\lambda + 4 + e^{-\lambda}) \) which is stable.

Example 4.2

Matrices \( A_0, \bar{A}_1 \) are same as in Example 4.1 but now \( b = \text{Colm}[0, 1] \). With similar argument and since condition CASE-1 C(v) is now satisfied \((4\lambda_4 - \lambda_2) = 7\). The \( \lambda_2/\lambda_4 \) slope gives a choice \( \lambda_4 = 0, \lambda_2 = -7 \). Selecting arbitrarily \( \lambda_1 = -4, \lambda_3 = -1 \) it can be easily found out that \( K_0 = [-13, -14], K_1 = [-9, -7], w_2 = -5/4, w_4 = -2 \), and so \( \bar{H} = (\lambda + 7) (\lambda + 4 + e^{-\lambda}) \) which is stable.

Example 4.3. Let

\[
\begin{align*}
A_0 &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \\
A_1 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \\
b &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
h &= 1.
\end{align*}
\]

This example is from Kamen [27] where a complicated control law involving distributed delays etc. places the closed loop eigenvalues at \((-1, -2)\). However since condition CASE-1 C is met it can be found out that with

\[
K_1 = [-1, -1], \quad P_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix},
\]

\( \lambda_3 = 0, \lambda_4 = -1 \) (arbitrary), \( \lambda_1 = -4, \lambda_2 = 0 \) (fixed),

\( K_0 = [-4, -5] \) and the CQP \( \bar{H} = (\lambda + 4) (\lambda + e^{-\lambda}) \). Thus the system has a DS \( \approx 0.303 \). But if \( \lambda_4 = -e^{-1} \) is chosen with \( \lambda_1 \) and \( \lambda_2 \) unchanged, greatest DS of 1 is achieved. Then \( \bar{H} = (\lambda + 4) (\lambda + e^{-1} \cdot e^{-\lambda}) \) and the first six roots are:

\(-1, (-3.08875 + j 7.4615), (-3.66448 + j 13.879), (-4)\).
Example 4.4. Let

\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
0 & 1 & 1
\end{bmatrix}, \quad \begin{bmatrix}
2 & 4 & 5 \\
3 & 1 & 2 \\
0.5 & 5
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 \\
0 & 2 \\
0 & 0
\end{bmatrix}.
\]

\((A_0, B)\) and \((A_1, B)\) are controllable. For the \((A,B)\) assumptions (Prop. 4.3) (i) and (ii) are automatically satisfied and (iii) is satisfied with \(\alpha = 0.5\). Choice of \(K_0\) so that \(A_0\) has arbitrarily chosen eigenvalues \(-2, -3\) and \(-4\), but at the same time \(A_{11}\) be diagonal and the element \((A_0)_13\) be zero gives

\[
\begin{bmatrix}
-4 & 0 & 0 \\
0 & -6 & -12 \\
0 & 1 & 1
\end{bmatrix}
\]

with \(P_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & -3 \\ 0 & 1 & 1 \end{bmatrix}\).

Solving for (4.49) - (4.50) so that condition (iv), (v) are also satisfied by making \(A_{11}\) diagonal, \(A_{12} = \alpha A_{02} = \text{Colm} 0.5 [0, -12]\) obtain

\[
\begin{bmatrix}
-3.5 & 0 & 0 \\
0 & -3 & -6 \\
0 & 0.5 & 5
\end{bmatrix}
\]

with eigenvalues \((-1, -1.5, -3.5)\).

Thus \(F_0^{-1} A_0 P_0\) and \(G_0^{-1} A_1 P_0\) are simultaneously diagonalised, and the CQP is factorised into

\[
H = (\lambda + 4 + 3.5 e^{-\lambda}) (\lambda + 3 + 1.5 e^{-\lambda}) (\lambda + 2 + e^{-\lambda})
\]

each factor of which is stable.
Example 4.5:

Consider the system in Example 3.3 of Chp. 3. It is shown there that the system is stabilisable by state feedback (4.2) if and only if $b_1 \neq -0.5$, $b_2 < 1$. This system is also seen from results of CASE-1 not to be stabilisable by $x(t)$ and $x(t-1)$ feedback whenever either

$b_1 \neq 0$ and $b_2 \neq 0$ or $b_1 = 0$ and $b_2 \neq 0$, but is stabilisable whenever $b_1 \neq 0$, and $b_2 = 0$. However Randolfi [35] showed that this system is even finite spectrum assignable by state feedback (4.2) when $b_1 = 0$ and $b_2 = 1$.

A further example in this connection is the stabilisation of Ross' example [85] solved as Example 2.3 in Chp. 2.

4.3 Observer Designs ([173] - [175])

In this section few observer designs for retarded DDE systems are suggested; those are also based on the ISA principle of stabilising a DDE system. Recent developments on observers for DDE system are reviewed in Sec. 4.1.2. It was emphasized there that due to inherent system complexities such as stabilisation, realisation etc., most of the available designs are either inadequately treated or restrictive in nature or difficult to implement, although theoretically elegant; and this is not surprising. To circumvent these difficulties a few special structures for system matrices are considered that give rise to simple and implementable designs.
4.3.1 Observer for Second Order DDE System [173]

This is a dual problem to CASE-1 of Sec 4.2.1 and analogous conditions are derived. Consider the LTI unforced DDE system of second order \( n = 2 \) with delay normalised to unity

\[
\begin{align*}
\dot{x}(t) &= A_0 x(t) + A_1 x(t-1), \quad t \geq 0, \\
x(\theta) &= \phi(\theta), \quad \theta \in [-1, 0], \\
y(t) &= H x(t) \quad (4.51)
\end{align*}
\]

where state \( x(.) \in \mathbb{R}^2 \), output \( y(t) \in \mathbb{R} \),

\[
A_0 = \begin{bmatrix} a_{01} & a_{02} \\ a_{03} & a_{04} \end{bmatrix}, \quad A_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{13} & a_{14} \end{bmatrix}, \quad H = [h_1 \ h_2],
\]

all elements \( a_{0i}, a_{1j}, \) and \( h_j \in \mathbb{R} \), the initial function \( \phi(\theta) \) is assumed continuous (but usually not precisely known).

(I) Full-order observer:

Referring to Sec. 4.1.2 remark (R01), the full order Luenberger-like observer for system (4.51) is given by

\[
\begin{align*}
\dot{z}(t) &= \bar{A}_0 z(t) + \bar{A}_1 z(t-1) + K y(t) + P y(t-1), \quad t \geq 0, \\
z(\theta) &= \xi(\theta), \quad \theta \in [-1, 0] \quad (4.52)
\end{align*}
\]

where \( z(.) \in \mathbb{R}^2 \), \( K = [K_1, K_2] \), \( P = [P_1, P_2] \); \( K_1, P_1 \in \mathbb{R} \).

Conditions for \( z(t) \) to estimate \( x(t) \) for \( t > 0 \) are

\[
\bar{A}_0 = A_0 + KH, \quad \bar{A}_1 = A_1 + H \quad (4.53)
\]

Due to the initial estimation bias (Sec 4.1.2, (R02)) the observation error \( e(t) = x(t) - z(t) \) is seen evolving from the DDE.
\[ \begin{align*}
\dot{e}(t) &= A_0 e(t) + A_1 e(t - 1), \quad t > 0, \\
e(\theta) &= \phi(\theta) - \xi(\theta), \quad \theta \in [-1, 0].
\end{align*} \] (4.54)

It is now essential that an arbitrary but proper degree of stability (remark (R04)) be given to error dynamics (4.54), and presently by choice of K and P alone.

Here the stabilising procedure followed is similar to (but dualisation of) that in Sec 4.2.1, CASE-1. Proceeding identically (with same terminologies as in CASE 1) to effect the CQR-decomposition of (4.54) as

\[ \det [\lambda I - A_0 - A_1 e^{-\lambda}] = (\lambda - \lambda_1 - \lambda_3 e^{-\lambda}) (\lambda - \lambda_2 - \lambda_4 e^{-\lambda}) \] (4.55)

where \( \lambda_i \in \mathbb{R} \), the following are the conditions analogous to (4.38)-(4.43):

\[ w_2 = (\lambda_1 K_2 - a_{01} K_2 + K_1 a_{03})/(\lambda_1 K_1 + a_{02} K_2 - K_1 a_{04}), \]

\[ w_2 = (\lambda_2 K_2 - a_{01} K_2 + K_1 a_{03})/(\lambda_2 K_1 + a_{02} K_2 - K_1 a_{04}), \] (4.56)

\[ w_4 a_{11} - a_{13} + q_3 h_1 + w_4 (w_4 a_{12} - a_{14} + q_3 h_2) = 0; \] (4.57)

\[ -w_2 a_{11} + a_{13} + q_4 h_1 + w_4 (-w_2 a_{12} + a_{14} + q_4 h_2) = \lambda_4 (w_4 - w_2); \] (4.58)

\[ w_4 a_{11} - a_{13} + q_3 h_1 + w_2 (w_4 a_{12} - a_{14} + q_3 h_2) = \lambda_3 (w_4 - w_2). \] (4.59)
where \( q_3 = w_4 p_1 - p_2 \), \( q_4 = -w_2 p_1 + p_2 \), \( K_1 \neq 0 \), \( K_2 \neq 0 \).

From (4.57) and (4.58) with nonzero \( p_1 \) and \( p_2 \),

\[
  w_4 p_1 (\lambda_4 - a_{14} + (p_2/p_1) a_{12}) = \\
  p_2 (\lambda_4 - a_{11} + (p_1/p_2) a_{13}) = 0. \tag{4.60}
\]

If choice of \( p_1 \) and \( p_2 \) are made such that

\[
  -a_{14} + (p_2/p_1) a_{12} = -a_{11} + (p_1/p_2) a_{13} = D_1. \tag{4.61}
\]

then from equation (4.60)

\[
  \lambda_4 = -D_1, \text{ when } w_4 p_1 - p_2 \neq 0. \tag{4.62}
\]

From this and from the solution of equation (4.61) for the ratio \( (p_1/p_2) \) to be real, following result is stated.

Proposition 4.4: The CQP of system (4.54) is transformable into the decomposable form (4.55) with arbitrary \( \lambda_1 \in \mathbb{R} \) if equation (4.61) is satisfied, and which necessitates choice of \( P = [p_1, p_2] \) such that \( (\lambda_1, P) \) be not controllable. Then a real set \( \{P\} \) exists if and only if

\[
  (a_{11} - a_{14})^2 + 4 a_{12} a_{13} \geq 0. \tag{4.63}
\]

As pointed out earlier the compatibility of \( \lambda_1 \) for stability may be verified from Hayes' result [1; p 444]; also Lemma 4.8 may be used. Following algorithm is suggested.
Algorithm 4.3:
Step-1. With $\lambda_4 = -D_1$ choose $\lambda_2$ so that $(\lambda - \lambda_2 \lambda_4 e^{-\lambda})$ has zeros with $\text{Re} \lambda \leq -\beta$, $\beta > 0$ is the desired degree of stability. Similarly choose $\lambda_1$ and $\lambda_3$ ($\lambda_3$ may be made preferably zero).
Step-2. With $\lambda_1$ and $\lambda_2$ known compute $W_2$, $W_4$, $K_1$ and $K_2$ from (4.56).
Step-3. With $(p_1/p_2)$ fixed from (4.51) find $p_1$ and $p_2$ from (4.59).

(II) Minimal-order observer:

Let for simplicity the output matrix be $H = [1, 0]$. The minimal order observer (MOO) of order one is derived, as in Sec 4.1.2 equations (4.15)-(4.22), to be the scalar DDE system:

$$
\dot{z}(t) = f_0 z(t) + f_1 z(t-1) + g_0 y(t) + g_1 y(t-1), \quad t \geq 0,
$$

$$
z(t) = T x(t), \quad t \in [-1, 0]
$$

and

$$
z(t) = T x(t) : = [t_1, t_2] x(t) \quad (4.64)
$$

where $z(.)$, $y(.)$, $f_i$, $g_i$, $t_i \in \mathbb{R}$. The transformation $T$ has to satisfy

$$
g_0 H + f_0 T = T A_0 ,
$$

$$
g_1 H + f_1 T = T A_1 \quad (4.65)
$$

and the observation error $e(t) := z(t) - T x(t)$, $e(.) \in \mathbb{R}$ evolves from the scalar DDE.
\[ \dot{e}(t) = f_0 e(t) + f_1 e(t-1) \]
\[ e(t) = \xi(t) - T \phi(t), \quad t \in [-1, 0]. \quad (4.66) \]

It was pointed out in remark (Ro3) of Sec. 4.1.2 that the M00 designs are restrictive in nature, because suitable choice of \((g_0, g_1), (f_0, f_1)\) and \((t_1, t_2)\) are to be made to satisfy (4.65) as well as stability conditions for (4.66). In the present case the existence conditions are obtained as relations between matrix elements \(\{a_{01}\}\) and \(\{a_{11}\}\). On expanding equations (4.65) it is seen that for these conditions to be satisfied the following relationship must hold:

\[
\begin{bmatrix}
a_{01} & f_0 & a_{03} & -1 & 0 \\
a_{02} & a_{04} & f_0 & 0 & 0 \\
a_{11} & f_1 & a_{13} & 0 & -1 \\
a_{12} & a_{14} & f_1 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
t_1 \\
t_2 \\
g_0 \\
g_1 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\end{bmatrix} \quad (4.67)
\]

For nontrivial solution of (4.67) the coefficient matrix should be singular which obtains the condition to be satisfied as

\[ a_{02} f_1 - a_{12} f_0 = a_{02} a_{14} - a_{04} a_{12} : = D. \quad (4.68) \]

The problem is thus to find \(\{f_0, f_1\}\) to satisfy (4.68) and at the same time to meet with the desired stability conditions for the observer dynamics; that is, the CQP \((\lambda - f_0 - f_1 e^{-\lambda})\) has all zeros with real parts less than a fixed negative quantity. Then a \(\{t_1, t_2, g_0, g_1\}\) from the solution of (4.67) is chosen.

The desired functional relationship between elements of \(A_0\) and \(A_1\) matrices available through (4.68) can be obtained
once equation (4.68) is compared with the expression for $D_2$ in the Proposition 4.1(B) where $D_2, \lambda_2, \lambda_4, a_0^2$ and $a_1^2$ in the latter is replaced by $D, f_0, f_1, a_0^2$ and $a_1^2$ respectively in the former. These are summarised below. However in addition to Lemma 4.5 being useful for this design, usually for better degree of stability (DS) $f_0 < 0$. In such case a further helpful result for selection of parameters is Lemma 4.8: Let the DDE system (4.66) be stabilised by choosing $f_0 < 0$ and $f_1 = 0$. Then an additional DS upto maximum of 1 may be obtained if $f_1 < 0$ and $f_1 \exp (-f_0) \to \exp (-1)$.

Proof: The simple proof is obtained from results of Chapter 3.

Proposition 4.5: Let $f_0 = -\beta, \beta > 0$ be chosen for better DS. Then the observer system (4.64)-(4.68) has a stabilising solution under the following conditions.

(i) If $a_0^2 \neq 0, a_{12} = 0 : f_1 = a_{14}$ and $\beta > |a_{14}|$.

(ii) If $a_0^2 = 0, a_{12} \neq 0 : f_1 = \text{arbitrary}, \beta = -a_{04}, So system is stable iff $a_{04} < 1$; and if $a_{04} = 0, f_1 \in (-\pi/2, 0)$.

(iii) If $a_0^2 = a_{12} = 0 : f_1 = a_{14}, f_0 = a_{04}$. So system is stable iff $(a_{04}, a_{14})$ are compatible for stability.

(iv) If $a_0^2 = a_{12} \neq 0$; let $K = a_{14} - a_{04}$. System is stable if $K \geq 0$ with maximum DS = $K + 1$ and minimum DS = $K$. In case $K < 0$ system is stable iff $|K| < \pi/2$. 

(v) If \( a_{02} \neq 0, a_{12} \neq 0, |a_{02}| > |a_{12}| \), following are the sub-cases:

a) \( D/a_{12} > 0 : \beta = D/a_{12}, f_1 = \text{arbitrary} \)
   
   when \( f_1 = 0, DS = \beta \).

b) \( D/a_{12} = 0 : \beta = -(a_{02}/a_{12}) f_1 \).

c) \( D/a_{12} < 0, a_{12} > 0, a_{02} > 0, D < 0 : \)
   
   \( \beta > D/(a_{12} - a_{02}), \ f_1 = -(|D| + \beta a_{12})/a_{02} < 0; \)
   
   for stability \(|f_1| < \beta\).

d) \( D/a_{12} < 0, a_{12} > 0, a_{02} < 0, D < 0 : \beta > D/(a_{02} + a_{12}), \)
   
   \( f_1 = (D - \beta a_{12})/a_{02} > 0, \ for \ stability \ f_1 < \beta. \)

e) \( D/a_{12} < 0, a_{12} < 0, a_{02} > 0, D > 0 : \beta > D/(a_{02} + a_{12}), \)
   
   \( f_1 = (D - \beta a_{12})/a_{02} > 0; \ for \ stability \ f_1 < \beta. \)

f) \( D/a_{12} < 0, a_{12} < 0, a_{02} < 0, D > 0 : \)
   
   \( \beta > D/(a_{12} - a_{02}), f_1 = (D - \beta a_{12})/a_{02} < 0; \ for \)
   
   stability \(|f_1| < \beta\).

---

Example 4.5: Consider the example by Bhatt and Koivo [37]

in a rearrangement form

\[
A_0 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 \\ 0 & \pi/2 \end{bmatrix}, H = [1, 0].
\]

Condition (4.68) gives \( f_1 = -\pi/2 \) and \( f_0 \) arbitrary. Thus existence of a stable observer is guaranteed. The set of simultaneous equations (4.65) with the selection of \( t_1 = 1, t_2 = -2/\pi \) give a solution: \( g_0 = 6, g_1 = \pi/2 \).

4.3.2 Observer Design When coefficient Matrices Commute[174]

This case is a dual to the CASE-6 of Sec 4.2.1. When the observer parameters are constrained so that its coefficient
matrices corresponding to delayed and non-delayed states commute, then an 'almost' desired degree of stability for it can be achieved. Referring to Sec. 4.1.2 and with a slight change of notation let an identity observer for the unforced system (taken for simplicity)

\[
\begin{align*}
\dot{x}(t) &= A_0 x(t) + A_1 x(t - 1), \, t \geq 0, \, x(.) \in \mathbb{R}^n, \\
x(t) &= \phi(t), \, t \in [-l, 0], \, \phi \in \mathcal{C}, \\
y(t) &= C x(t), \, y \in \mathbb{R}^p
\end{align*}
\]

be given by

\[
\begin{align*}
\dot{z}(t) &= \tilde{A}_0 z(t) + \tilde{A}_1 z(t - 1) + 3 y(t) + H y(t - 1), \, t \geq 0, \\
z(t) &= \tilde{\xi}(t), \, t \in [-l, 0], \, z \in \mathbb{R}^n, \, \tilde{\xi} \in \mathcal{C}
\end{align*}
\]

(4.69)

where the matrices are real and appropriately dimensioned.

The observation error \( e(t) = x(t) - z(t) \) evolves from the DDE

\[
\begin{align*}
\dot{e}(t) &= \tilde{A}_0 e(t) + \tilde{A}_1 e(t - 1), \, t \geq 0, \\
e(t) &= \phi(t) - \tilde{\xi}(t), \, t \in [-l, 0]
\end{align*}
\]

(4.70)

and where \( \tilde{A}_0 = A_0 - GC, \tilde{A}_1 = A_1 - HC \). Now, using Lemma 4.7 the following result can be stated based on Proposition 4.3.

**Proposition 4.5**: Let for the system (4.69), \((A_0, C)\) and \((A_1, C)\) be observable. Let there exist a set of \((n,p)\) matrices \( G = (g_{ij}) \) and \( H = (h_{ij}) \), \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, \ldots, p \), such that matrices \( \tilde{A}_0 \) and \( \tilde{A}_1 \) commute. Then an observer given by equation (4.70) is constructible having error evolving from equation (4.71) with an 'almost' arbitrary decay rate, by which is meant that decay rate can be varied over a reasonably wide range.
As discussed earlier, whenever the commuting relation
\( \tilde{A}_0 \tilde{A}_1 = \tilde{A}_1 \tilde{A}_0 \) is satisfied, a nonsingular transformation \( P_0 \) (or \( P_1 \)) diagonalises simultaneously both matrices with their respective eigenvalues \( \sigma(\tilde{A}_0) = \{\lambda_{0i}\} \) and
\( \sigma(\tilde{A}_1) = \{\lambda_{1i}\}, i = 1,2, \ldots, n \) as diagonal elements.

Then the required decomposition of the CQP \( \tilde{H} = \text{det} [\lambda I - \tilde{A}_0 \tilde{A}_1 e^{-\lambda}] \) is effected. The control is exercised through \( G \) and \( H \) alone, but their 'working ranges are limited' due to the nature of the problem. However, the observability conditions and special structures of system matrices to be chosen analogous to those described in Proposition 4.3, can still allow a good freedom (if not arbitrary) in the choices of \( \sigma(\tilde{A}_0) \) and \( \sigma(\tilde{A}_1) \) which are compatible with the overall stability requirements (Defn. 4.7).

The transformed error equation is now
\[
\dot{w}(t) = D_0 w(t) + D_1 w(t - 1)
\]
(4.71a)

where \( D_0 = \text{diag} (\lambda_{01}, \lambda_{02}, \ldots, \lambda_{0n}) \), \( D_1 = \text{diag} (\lambda_{11}, \lambda_{12}, \ldots, \lambda_{1n}) \) and \( e(t) = P_0 w(t) \) (or \( = P_1 w(t) \)). With trial governed by the following suggested algorithm, at least the most dominant zero of each scalar factor of \( \tilde{H} \) (and so most dominant system eigenvalue) can be 'almost arbitrarily' fixed.

Algorithm 4.4 (after Algorithm 4.2):
Step-1. For satisfaction of \( \tilde{A}_0 \tilde{A}_1 = \tilde{A}_1 \tilde{A}_0 \) find the relationship between \( (g_{ij}) \) and \( (h_{ij}) \).
Step-2. From the characteristic equations of $\tilde{\mathbf{A}}_0$ and $\tilde{\mathbf{A}}_1$ in the respective indeterminate $(g_{ij})$ and $(h_{ij})$ find their ranges such that (i) all coefficients are positive, (ii) coefficients concerning $\tilde{\mathbf{A}}_0$ are much larger than corresponding ones of $\tilde{\mathbf{A}}_1$ (a desirable condition for better result).

Step-3. Choose 'arbitrary' $\{\lambda_{0i}\}$ to be large negative reals or $\{\lambda_{1i}\}$ to be small negative reals (the real eigenvalue conditions are naturally satisfied if either set is chosen real).

Step-4. Form the $n$ number of scalar CQPs $h_i = \lambda + \lambda_{0i} + \lambda_{1i} e^{-\lambda}, i = 1, 2, \ldots, n$, and compute eigenvalues (Chap. 3 Appendix). If the desired degree of stability is obtained, stop. If not, go to Step 3 and iterate.

Example 4.6: Let $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$

$$
\tilde{\mathbf{A}}_0 = \begin{bmatrix} g_1 & 0 & 2 \\ g_2 & -2 & 0 \\ g_3 & 4 & -2 \end{bmatrix}, \quad \tilde{\mathbf{A}}_1 = \begin{bmatrix} h_1 & 0 & 1 \\ h_2 & 2 & 0 \\ h_3 & 2 & 2 \end{bmatrix}
$$

where $g_i$ and $h_i \in \mathbb{R}, i = 1, 2, 3$ are elements of vectors $\mathbf{G}$ and $\mathbf{H}$ respectively. Conditions to be satisfied for $\tilde{\mathbf{A}}_0$ and $\tilde{\mathbf{A}}_1$ to compute are found to be $g_1 = 2h_1 - 6, g_2 = 2h_2, g_3 = 2h_3$. For the characteristic equation of $\tilde{\mathbf{A}}_0$ and $\tilde{\mathbf{A}}_1$ to have all positive coefficients, the ranges are $h_1 < -4, h_2 < -12, h_3 < -20$. Let $\sigma(\tilde{\mathbf{A}}_1)$ be chosen as $(-.5, -1.0, -1.5)$ which gives $h_1 = -7, h_2 = -13.125, h_3 = -26.75,$ and
so $g_1 = -20$, $g_2 = -26.25$, $g_3 = -53.5$. Thus, $\sigma(\tilde{A}_0) = (-7, -8, -9)$. The corresponding first six zeros of each CQP (i.e., first 18 eigenvalues of error equation) are approximately

$$(-2.3664 \pm j 2.626, -2.924 \pm j 8.31, -3.361 \pm j 14.38);$$
$$(-1.898 \pm j 2.722, -2.333 \pm j 8.45, -2.748 \pm j 14.485);$$
$$(-1.65 \pm j 2.78, -1.993 \pm j 8.541, -2.358 \pm j 14.565).$$

So a degree of stability of 1.65 is achieved.

A second choice of $h_1 = -4.7633$, $h_2 = -5.663238$, $h_3 = -15.195$ for $\sigma(\tilde{A}_1) = (-0.0283, -0.235, -0.5)$ gives $\sigma(\tilde{A}_0) = (-5.965 - 6.7384, -6.823139)$. Corresponding zeros of the CQP are $(-4.521 \pm j 2.16, -5.6012 \pm j 7.9, -6.207 \pm j 14.12); (-2.9623 \pm j 2.548, -3.623 \pm j 8.216, -4.206 \pm j 14.318); (-2.339 \pm j 2.6138, -2.90768 \pm j 8.295, -3.3846 \pm j 14.32);$ and a degree of stability of 2.339 is achieved.

4.3.3. An Approximate Observer [175]

Another set of structural constraints on DOE system matrices are found out that allow the required factorisation of the closed-loop CQP, and thereby guaranteeing existence of a simple and easily realisable observer with arbitrary degree of stability. Then, as in earlier cases the complete system spectrum is computable. This in turn motivates construction of a finite-dimensional (delay free) system based on the SDT of Chp. 3. However, this approximate-observer is used to approximately estimate instantaneous states $x(.)$ of varied delayed arguments; and in
many simple cases the true observer (a DDE system) and the
Approximate Observer (AO) may be made arbitrarily close in
performances. The AO has only limited utility in which case
an AO-based controller is suggested.

Consider the DDE'system (4.69). Let
(a_1) (A_0, C) be observable.
(a_2) A_1 = LC + a I_n for some \( L \in \mathbb{R}^{n \times p} \), \( a \in \mathbb{R} \).
(a_3) System be stabilisable by \( x(t) \) and \( x(t-1) \) feed-
backs, \( i \in (0, 1] \).

The assumptions (a_1) and (a_2) are necessary for construction
of the observer while (a_3) will be necessary for the control
problem based on the estimated states. Further, some degree
of arbitrary spectrum assignment is possible by the feedback
of \( x(t - j) \), \( j = 1, 2, \ldots \) if (in addition) system (4.69)
is weakly controllable. Let \( \tilde{\Lambda}_0 := A_0 - GC \), \( \tilde{\Lambda}_1 := A_1 - HC \),
\( P_0 \) = modal matrix of \( \tilde{\Lambda}_0 \), \( \sigma(S) \) represents the system spectrum,
and \( \sigma(h_i) \) represent the zeros of \( \lambda - \lambda_1 - ae^{-\lambda} \).

Lemma 4.9: Let assumptions (a_1) and (a_2) be satisfied. Then
there exist real \((n, p)\) matrices \( G \) and \( H = L \) such that
\( \sigma(\tilde{\Lambda}_0) = \{\lambda_1\} \), \( \lambda_1 \in \mathbb{R} \) are arbitrary, \( \tilde{P}_0^{-1} \tilde{\Lambda}_0 \tilde{P}_0 =
diag[\lambda_1, \lambda_2, \ldots, \lambda_n] \), and \( \tilde{\Lambda}_1 = \alpha I_n = \tilde{P}_0^{-1} \tilde{\Lambda}_1 \tilde{P}_0 \).

Further, the closed loop CSSP is factorised into

\[ H(\lambda, e^{-\lambda}) = \prod_{i=1}^{n} (\lambda - \lambda_1 - ae^{-\lambda}) \]

and \( \sigma(S) = \bigcup_{i=1}^{\tilde{n}} \sigma(h_i) \) where \( \bigcup \) denotes the union with
any common elements repeated.
The proofs of above results are straightforward.

Now consider the observer equations (4.70) and the corresponding error dynamics evolving from the DDE (4.71). In view of above lemma following is the existence result.

Proposition 4.7: Under conditions of Lemma 4.2 an observer (4.70) can be constructed with estimation error dynamics evolving from the DDE (4.71) such that the error has an arbitrary decay rate.

Proof: Since \( a \) is known a set of \( \lambda_i^* \subseteq \mathbb{R} \) can be chosen on the basis of Lemma 4.5, so that \( h_i = \lambda - \lambda_i^* - ae^{-\lambda} \) for each \( i = 1, 2, \ldots, n \) has the largest zero arbitrarily (but 'properly'; as discussed in Sec 4.1.2 (R04)) fixed. As \((A_o, C)\) is observable these \( \lambda_i^* \) can be assigned as eigenvalues of \( \tilde{A}_o \). Thus the transformed \((w := P_o^{-1} e)\) error system is given by

\[
\dot{w}(t) = [\text{diag}(\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*)]w(t) + a I_n w(t-1),
\]

which is SSA.

Example 4.7 [37]:

\[
A_o = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} -\pi/2 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = [0, 1].
\]

Choosing \( H = \text{Colm}[0, \pi/2] \) and splitting

\[
\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & \pi/2 \end{bmatrix} + (-\pi/2) I_2 \text{ obtains } \tilde{\Lambda} = (-\pi/2 I_2).
\]

Since \((A_o, C)\) is observable arbitrary eigenvalues can be assigned to \( \tilde{A}_o \). Thus \( H = (\lambda - \lambda_1^* + \pi/2 e^{-\lambda}, \lambda - \lambda_2^* + \pi/2 e^{\lambda}) \).

Specifying the desired degree of stability, choice of
\( \lambda_1^* \) and \( \lambda_2^* \) (and so the matrix \( G \)) may be made following Chap. 3, Appendix. For example the largest zero \( \hat{\zeta}_0 \) of 
\( (s - a_0 + \pi/2 e^{-s}) \) has real parts: 0, -0.2723, -1.087, -1.7196, -2.136 respectively for \( a_0 = 0, 1, -5, -10, -15 \).

**Approximate realisation:**

Once a proper selection of \( G \) and \( H \) is made for designing an asymptotic observer \((4.70)\) an 'approximate' realisation scheme can be considered. The observer, even if it is a simple DDE system (compared with the integro-differential equation of Bhat and Koivo [37]), needs delay lines and memories for its construction. Taking into account the assumption \((a_3)\) an approximate realisation scheme estimating instantaneous states \( x(t - i/M) \), \( i = 0, 1, 2, \ldots \), \( M \) may be reasonably accepted in certain cases ([19], [85]). The principle adopted is the spectral decomposition technique of Chap. 3.

In the present case the observer system eigenvalues \( \{ \tilde{\lambda}_1 \} \) are 'completely' known, and moreover those of them to be handled in the SDT are stable ones (through the choice of \( G \) and \( H \)). Following the notations of Chap. 3 Sec. 3.3, the observer DDE system \((4.70)\) when projected onto the finite-dimensional subspace \( \mathcal{L}_P \) obtained through the spectral decomposition of the state space \( \mathcal{L} \) by \( \lambda = \{ \tilde{\lambda}_1, \tilde{\lambda}_2, \ldots \} \), satisfies the ODE (and is the approximate model of the observer)

\[
\dot{\xi}(t) = A_N \xi(t) + \bar{V}_N(0) \left[ G y(t) + H y(t-1) \right]. \quad (4.72)
\]
The value of $N$, that governs the dimension of the approximate model, is computed using the algorithm suggested in Sec 3.5. The estimated values of the observer instantaneous states $\hat{z}(t - i/M)$, $i = 0, 1, \ldots, M$ are given by

$$\hat{z}(t) = \phi_N(0) \xi(t),$$

$$\hat{z}(t - j) = \phi_N(-j) \xi(t),$$

$$\hat{z}(t - l) = \phi_N(-l) \xi(t).$$

(4.73)

It is to remark that the Repin-type approximation scheme discussed in Chaps. 1 and 2 also needs 'auxiliary' states those approximate $x(t - ih/M)$ for control. In that case the above type observer of relatively low order can be used.

Example 4.8:

Consider as a simple case the scalar system in Examples 3.1 and 3.2, i.e. $x(t) = \frac{\pi}{2} x(t - 1), x(t) = 1, t \in [-1, 0]$. Retaining first 4 eigenvalues ($N = 4$) the actual values $x(t)$ and their estimated values $\hat{x}(t)$ obtained from approximations (4.73) for various values of $t$ are given by

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x(t)$</th>
<th>$\hat{x}(t)$</th>
<th>$\hat{x}(t-1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0.95641</td>
<td>-</td>
</tr>
<tr>
<td>0.1</td>
<td>-0.5707963</td>
<td>-0.5716941</td>
<td>0.917416</td>
</tr>
<tr>
<td>2</td>
<td>-0.9078921</td>
<td>-0.9076631</td>
<td>-0.5745222</td>
</tr>
<tr>
<td>3</td>
<td>0.5764491</td>
<td>0.5764626</td>
<td>-0.90655763</td>
</tr>
<tr>
<td>4</td>
<td>*</td>
<td>*</td>
<td>0.5768135</td>
</tr>
</tbody>
</table>
4.3.3 1 An approximate-observer Based Controller:

When the observer system is approximately realised as given by (4.72) -- (4.73) its output may be thought of being 'applied' to an identically approximated system model obtained via the SDT described in Chp. 3. Let the reduced order model of the system and the observer be of dimensions $N_s$ and $N_o$ respectively. Preferably $N_s$ is assumed large for better approximation and $N_o$ is small, since the observer is to be 'physically constructed'. Further, as assumed earlier only RF control is being used for the DDE system and let it be of the form

$$u(t) = \sum_{i=0}^{q} K_i x(t - \tau_i), \tau_i \in [0, 1], \quad K_i \in \mathbb{R}^{m \times n}_i$$ (4.74)

Following the notations of Chp. 3, let the control law for the ROM, for example that obtained by solution of a state regulator problem in $\mathcal{G}_p$ of dimension $N_s$, be of the form (Eqn. (3.48))

$$u_s(t) = K Y_N(t), \quad K \in \mathbb{R}^{m \times N_s}_s$$ (4.75)

Again, when approximated in $\mathcal{G}_p$ the control law (4.74) obtained from equation (3.43a) is

$$u_a(t) = \sum_{i=0}^{q} K_i \phi_{N_s}(-\tau_i) Y_N(t) := K_N \bar{\phi}_N(t)$$ (4.76)

where $K_N = [K_0, K_1, \ldots, K_q]$ and $\bar{\phi} = \text{Colm} \left[ \phi_{N_s}(-\tau_0) \right. \left., \ldots, \phi_{N_s}(-\tau_q) \right]$ are real matrices of dimension $(m \times n(q+1))$ and $(n \times (q+1), N_s)$ respectively. To be compatible equations
(4.75) and (4.76) satisfy $K_N \tilde{\Phi} = K$. Since $\tilde{\Phi}$ is $(n(q+1), N \! s)$ matrix whose columns are eigenfunctions, above equation has a solution for $K_N$ if $n(q+1) = N \! s$; that is $N \! s$ has to be an integral multiple of $n$. In that case $K_N = K \tilde{\Phi}^{-1}$. Further, depending on system eigenvalues $N \! s$ may be even or odd.

Therefore once $N \! s$ is known, starting with (say) $q = 1$, values of $N \! s$ and $q$ may be increased until $n(\tilde{q} + 1) = \tilde{N} \! s$ is satisfied where $\tilde{N} \! s$ and $\tilde{q}$ are respective final values.

Example 4.9: Let $n = 3$, $N \! s = 10$. If system eigenvalues $\lambda_{11}$ and $\lambda_{12}$ are a complex conjugate pair then $\tilde{N} \! s = 12$ and $\tilde{q} = 3$.

In case $\lambda_{11}$ is real and $\lambda_{12}$ to $\lambda_{15}$ are complex and pairwise conjugate then $\tilde{N} \! s = 15$ and $\tilde{q} = 4$.

The observer instantaneous states $\hat{z}(\cdot)$ were shown to be approximated by relations (4.73) where $\xi(t)$ is the observer-terminated system states. Now with renewed $\tilde{N} \! s$ and $\tilde{q}$ fresh values of $K$ and $\tilde{K}_N$ can be computed in which case the approximated control law based on this observer is given by

$$\tilde{u}_a(t) = \tilde{q} \sum_{i=0}^{\tilde{q}} \left[ \tilde{K}_i \tilde{\Phi}_N \operatorname{sign}(i) \right] \xi(t) \quad (4.77)$$

where $\tilde{K}_i$ are constituent blocks of $\tilde{K}_N$.

Few simple examples are given below to illustrate the above procedure. The system is the scalar DDE in Example 4.8 and the curtailed system is of second order as in Example 3.4, retaining only the eigenvalues $\pm j\pi/2$. In what follows $\lambda_i$ and $\tilde{\lambda}_i$ will represent respectively the DDE system and the curtailed system eigenvalues, the given values
of which are only approximate.

Example 4.10: Consider as stabilising process the scalar system $\dot{x}(t) = -\frac{\pi}{2} x(t - 1) + u(t)$. Its curtailed system derived in Example 3.4 is given by

$$A_N = \begin{bmatrix} 0 & -\pi/2 \\ \pi/2 & 0 \end{bmatrix}, \quad B_N = \begin{bmatrix} \pi d \\ 2d \end{bmatrix}, \quad d = (1 + \pi^2)^{-1}.$$

Since $(A_N, B_N)$ is controllable, the control law $u_S(t) = K Y_N(t)$ for placing poles of the curtailed system (say) at $(-2, -3)$ is found and given by $K = \begin{bmatrix} -2.80253, -4.2553 \end{bmatrix}$. Then with $Y(0) = \begin{bmatrix} 0 \end{bmatrix}$ from Example 3.2,

$$K_N = K \Phi^{-1} = K \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -4.2553, 2.80253 \end{bmatrix}. \quad \text{Then with } \Phi(t)$$

The control law for DDE system is then $u(t) = -4.2553 x(t) + 2.80235 x(t - 1)$ giving first three closed loop poles at

$$\lambda_1 = -0.9773, \quad \lambda_{2,3} = -1.556 \pm j 5.191.$$

In case $\lambda_1 = -4$ and $\lambda_2 = -5$ are taken, the gains are given by $K_N = \begin{bmatrix} -13.2663, -0.08302 \end{bmatrix}$ giving rise to first four closed poles at $\lambda_{1,2} = -4.7063 \pm j 2.823, \quad \lambda_{3,4} = -4.9741 \pm j 8.62$.

Example 4.11: Consider the control of above system by the law $u(t) = -2x(t) + 0.5 x(t - 1)$. The closed loop system is stable with first four eigenvalues at $\lambda_{1,2} = -0.80556 \pm j 2.09$, $\lambda_{3,4} = -2.0003 \pm j 7.86$. Then for the curtailed system $K = \begin{bmatrix} -2 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -0.8033 & -2 \end{bmatrix}$ which gives $\lambda_{1,2} = -0.8033 \pm j 1.5649$.

Interestingly, in case $u(t) = -x(t) + \frac{\pi}{2} x(t - 1)$ the closed loop DDE system is delay free and with $\alpha = 10$ its
eigenvalue is $-10$. Thus, for the curtailed system $K = [-10, 0 1]$, $\pi/2]l_{-1 0} = [-\pi/2, -10]$, and so its eigenvalues are only at $\tilde{\lambda}_{1,2} = -3.5956 \pm j1.5322$. Similarly for $\alpha = 4$, $\tilde{\lambda}_{1,2} = -1.8652 \pm j1.805$.

To see how even a small coefficient of $x(t-1)$ in the closed loop system affects the difference ($\lambda - \tilde{\lambda}$) let $u(t) = -10 x(t-1) + (\pi/2 - 0.001) x(t-1)$. Then first two roots are $\tilde{\alpha}_{1,2} = -8.01434 \pm j2.285$ (instead of $\lambda = -10$), whereas the corresponding curtailed system roots remain almost unchanged; i.e., $\tilde{\tilde{\lambda}}_{1,2} = -3.5951505 \pm j1.53562$.

However, since the second order curtailed system is in fact not a good approximation, the above discrepancies may be reduced by increasing its dimension.

4.4 Dynamic Compensator for Second Order DDE System [176]

Dynamic compensation of general infinite-dimensional system (IDS) with a view for application to time-delay systems (TDS) were discussed in Sec 4.1.1 and Sec. 4.1.3. The up to date theoretical developments in this regard were also pointed out. As emphasised earlier, implementation of the state feedback law which is described as an operator involving integrals of $x(t + \theta)$ over $\theta \in [-h, 0]$, or of controllers of infinite order is often inconvenient. The easiest approach for a practicable controller is to replace the IDS by a finite-dimensional (FD) 'reduced order model', and then applying the standard techniques to obtain an FD compensator to 'control' the IDS. The FD controller should also stabilise the IDS.
Other associated problems are emphasised by Halas [81] and Schumacher [117].

Recently there have been a few attempts in developing FD compensators for IDS not based on reduced order models where applications to TDS are pointed out. Notable among them are the works of Rohjolainen [117A] and Schumacher [117]. In this section design of a dynamic compensator is presented for a second order DDE system. The method is based on the SDT of Chp. 3, and on suggestions for a stabilising control in [34], [35], and [66]. Since the SDT is basically dependent on the locations of system zeros, the designs based on this may as well be sensitive to parameter uncertainty. Therefore, for the sake of theoretical clarity it will be assumed that the DDE system to be controlled is precisely known. Motivation for choosing the second order system is that by means of the feedbacks of instantaneous states $x(t)$ and $x(t - h)$ at least the CQP can be factorised into two scalar quasi-polynomials, the zeros of which are easily computed.

Consider the system (4.35)

$$
\dot{x}(t) = \begin{bmatrix} a_{01} & a_{02} \\ a_{03} & a_{04} \end{bmatrix} x(t) + \begin{bmatrix} a_{11} & a_{12} \\ a_{13} & a_{14} \end{bmatrix} x(t-h) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t), t \geq 0,
$$

(4.78)

where $a_{ij}, a_{1i}$ and $b_j \in \mathbb{R}$. Let (i) point states $x(t)$ and $x(t - h)$ the directly available for feedback, or the output $y = H x \in \mathbb{R}$ in equation (4.51) be such that these can be
estimated (Sec. 4.3.1); (ii) the system be stabilisable
(i.e., it satisfies Randolfi's conditions [35]) but not by
feedbacks of \(x(t)\) and \(x(t-h)\) alone (i.e., conditions of
Sec 4.2.1, CASE-1 are not satisfied); (iii) the control
\[ u(t) := u_1(t) + u_2(t) \]
where
\[ u_1(t) = K_0 x(t) + K_1 x(t-h), \quad K_0, K_1 \in \mathbb{R}^2, \tag{4.79} \]
and structure of \(u_2(t)\) (the dynamic part) will be decided
later. Then by straight forward application of \(u_1(t)\) with
proper \(K_0\) and \(K_1\) both elements of either row of \(A_c\) and \(A_1\)
matrices can be assigned arbitrary values so that off-
diagonal elements of the manipulated rows in both matrices
are zero and diagonal elements are compatible for stability
(Defn. 4.5). Thus for instance let
\[ \bar{A}_0 := \begin{bmatrix} \bar{a}_{01} & 0 \\ \bar{a}_{03} & \bar{a}_{04} \end{bmatrix}, \quad \bar{A}_1 := \begin{bmatrix} \bar{a}_{11} & 0 \\ \bar{a}_{13} & \bar{a}_{14} \end{bmatrix} \]
where \(\bar{A}_0 = A_0 + b K_0\), \(\bar{A}_1 = A_1 + b K_1\), and the system (4.78)
transforms into
\[ \dot{x}(t) = \bar{A}_0 x(t) + \bar{A}_1 x(t-h) + b u_2(t), \quad t \geq 0. \tag{4.80} \]
Now both system matrices are in identically triangular
form so that the CQP is
\[ \bar{H} = (\lambda - \bar{a}_{01} \bar{a}_{11} e^{-h \lambda}) (\lambda - \bar{a}_{04} - \bar{a}_{14} e^{-h \lambda}). \tag{4.81} \]
In (4.81) \(\bar{a}_{01}\) and \(\bar{a}_{11}\) (which can as well be made zero) are
arbitrary so that the first quasi-polynomial is stable,
and all the unstable system eigenvalues appear as the zeros
of the second one. The zeros of \((\lambda - a_{04} - a_{14} e^{-\lambda})\) can be easily computed (Appendix of Chap. 3) and let the unstable ones be the set \(S = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}\), \(\Re{\lambda_i} > 0, i = 1, 2, \ldots, N\). Following the notations of Sec. 3.3 on SDT, the state space is decomposed by \(S\) and the system (4.80) satisfies (Eqn. (3.48)) in \(\mathcal{G}_p\) the ODE

\[
\dot{Z}(t) = A_N Z(t) + b_N u_2(t), \quad t > 0
\]

where \(A_N\) is in Jordan canonical form with its eigenvalues the set \(S\), \(b_N = \overline{\Psi}_N(0) b\), and \(Z \in \mathbb{R}^N\) is defined by the expressions (3.42) and (3.20) as

\[
Z(t); = < \overline{\Psi}_N, x_t > = \overline{\Psi}_N(0) x(t) + \int_{-h}^{0} \overline{\Psi}_N(\theta + h) A_1 x(t + \theta) d\theta.
\]

(4.83)

From the conclusions of Proposition 3.3, the reduced system \((A_N, b_N)\) is controllable as the actual system is stabilisable by assumption. Therefore the following result is stated [17].

Proposition 4.8: Let the system (4.80) be stabilisable. Then the stabilising control law is given by

\[
u(t) = F Z(t)
\]

(4.84)

where

\[
Z(t) = \overline{\Psi}_N(0) x(t) + V(t),
\]

(4.85)

\[
V(t) = \int_{-h}^{0} \overline{\Psi}_N(\theta + h) A_1 x(t + \theta) d\theta,
\]

(4.86)

and \(V(t)\) satisfies the ODE
\[ \dot{V}(t) = A_N V(t) + \left[ \exp(-A_N) \right] \dot{y}_N(0), A_1 \right] x(t) - \left[ \dot{y}_N(0), A_1 \right] x(t - h). \tag{4.87} \]

Thus the complete stabilising control law is

\[ u(t) = u_1(t) + u_2(t) = \left[ K_0 + F \dot{y}_N(0) \right] x(t) + K_1 x(t - h) + F V(t). \tag{4.88} \]

Proof (Outline). The stabilisability of system (4.80) implies controllability of \((A_N, b_N)\) in (4.82), and so by suitable choice of \(F\) matrix \((A_N + b_N F)\) is stable. Then the \(Z(t)\) feedback law (4.84) is also a stabilising one for (4.80).

Expressions (4.85) and (4.86) are derived from (4.33), and (4.87) is obtained by differentiating (4.86). The complete law (4.88) for the DDE (4.78) is then obtained from (4.79) and (4.84).

Thus, once the feedback gain \(F\) is found out by any standard FD techniques, the dynamic part of the compensator is provided by the ODE system (4.87). As mentioned earlier systems with this theoretically elegant feedback might be sensitive to errors in parameters. In this regard the observations of Manitius and Olbrot [140] might be helpful in designing an 'insensitive feedback'. Further, although the difficulties in computation of system eigenvalues compelled the present specialisation to a second order case, the procedure can be well adopted for higher order systems where this difficulty can be overridden as there have been
a few good computational methods for finding roots of DDB system. Particularly for systems with rank \( B = n - 1 \) the suggested procedure can be applied \textit{per se}.

The suggested algorithm is as follows.
Algorithm 4.5:
(i) Check for stabilisability of the system by Bhadolfi's rank condition. If stabilisable, 
(ii) Check for stabilisability with \( x(t) \) and \( x(t - 1) \) feedback by conditions of Sec. 4.2.1, CASE-1.
(iii) Check for the existence of observer through Sec. 4.3.1. If \( x(t) \) and \( x(t-1) \) are available but system is not stabilisable by their feedback (from step (ii))
(iv) find \( u_\perp \) in (4.79). Compute zeros of the unstable scalar quasi-polynomial. Form the set \( S \). (Few more stable \( \phi \)'s may be included in \( S \) to achieve a degree of stability. Algorithm of Chap. 3 may also be used to get a closely approximating curtailed system (4.82)). Find corresponding \( \phi \)'s and \( \Psi \)'s.
(v) Find the stabilising gain \( F \) for (4.82) by a suitable method of finite-dimensional theory.
(vi) Obtain the compensator.

Example 4.12: Consider the example of Schumacher [117; p 122] who found a stabilising FD compensator by a different method.

\[
\begin{align*}
\dot{x}_1(t) &= -(\pi/2) x_\perp(t - 1) + x_2(t) \\
\dot{x}_2(t) &= u(t) \\
y(t) &= x_\perp(t).
\end{align*}
\]
Thus \( A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \), \( A_1 = \begin{bmatrix} -\pi/2 & 0 \\ 0 & 0 \end{bmatrix} \), \( b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), \( H = [1, 0] \)

and \( h = 1 \).

The open loop CQP \( H = \lambda (\lambda + \pi/2 e^{\lambda}) \) having zeros at 0, \( \pm j \pi/2 \), and at infinitely many other points in the LHP. Rudolfi's rank conditions are satisfied at \( \lambda_1 = 0 \) and \( \lambda_{2,3} = \pm j \pi/2 \), and so system is stabilisable. Checking with conditions in Sec. 4.2.1 the system is not stabilisable by \( x(t) \) and \( x(t-1) \) feedbacks. From conditions in Sec. 4.3.1 existence of the observer is guaranteed and so \( x(t) \) and \( x(t-1) \) can be estimated. Let \( u_1 = -5x(t) \) so that \( \overline{A}_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) and \( \overline{A}_1 = A_1 \). Thus unstable eigenvalues are contained in the zeros of \( \lambda + \pi/2 e^{\lambda} \) which are \( (\pm j \pi/2) \). Computation of normalised eigenfunctions and adjoint eigenfunctions for these roots give

\[
\phi_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \phi_2(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \Psi_1(0) = \begin{bmatrix} 3.7551711, 0.3462411 \end{bmatrix}, \\
\Psi_2(0) = \begin{bmatrix} 5.898609, 1.2984969 \end{bmatrix}.
\]

Now forming (4.82) with

\[
A_N = \begin{bmatrix} 0 & -\pi/2 \\ \pi/2 & 0 \end{bmatrix} \quad \text{and} \quad b_N = \begin{bmatrix} \Psi_1(0) \\ \Psi_2(0) \end{bmatrix} \quad b, \quad \text{the gain}
\]

\[
F = -[0.789102, 2.09995] \quad \text{for the eigenvalues of the curtailed system to be (say) } s_1 = -1, \ s_2 = -2. \quad \text{The compensator is thus formed and the closed loop CQP is found to be}
\]

\[
\tilde{H} = (\lambda + 5.6135)(\lambda + 1.19325 + j 1.731066) \cdot \\
(\lambda + 1.19325 - j 1.731066)(\lambda + \pi/2 e^{\lambda})(\lambda^2 + \pi^2/4)^{-1},
\]

which can be verified to possess stable zeros.
Example 4.13: Consider the system with

$$ \Lambda_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ \Lambda_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \ b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ H = [1, 0] $$

and $h = 1$. The open loop CQP is $(\lambda + 1) \cdot (\lambda - 1 - e^{-\lambda})$ whose zeros are: $\lambda_1 = 1.2788$, $\lambda_2 = -1$, $\lambda_3, 4 = -1.5898 \pm j 4.155$, $\lambda_5, 6 = -2.4113 \pm j 10.6867$ etc. Randolfi's conditions are satisfied for $\lambda_1$. The minimal order observer exists and is given by (Sec 4.3, eqn. (4.64)):

$$ t_1 = -1, \ t_2 = 2, \ \gamma_0 = -2 \quad \text{and} \quad \gamma_1 = -1.5. $$

The system is also checked not be stabilisable by $x(t)$ and $x(t - 1)$ feedbacks.

Suppose a degree of stability of 2 is desired for the compensator. Then let $S = \{ \lambda_1, \lambda_2, \lambda_3, \lambda_4 \}$ and correspondingly the normalised $\phi(o) = [1, 0]'$, $\phi_2(o) = [1.01778, -1.73576]'$, $\phi_3(o) = [1, 0]'$, $\phi_4(o) = [0, 0]'$, $\Psi_1(0) = [0.78213, 0.09554]$, $\Psi_2(0) = [0, 1]$, $\Psi_3(0) = [0.08035, -0.013215]$, $\Psi_4(0) = [0.20956, 0.013215]$. Assuming the curtailed system has eigenvalues $\{-2, -2, -2, -2\}$, the feedback gain is found as $F = [ \tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{f}_4 ]$ where $\tilde{f}_1 = 102.4461$, $\tilde{f}_2 = 4.700312$, $\tilde{f}_3 = -453.238994$, $\tilde{f}_4 = 569.94002$. Then the compensator is formed.
5.1 Introduction

Systems with time delays in the input-vector are more difficult to control than those with delays in the state. Such systems already introduced in Chapter 1 are repeated below.

(S2) is (S - CDS): \[ \dot{x}(t) = A_0 x(t) + B_0 u(t) + B_1 u(t - \tau), \quad B_1 \neq 0, \quad t \geq 0. \]

(S4) is (M - CDS): \[ \dot{x}(t) = A_0 x(t) + \sum_{j=0}^{M} B_j u(t - g_j), \quad t \geq 0, \quad 0 = g_0 < g_1 < \ldots < g_M = \tau < \infty. \]

Similarly the pure-delay system is defined as

(S7) is (P - CDS): \[ \dot{x}(t) = A_0 x(t) + B_1 u(t - \tau), \quad t \geq 0, \quad (5.1) \]

where \( x(t) \in \mathbb{R}^n, \ u(.) \in \mathbb{R}^m, \ A_0 \in \mathbb{R}^{n \times n}, \ B_j \in \mathbb{R}^{n \times m}; \ t, \ g_j \in \mathbb{R}. \)

Depending on the initial data above models represent various physical systems \([9],[30]\). Systems (S2) and (S4) represent interesting cases only if \( u(t) \) is an unknown function to be determined. Otherwise substitution of \( V(t) = \sum_{j=0}^{M} u(t - g_j) \) reduces these to ODE cases. Thus the problem with control delays is nontrivial only if both \( u(t) \) and \( u(t - \tau) \) appear simultaneously in the equation. Another situation arises for the following representation

(S8) \[ \dot{x}(t) = A_0 x(t) + B_0 u(t) + B_1 V(t - \tau) \]

where the set of components of \( u \) entering nontrivially into \( B_0 u \) is disjoint from the set entering into \( B_1 V(.) \); the delay separability condition \([45]\). Such situations have been
considered by Mee [144] for LQOC problem and will also be considered here.

A 'dead-time' compensation scheme utilizing the mathematical model of the process was suggested by Smith [145], which is called a Smith Predictor (SP) and eliminates the delay transfer term \( \exp(-gs) \) from the closed loop characteristic equation. The SP was prone to practical system instabilities due to 'mismatch' between parameters of the plant and its model, and had little industrial use. Palmer and Powers [150] further improved the SP so as to improve the regulating properties (e.g., load or input disturbance compensation) besides enhancing servomechanism properties of delay system. Further improvements in SP and its wider field of applications are described in [144], [146]-[151].

Analogous to state delayed system, the true or complete state of a control delay system is given by (Ichikawa in [9])

\[
z(t) = \{ x(t), u_t(s) : = u(t + s), s \in [-g, 0] \}.
\]

The state space \( Z = \mathbb{R}^n \times L_2([-g, 0]; \mathbb{R}^m) \) is infinite-dimensional in nature. Lack of appreciation regarding the complete state has given rise to errors in earlier works as pointed out by Fuller [147]. On the basis of this complete state concept, Lewis [45] and others [46],[94], [99] have obtained conditions for absolute controllability, stabilisability etc. Making use of certain transformations, control delay systems can be rendered into a delay-free ordinary model (an ODE) for which control law for stabilisation, arbitrary
pole placement and various optimisation problems can be derived ([45], [46]). Such reduction type schemes are also employed implicitly in [39], [44], [140]. Some results of interest are:

(I) Consider system (S2). Let
\[ z(t) = x(t) + \int_{t-g}^{t} (\exp(t-s-g)A_0)B_1 u(s) \, ds. \] (5.2)
Then \((x(t), u(t))\) is admissible for (S2) iff \((z(t), u(t))\) is admissible for
\[ z(t) = A_0 z(t) + (B_0 + \exp(-gA_0)B_1)u(t). \] (5.3)

(II) Consider the time varying version of (S4); i.e.
\[ \dot{z}(t) = A_0(t)z(t) + \sum_{j=0}^{M} \Phi(t, t+g_j)B_j(s+g_j)u(s) \, ds. \] (5.4)
Then \((x(t), u(t))\) is admissible for (S4) iff \((z(t), u(t))\) is admissible for
\[ \dot{z}(t) = A_0(t)z(t) + \sum_{j=0}^{M} \Phi(t, t+g_j)B_j(t+g_j)u(t). \] (5.5)
when \(\Phi(t, s)\) is the fundamental matrix solution of \(\dot{x}(t) = A_0(t)x(t)\).

Many stabilising control schemes for systems with control delays can be found in literature. Notable among them are in Olbrot ([32A], [39A]), Lee and Manitius [132], Manitius and Olbrot [140], Gabasov et al [152], Solodovnikov et al [149]. Few interesting studies on stability can be also seen in [45], [46], [65], [153]. Genericity results of Lee and Olbrot [47] are discussed in Chp. 4.
For optimal control problems of control delay systems (CDS) the necessary conditions for $\mathbb{R}^n$-target sets were given in late sixties (e.g., [30], [154] and refs. therein). Budelis and Bryson (in [9]) seem to be the first workers to treat problems with delayed and non delayed controls appearing together. A survey and some new results like derivation of Riccati equation and feedback law (FCL) both for state and co control delay were given by Manitius [95], works of Koivo-Lee are Soliman-Ray are in [9],[83],[96C]. Wise [144] showed that Feedback Control has the same structure as the optimal controller but has suboptimal feedback coefficients. Lee and Manitius [38],[123],[132] dealt with QOCP together with controllability, observability, stability, canonical forms etc., and the possibilities of an incomplete state feedback not involving the integral part of the law. Other relevant developments are documented in [9], [75]. With the introduction of product space concept to QOCP (e.g., [16]) Delfour extended the earlier works and indicated in [11], [21] special applications to CDS. Koivo-Lee [96C] showed that QOC law for S-SCD systems has feedback terms in $x_t$ as well as in the past control $u_t$. Olbrot [394] gave a stabilising control law for systems having same delay in state and control, and also gave algebraic stabilising conditions. However Kwong [99] proved that the stabilising control law also needs feedback on entire past history $x(s), s \in [-h, t]$; i.e.,

$$u(t) = -K_0 x(t) - \int_{-h}^{0} K_1(\theta) x(t+\theta)d\theta - \int_{-h}^{t} K_2(t-s)x(s)ds$$

(5.6)
where $K^1(\cdot)$ and $K^2(\cdot)$ are measurable, essentially bounded over $[-h, 0]$ and $[0, \infty)$ respectively. He also gave explicit control law and finite cost function for infinite-time QOCP. The law to be stabilising naturally needs $(A_0, A_1, B_0, B_1)$ be stabilisable, and $(A_0, A_1, H)$ detectable where the state weighting matrix $Q = H'H$.

There were also attempts in sixties on approximations of optimal control problem leading to certain type of suboptimal or near-optimal control [75]. Day-Hsia obtained a discrete-time controller which was later improved in [75] to give a good suboptimal control. Fuller [147] considered QCP for CDS where the plant consisted of a pure delay followed by a lumped parameter system, which was later extended to case of several delays by Reeve (in [9]). Attempts were also made to expand the delayed terms in Taylor or Padé series (e.g., [79]). Soliman and Ray [83] approximated the system by a set of coupled-ODEs by discretising the delay interval (Repin-type) and obtained the approximate QOC law of the form

$$u(t) = K_0 x(t) + \sum_{i=1}^{N} K_{1i} x(t-h_i) + \sum_{i=1}^{M} K_{2i} u(t-g_i). \quad (5.7)$$

Mercier [128] used the SDT (Chp 3) to derive necessary and sufficient conditions for designing LQOC law for M-SCD systems. Lewis [45] conjectured that the QOC law has the same form as a stabilising law given by $u(t) = K z(t)$, which is obtained through (5.2)-(5.3). This gain $K$ may be taken as a suboptimal gain or to initialise the actual OCP.
5.2. Suboptimisation of Control Delay Systems With Output Feedback [177]

5.2.1. Problem Formulation:

Consider the LTI system (S2) with a constant time delay in the control

\[
\dot{x}(t) = A x(t) + B_0 u(t) + B_1 u(t-T), t \geq 0,
\]

\[
x(0) = x_0 ; u(s) = \eta(s), s \in [-T, 0],
\]

\[
y(t) = C x(t)
\]

(5.8)

where \( x(t) \in \mathbb{R}^n \) is the state, \( u(.) \in \mathbb{R}^m \) are the controls or inputs, \( y(t) \in \mathbb{R}^p \) is the output; \( A, B_0, B_1, C \) are real constant matrices of compatible dimensions; \( T > 0 \) is the delay, the initial data \( \eta \) is a vector valued function assumed to be in \( L_2 ([0, T]; \mathbb{R}^n) \), \( C \) is of full rank \( r \), the initial time is taken to be zero. The problem is to minimise the quadratic cost functional

\[
J = \frac{1}{2} \int_0^\infty [x'(t) Q x(t) + u(t) R_0 u(t) + u'(t - T) R_1 u(t - T)] dt
\]

(5.9)

subjected to the system constraint (5.8) and the controller constraint that only the output \( y(t) \) is available for feedback, where symmetric matrices \( Q > 0 \in \mathbb{R}^{n \times n} \) and \( R_0, R_1 > 0 \in \mathbb{R}^{m \times m} \).

Such a problem using full \( x(t) \) has been solved (e.g., [45], [99]) and it is known that when the system is controllable (or stabilisable) the stabilising control law requires feedback from the complete state \( \{x(t), u_\ell(s); u(t + s), s \in [-T, 0]\} \). The law is quite complicated and the feedback gains are found out by solving coupled Riccati equations.
(e.g., [45]) derivable from those for finite-time optimisation [96C], [83].

For this section it is assumed that the set of components of \( u \) entering nontrivially into \( B_0 u \) (acting in the direct channel) is disjoint from the set entering into \( B_1 u(\cdot) \) (acting in the delayed channel). In what follows, these controls are now symbolised as \( u_0 \) and \( u_1 \) respectively, with \( u_1(t - T) := V_1(t) \). Under this "delay separability" condition, Mee [144] solved the LQOCP for systems with different delay lengths that generalises the work of Fuller [147]. His work has essential features of solving only non-delayed problems. The two control laws obtained gave together the needed time invariant FBL which has a Smith \textit{Predictor}-type structure. The laws are also quite complicated and involve again the complete state, performing a linear operation on it. However, the problem with the whole of the state \( x(t) \) being not available and its estimation for the CDS ([87], [39A]) ruled out has not yet been tackled. Works on output feedback control of systems with delay are only a few; also the constant output feedback stabilisation of TDS is more stringent and is not guaranteed.

Thus to start with, it is assumed that the system is stabilisable by feedbacks from the output vector \( y(t) \) plus the complete information on the past controls generated through this. Also let, as usual (e.g., [39], [45]), \( u \in L^2 ([0, \infty); \mathbb{R}^m) \) be assumed. In view of the assumed delay separability the problem is reformulated as follows.
Minimise \( J = \frac{1}{2} \int_0^\infty \left[ x'(t) Q x(t) + u'_0(t) R_0 u_0(t) \\
+ V'_1(t) R_1 V_1(t) \right] dt \) \hspace{1cm} (5.10)

by the choice of controls \( u_0, u_1 \in L_2([0, \infty); \mathbb{R}^m) \), subject to system constraint

\( \dot{x}(t) = A_0 x(t) + B_0 u_0(t) + B_1 V_1(t), \; t \geq 0, \) \hspace{1cm} (5.11)

and controller constraint that output \( y(t) = C x(t) \) is available for feedback as the only contribution from the state.

5.2.2. Solution

Here \( u_0 \) and \( u_1 \) stand for non-delayed and delayed controls respectively. Variables \( u_0 \) have immediate effect whereas \( u_1 \) cannot affect the states until after a delay time \( T \). It is proposed to find a solution to the above problem using feedback from the 'partial state'

\( \{ y(t), (u_0(t), u_1(t + s)) \}, \; s \in [-T, 0] \).

The cost functional (5.10), dropping the argument \( t \), becomes

\[
J = \frac{1}{2} \int_0^T (x' Q x + u'_0 R_0 u_0) dt \\
+ \frac{1}{2} \int_T^\infty (x' Q x + u'_0 R_0 u_0 + V'_1 R_1 V_1) dt
\]

\( = J_1 + J_2 \).

It can be observed that

\[
\min_{u_0, u_1} J = \min_{u_0(0,T)} \left[ J_1 + \min_{u_0(T,\infty)} J_2 \right]. \hspace{1cm} (5.13)
\]

Therefore, in the interval \([T, \infty)\), the problem is an infinite-time state regulator one with optimal feedback controls given by
\[ u_0(t + T) = F_0 y(t + T) = F_0 C x(t + T) \]

and

\[ u_1(t) = V_1(t + T) = F_1 y(t + T) = F_1 C x(t + T) \] \hspace{1cm} (5.14)

where \( F_0 \) and \( F_1 \) are real \((m \times r)\) constant gain matrices. For the problem to be meaningful the system is assumed to be asymptotically stable under the above control actions, so that the cost remains finite. Now expressing \( J_2 \) in a general form and substituting (5.14) into it

\[
J_2 := g_2 \left[ x(t + T) \right] = \frac{1}{2} \int_{t+T}^{\infty} \left[ x'(\tau) \right] Q x(\tau) \, d\tau + u'_0(\tau) R_0 u_0(\tau) + V'_1(\tau) R_1 V_1(\tau) \, d\tau \\
= \frac{1}{2} \int_{t+T}^{\infty} x'(\tau) \bar{Q} x(\tau) \, d\tau
\] \hspace{1cm} (5.15)

where \( \bar{Q} = C' \left[ F'_0 \, R_0 \, F_0 + F'_1 \, R_1 \, F_1 \right] C + Q \).

The minimum cost under the optimal control action is assumed to be in quadratic form as follows

\[
g_{2\min} \left[ x(t + T) \right] = \min_{F_0, F_1} g_2 \left[ x(t + T) \right] \\
= \frac{1}{2} x'(t + T) \, S \, x(t + T)
\] \hspace{1cm} (5.16)

where \( S \) is a symmetric p.d. \((n \times n)\) matrix. Infinite-time optimisation over the interval \([t + T, \infty)\) will now be carried out using the concept of continuous dynamic programming [156] which gives the necessary conditions for optimality.

Proposition 5.1 (LTI output-feedback):

Consider the system (5.11) for \( t \geq T \), and the cost functional \( J_2 \) in (5.15) to be minimised over \([T, \infty)\). Let the
control laws (5.14) stabilise the system. Then for the feed-
back gains $F_0$ and $F_1$ to be optimal, it is necessary that

$$F_0 = -R_0^{-1} B'_o S M C'[C M C']^{-1}$$

$$F_1 = -R_1^{-1} B'_1 S M C'[C M C']^{-1}$$

where the $(n \times n)$ matrices $M$ and $S > 0$ satisfy

$$S \bar{A} + \bar{A}' S + C' F_0 R_0 F_0 C + C' F_1 R_1 F_1 C + Q = 0$$

$$\bar{A} M + MA' + x(T) x'(T) = 0$$

$$A = A + B_o F_0 C + B_1 F_1 C ,$$

Derivation: The derivation of the above results is straight-
forward and outlined as follows (e.g., [112A; p 75]). For a
very small positive quantity $\Delta$ the functional $g_2$ of (5.15)
can be expressed as

$$g_2 [x(t+T)] = \frac{1}{2} \int_{t+T}^{t+T+\Delta} x'(\tau) \bar{Q} x(\tau) d\tau$$

$$+ \frac{1}{2} \int_{t+T+\Delta}^{\infty} x'(\tau) \bar{Q} x(\tau) d\tau .$$

Now using (5.16) the second part of (5.22) has the minimum
value as $(1/2) x'(t + T + \Delta) S x(t + T + \Delta)$. Let the inte-
grand $(x'(\tau) \bar{Q} x(\tau))$, represented by $p(x(\tau), u(\tau) , \tau)$, be
assumed smooth over the interval $(t + T) \to (t + T + \Delta)$
under optimum action. Then with $\Delta$ sufficiently small, the
minimum value of the first part of cost (5.22) can be put as
\[
\min_{F_0, F_1} \frac{1}{2} \int_{t+T}^{t+T+\Delta} (x'(\tau) - u(\tau)) \, d\tau
\]
\[
= \min_{F_0, F_1} \frac{1}{2} \{ \Delta \cdot \rho [x(t + T + \alpha \Delta),
\quad u(t + T + \alpha \Delta), t + T + \alpha \Delta] \}, \quad 0 < \alpha < 1
\]
\[
= \min_{F_0, F_1} \frac{1}{2} \Delta \cdot \left[ x'(t + T) \bar{Q} x(t + T) \right] + \varepsilon_1(\Delta). \quad (5.23)
\]

Again using Taylor series expansion about \( \Delta = 0 \),
\[
x(t + T + \Delta) = x(t + T) + \Delta \dot{x}(t + T) + \varepsilon_2(\Delta) \quad (5.24)
\]

where \( \varepsilon_1(\Delta) \) and \( \varepsilon_2(\Delta) \) are higher order terms (including \( \Delta^2, \Delta^3, \ldots \)) in \( \Delta \) and are such that
\[
\lim_{\Delta \to 0} \varepsilon_1(\Delta)/\Delta = 0, \quad \lim_{\Delta \to \infty} \varepsilon_2(\Delta)/\Delta = 0.
\]

Using equations (5.11), (5.17) and (5.18) to express \( \dot{x}(t + T) \) in (5.24) and expression for \( \bar{Q} \) from (5.15), application of the principle of optimality gives (with small manipulation)
\[
\varepsilon_2 \min \left[ x(t+T) \right] = \frac{1}{2} x'(t + T) S x(t + T)
+ \frac{1}{2} x'(t + T) \left[ A' S + C' F_1 F_0 B_0 S + C' F_1' F_1 B_1 S
\right.
+ S A + S B_0 C + S B_1 F_1 C + Q
+ C' F_0' R_0 F_0 C + C' F_1' R_1 F_1 C \right] x(t + T) \Delta + \varepsilon_1(\Delta) + \varepsilon_2(\Delta). \quad (5.25)
\]

Since by assumption (5.16) this minimum value is also equal to \((1/2) x'(t + T) S x(t + T)\), the second and onward terms of (5.25) are identically zero. Therefore dividing
these terms by $\Delta$ and letting $\Delta \to 0$, the constraint obtained under the optimum condition is that the terms under the square bracket in the RHS of (5.25) must be identically equal to zero; i.e.,

$$\hat{f} = C' F' R_0 F_0 + C' F'_1 R_1 F_1 C + A'S + S \bar{A} + Q = 0 \quad (5.26)$$

where $\bar{A}$ is as in (5.21). Finally the problem now reduces to finding gain matrices $F_0$ and $F_1$ so as to minimise (from (5.12))

$$g_2 [x(T)] = J_2 = \frac{1}{2} x'(T) S x(T) \quad (5.27)$$

under equality 'system' constraint (5.26). A Hamiltonian $h$ is formed by adjoining the cost $g_2$ to $\hat{f} = \{f_{ij}\}$ by the $(n \times n)$ symmetric Lagrange multiplier matrix $M = \{m_{ij}\}$

$$h[x(T)] = \frac{1}{2} x'(T) S x(T) + \sum_{i=1}^{n} \sum_{j=1}^{n} (m_{ij}) (f_{ij})$$

$$= \text{Trace} \left[ \frac{1}{2} 3 x(T) x'(T) + M \hat{f} \right] \quad (5.28)$$

The above parameter optimisation problem is solved by minimising $h [x(T)]$ w.r.t. the parameters $F_0, F_1, M$ and $S$. On taking partial derivatives of (5.26) w.r.t. these parameters and then equating them to zero, the set of necessary conditions for optimality given by Eqns (5.17) - (5.20) are obtained.

5.2.21 Determination of $u_0(t)$ within $[0, T]$:

Control $u_1(t)$ is not effective until after a time $(t_0 + T)$ whereas $u_0(t)$ is immediately effective. Further, determination of $u_0$ and $u_1$ in $[T, \infty)$ depends on the state $x(t+T)$
(which in turn depends on \( x(0) \)) and it is not known yet. From (5.13) and optimisation results on \([T, \infty)\),

\[
\min_{u_0, u_1} J = \min_{u_0(0,T)} \left[ J_1 + \frac{1}{2} x'(T) S x(T) \right].
\] (5.29)

Thus, a minimisation of (5.29) is to be performed in the interval \([0, T]\) taking into account that \( V_1(0,T) \), the initial delay storage, enters as a known disturbance in real time. The problem is then to find a control \( u_0(t) \), \( t \in [0, T] \) which will reduce the 'disturbed state' to some desired state \( x(t) \) with \( x(t = T) = x(T) \), simultaneously minimising \( J_1 \) over \([0, T]\).

This involves the solution of a linear quadratic tracking problem with output feedback. The method of Levine and Athans \([101]\) for the time varying output feedback gain will be essentially adopted, suitably modified for the tracking problem. Following preliminary result is stated.

Proposition 5.2 (Linear tracking problem in \([0, T]\)):

The control law \( u_0(t) \), \( t \in [0, T] \) that minimises an average value of \( J_1 \) in (5.12), simultaneously reducing the disturbance response due to initial delay storage \( V_1(0,T) \) identically to zero, subject to system (5.11) is given by

\[
u_0(t) = K_1(t) C x(t) - R_0^{-1} B_0' q(t) \] (5.31), (5.44)

where the \((m \cdot r)\) gain matrix

\[
K_1(t) = - R_0^{-1} B_0' L(t) \Psi(t) [C \Psi(t)]^{-1},
\]

\[
\Psi(t) = [\Phi(t) x_0 \Phi'(t) C]',
\]

\[
x_0 = E \{ x_0 x_0' \}. \] (3.43)
and the system transition matrix $\Phi(t)$, the $(n \times n)$ symmetric p.d. matrix $L(t)$, the $(n \times 1)$ vector $q(t)$ satisfy:

\[
\begin{align*}
\dot{\Phi}(t, t_0) &= [A + B_o K_1(t) C]\Phi(t, t_0) = \dot{\Phi}(t, t_0), \\
\Phi(t_0, t_0) &= I_n; \\
L(t) + L(t) A + A^T L(t) + (Q + C^T K_1(t) R_o K_1(t) C) &= 0, \\
L(T) &= S, \\
q(t) &= [A - B_o R_o^{-1} B_o^T L(t)] q(t) + L(t) B_1 V_1(0,T) = 0, \\
q(T) &= 0.
\end{align*}
\] (5.45)

the $(n \times n)$ symmetric p.d. matrix $S$ is as in (5.19);

\[
\begin{align*}
\dot{q}(t) &= [A - B_o R_o^{-1} B_o^T L(t)] q(t) + L(t) B_1 V_1(0,T) = 0, \\
q(T) &= 0.
\end{align*}
\] (5.46)

**Derivation:**

Let $x(t)$ and $x_d(t)$, $t \in [0, T]$ be respectively the desired state response and the response due to the 'disturbance' $V_1(0, T)$. Then the cost functional to be minimised may be modified to accommodate this, and is assumed to be of the form:

\[
\begin{align*}
\hat{J}_1 &= \frac{1}{2} [x(T) - x_d(T)] S [x(T) - x_d(T)] \\
&+ \frac{1}{2} \int_0^T \{[x(t) - x_d(t)] q[x(t) - x_d(t)] + u'(t) R_o u_o(t)\} dt \\
\hat{J}_1 &= \hat{J}_T + \int_0^T J_T dt
\end{align*}
\] (5.30)

where the weighting matrix $S$ in the terminal cost is as in (5.19). In fact the aim is to minimise $x_d(t)$, $t \in [0, T]$ by application of an additional input (besides the one required to optimise $x(t)$) and finally make it zero for all
time. This will be necessary for obtaining a realisable control law. Following the procedure of the usual tracking problem (e.g., [112A; p 96]) control law can be assumed to be

\[ u_0(t) = K_1(t) C x(t) - w(t) \quad (5.31) \]

where the gain matrix \( K(t) \) is \((m-r)\) and the 'external input' \( w(t) \) is \((m-1)\). Then the closed loop system (5.11) is

\[ \dot{x}(t) = [A + B_0 K_1(t) C] x(t) + B_1 \psi_1 (0, T) \]

\[ - B_0 w(t), \ t \in [0, T] . \quad (5.32) \]

It is required to obtain a 'matrix-differential system' representation of (5.32), only to be utilised for formation of a Hamiltonian, so that the necessary gradients for optimality can be evaluated using the matrix minimum principle [103]. Comparing with similar situation in Levine-Athans approach [101], the present case appears to be 'nonstandard' and is based on some reasonable assumptions. The manipulation is as follows. Let the \((n-n)\) transition matrix \( \Phi(t, t_0) \) of (5.32), subsequently identified as \( \Phi(t) \) with \( t_0 = 0 \), be given by

\[ \dot{\Phi}(t) = [A + B_0 K_1(t) C] \Phi(t) = \Phi(t), \ \Phi(0) = I. \quad (5.33) \]

With nonzero initial data \( x_0 \), let \( K_3 \) be the \((1-n)\) vector such that

\[ K_3 x_0 = 1, \ x_0 \neq 0. \quad (5.34) \]

Under further assumption that the ultimate optimal response be disturbance-free, the state vector \( x(t) \) in (5.32)
as well as in (5.30) may be replaced by $\hat{\Phi}(t)x_0$, the desired disturbance free response with $B_1V_1(0, t)$ compensated by $B_0w(t)$. Therefore, the 'new' state constraint, from (5.32), (5.33) and (5.34) is the matrix differential system

$$\dot{\Phi}(t) = \hat{A}\Phi(t) + [B_1V_1(0, t) - B_0w(t)]K_3 \tag{5.35}$$

and the 'new' cost functional from (5.30) is given by

$$J_T = \frac{1}{2} \text{Trace} \left[ \Phi'(T) S \Phi(T)x_0x_0' - \Phi'(T)Sx_d(T)x_0' - \Phi'(T)Sx_d(T)x_d'(T) \right] + \frac{1}{2} \text{Trace} \left[ R_0w(t)w'(t) \right], \tag{5.36}$$

$$J_T = \frac{1}{2} \text{Trace} \left[ \Phi'(T) S \Phi(T)x_0x_0' - \Phi'(T)Sx_d(T)x_0' - \Phi'(T)Sx_d(T)x_d'(T) \right] \tag{5.37}$$

where the property $x'Ax = \text{Trace} \left[ A x x' \right]$ has been used.

Eqns (5.36) and (5.37) bring into focus the dependence of the cost upon $x_0$, $K_1(t)$ and $w(t)$. As explained in Chap. 2 (Secs. 2.2, 2.3) to 'get desired optimal values of $K_1(t)$ and $w(t)$ independent of initial states, the performance is averaged for a linearly independent set of initial states. However the values obtained thus are optimal only in an average sense.

Following Levine-Athens [101], $x_0$ is assumed to be a random variable uniformly distributed over the surface of an $n$-dimensional sphere such that $E\{x_0\} = 0$ and $E\{x_0 x_0'\} = X_0$, a nonsingular matrix which can be used as a design
parameter if there is some a priori knowledge of which states are likely to be disturbed. For simplicity it will be assumed that $X_0 = \alpha \mathbf{1}_n$, for some constant $\alpha \neq 0$ (for example $\alpha = \frac{1}{n} \| x_0 \|$). Then the expected value of the cost is given by $\hat{J}_1 = \hat{J}_T + \hat{J}_t$ where $\hat{J}_T$ is obtained from $J_t$ with $(x_0, x_0')$ replaced by $X_0$. Now the 'averaged' Hamiltonian $\hat{H}$ is formed by adjoining system constraint (5.35) to $\hat{J}_t$ through an $(n \times n)$ 'matrix-Lagrange multiplier' $R(t)$ as

$$
\hat{H} = \frac{1}{2} \text{Trace} \left[ \dot{\Phi}(t) \left( R + C'K_1(t) R_0 K_1(t) C \right) \Phi(t) x_0 \right] \\
- \dot{\Phi}(t) C K_1(t) R_0 w(t) x_0' - \dot{\Phi}(t) x_0 x_0' + Q x_0 x_0' \\
- \dot{\Phi}(t) C'K_1(t) R_0 w(t) x_0' - \dot{\Phi}(t) x_0 w'(t) R_0 K_1(t) C \\
+ \frac{1}{2} \text{Trace} \left[ R_0 w(t) w'(t) \right] + \text{Trace} \left\{ A + B_0 K_1(t) C \right\} \Phi(t) \Phi'(t) \right].
$$
(5.38)

The scalar $\hat{H}$ is a quadratic function of $K_1(t)$ (as well as of $w(t)$), and the necessary optimality conditions that these minimise $\hat{H}$ can be derived using matrix minimum principle [103], where the gradient matrices w.r.t. $K_1(t)$ and $w(t)$ must vanish. As it happened both these resulted in identical conditions. Therefore, evaluating for $K_1(t)$,

$$
\frac{\partial \hat{H}}{\partial K_1(t)} K_1(t) = R_0 K_1(t) C \Phi(t) x_0 \Phi'(t) C' - R_0 w(t) x_0' \Phi'(t) C' + B_0 F(t) \Phi'(t) C' = 0.
$$
(5.39)

Also, $F(t)$ satisfies
\[ \dot{E}(t) = -\frac{\partial}{\partial t} \hat{\Phi}(t) = -(Q + C'K_1(t)R_0K(t)C)\hat{\Phi}(t)x_0 \]
\[ + Qx_d x_0' + (C'K_1(t)R_0w(t)x_0' - (A+E_0K_1(t)C)\hat{\Phi}(t) \]
\[ (5.40) \]

with the terminal condition
\[ P(T) = \frac{\partial}{\partial t} \hat{\Phi}(T) = S\hat{\Phi}(T)x_0 - Sx_d(T)x_0'. \]  
\[ (5.41) \]

For a closed loop control law, the following linear relationship between 'state' and 'costate' \( P(t) \) is assumed
\[ P(t) = L(t)\hat{\Phi}(t)x_0 + q(t)x_0' \]  
\[ (5.42) \]

where \( L(t) \) is \((n\times n)\) symmetric p.d. matrix that satisfies a matrix Riccati differential equation, and \( q(t) \) is \((n\times 1)\) 'state vector that generates external control \( w(t) \)' satisfying a linear vector-differential equation. Now, substituting (5.42) in (5.39) and equating separately terms containing \((\hat{\Phi}x_0 \hat{\Phi}'C)\) and \((x_0' \hat{\Phi}'C')\) obtain
\[ K_1(t) = -R_0^{-1} B_0' \left[ L(t)\hat{\Phi}(t) [C\hat{\Phi}(t)]^{-1} \right] \]  
\[ (5.43) \]
\[ w(t) = R_0^{-1} B_0'q(t) \]  
\[ (5.44) \]

where \( \hat{\Phi}(t) = [\hat{\Phi}(t)x_0 \hat{\Phi}'(t)C] \) and the indicated inverse exists as \( X_0 \) is assumed nonsingular, rank \( C = r \leq n, \hat{\Phi} \) is the transition matrix. Thus the time derivative of (5.42), with \( \hat{\Phi}(t) \) from (5.35) along with (5.34), when equated to (5.40) given rise to
\[ \dot{L}(t) + L(t) \hat{\Lambda} + \hat{\Lambda}'L(t) + (Q + C'K_1(t)R_0K(t)C) = 0, \]  
\[ (5.45) \]
\[ L(T) = S ; \]
\[ q(t) + L(t) \left( B_1 V_1(0,T) - B_0 w(t) \right) \]
\[ = -A'q(t) + C'K_1(t) R_0 w(t) + J q_d(t), \]
\[ q(T) = -S x_d(T) \] (5.46a)

where \( \hat{A} \) is as in (5.33). In practice, usually the control law (5.31) is computationally unrealisable since it involves \( q(t) \) which is the solution of (5.46a), and this equation must be solved backward in time from \( T \) to 0. This requires knowledge of \( V_1(0, T) \) and \( x_d(t) \), \( t \in [0, T] \) which is usually not available in the beginning. If known a priori, \( V_1(0,T) \) may be obtained from a delay storage. For simplicity it is assumed (and desired) that \( x_d(t) \) be identically zero over \( t \in [0, T] \). Then (5.46a) with (5.44) gives

\[ \dot{q}(t) + \left[ A - B_0 R_0^{-1} B_0' L(t) \right] q(t) + L(t) B_1 V_1(0,T) = 0, \]
\[ q(T) = 0. \] (5.46)

5.2.22 Determination of \( u_0(t) \):

Now, let \( \hat{\Phi}(t, \tau) \) be the transition matrix of the time varying system (5.46) in the interval \( \tau \) to \( T \) given by

\[ \frac{d}{dt} \hat{\Phi}(t, \tau) = [A - B_0 R_0^{-1} B_0' L(t)] \hat{\Phi}(t, \tau), \]
\[ \hat{\Phi}(\tau, \tau) = I_n \] (5.47)

Then taking the adjoint equation to (5.47) it can be seen that \( q(t) \) has an explicit solution

\[ q(\tau) = \int_{\tau}^{T} \hat{\Phi}'(s, \tau) L(s) B_1 V_1(s) \, ds \] (5.48)

and in particular
Using (5.43) and (5.44) in (5.32)
\[ u_0(0) = -R_0^{-1} B'_0 [L(0) \Psi(0) (C \Phi(0))^{-1} C x_0 \]
\[ + \int_0^T \hat{\phi}'(s,0) L(s) B_1 V_1(s) ds] . \]

Since zero point is completely arbitrary, for a general starting time \( t \in [0, \infty) \)
\[ u_0(t) = -R_0^{-1} B'_0 [L(0) \Psi(0) (C \Phi(0))^{-1} C x(t) \]
\[ + \int_0^T \hat{\phi}'(s,0) L(s) B_1 u_1(t + s - T) ds] . \] (5.49)

5.2.23 Determination of \( u_1(t) \)

Computation of \( u_1(t) \) needs the value of \( x(T) \) which can be available now. From (5.32), (5.33), (5.44) and (5.48) the state at \( t = T \) is
\[ x(T) = \hat{\phi}(T) x_0 + \int_0^T \hat{\phi}(T, \tau) B_1 V_1(\tau) d\tau \]
\[ - \int_0^T \hat{\phi}(T, \tau) B_0 R_0^{-1} B'_0 \left[ \int_\tau^T \hat{\phi}'(s, \tau) L(s) B_1 V_1(s) ds \right] d\tau . \] (5.50)

Changing the order of integration and defining
\[ \lambda(\tau) = \int_0^\tau \hat{\phi}'(T, s) B'_0 R_0^{-1} B'_0 \hat{\phi}'(\tau, s) ds, \] (5.51)
\[ x(T) = \hat{\phi}(T) x_0 \int_0^T [\hat{\phi}(T, \tau) + \lambda(\tau) L(\tau)] B_1 u_1(\tau - T) d\tau . \] (5.52)

This value can be used in the computation over \([T, \infty)\) to determine \( S, P_0 \) and \( F_1 \) as in Eqns. (5.17)-(5.21). Therefore, for a general starting time \( t \), from (5.14)
Thus, Eqns. (5.49) and (5.53) together give an LTI suboptimal output feedback control law, which is optimal only in the average sense. It can be seen that when the full state $x(t)$ is available (i.e., $C = I_n$) these are identical to that found by Mee [144].

As expected, the control laws found are quite complicated although theoretically only finite dimensional (delay free) equations are solved; and so are difficult to implement. However, instead of measuring the delay storage the laws may be implemented by simulating them so that the same stability properties are maintained. Identical to the case of state delayed systems discussed in Ch. 2, the system with control delays may be approximated by a system of ODEs either through Rabin-type or Rado-type scheme (as derived in Sec. 1.6.3) and then standard procedures adopted. This has been done in [178], and is not included here for reason of space.

5.2.3. Computational Procedure

The existence of output feedback stabilising control for the system cannot be ascertained at this stage. However, intuitively certain preconditions for stabilisation may be assumed as follows:

1) There exists $K_0 \in \mathbb{R}^{m \times n}$ such that the system \( \dot{x}(t) = (A + B_0 K_0) x(t) + B_1 u_1(t - T) \) is asymptotically stabilisable (e.g., through Eqns. (5.2) - (5.3)).
ii) There exist $F_q, F^1 \subseteq \mathbb{R}^{n\times n}$ such that \((A + B_0 F_q C + B_1 F^1 C)\) is stable.

Then, the following algorithm is suggested.

Steps:

1. Take $F_q(1), F^1(1)$ such that \((A + B_0 F_q(1) C + B_1 F^1(1) C)\) is stable, $i = 0$.
2. Solve (5.19) for $S(i), i = 0$.
3. Solve (5.45), (5.33) and (5.43) simultaneously as:
   a) choose $L(i)(t) = S(i)$, $\phi(i)(t, t_0) = I$ and find $K(i)_{1}$ from (5.43), $i = 0$;
   b) solve the linear matrix equation (5.45) with $L(i+1)(t) = S(i)$ and $K(i)_{1}$, and call it $L(i+1)(t), i = 0$;
   c) solve the nonlinear matrix Eqn. (5.35) to get $\phi(i+1)(t)$ in terms of $K(i+1)_{1}$, and simultaneously $K(i+1)_{1}$ in terms of $\phi(i+1)(t)$ from (5.43) with $L(i+1)(t), i = 0$;
   d) iterate steps (b), (c) for $i = i + 1 \ldots \ldots$ giving sequence \(\{L(p)(t)\}, \{\phi(p)(t)\} \text{ and } \{K(p)(t)\}\),
   \(p = 1, 2, \ldots \ldots\)
4. Find $q(j)(t)$ with $L(p)(t)$ from (5.48), $j = p$.
5. Find $u(j)(0)$ from (5.33) (with given initial data for $u_1$ and $x$).
6. Solve the state Eqn. (5.32)-(5.44) (with $q(j)$) to get $x(j)(t)$ in terms of $u_1(j)(t)$; and solve (5.53) to get $u_1(j)(t)$ in terms of $x(j)(t)$. Find $x_1^j(t)$ and $u_1(j)(t)$.
7. Solve for $u_0(j)(t)$ from (5.49).
(8) Test for stability of closed loop system (5.11). If stable, stop. If not, iterate steps (4) to (8) for \( j = j + 1 \).

If stability is not being approached go to step (9).

(9) Find \( x_{i+1}^{(T)} \) for any \( j \) from step (6) giving minimum \( \frac{1}{2} x^{(j)}(T) S^{(i)} x^{(j)}(T) \), and solve (5.20) with \( F_{0}^{(i)} \), \( F_{1}^{(i)} \) and \( x_{i+1}^{(T)} \) to get \( M^{(i)} \), \( i = 0 \).

(10) Solve (5.17) and (5.18) to get \( F_{0}^{(i+1)} \), \( F_{1}^{(i+1)} \) and solve (5.19) for \( S^{(i+1)} \), \( i = 0 \).

(11) Go to step (3) and iterate for \( i = i + 1 \).

Example 5.1:

This first order example is borrowed from Salimian and Ray [83] just for an illustration, but modified for infinity time QOCP with added weightage on delayed control. Further, the nondelayed and delayed controls are assumed non-interacting.

\[
\text{Minimise } J = \frac{1}{2} \int_{0}^{\infty} [x^2(t) + 0.5 u_0^2(t) + u_1^2(t-2/3)] dt
\]

subject to:

\[
\dot{x}(t) = -x(t) + u_0(t) - 0.5 u_1(t-2/3),
\]

\[
x(0) = 1, \quad u_1(s) = 0, \quad s \in [-2/3, 0);
\]

\[
x, \ u_0, \ u_1(.) \in \mathbb{R}.
\]

Here \( Q = 1, \ R_0 = 0.5, \ R_1 = 1, \ A = -1, \ \\
\ R_0 = 1, \ R_1 = -0.5, \ T = 2/3, \ C = 1. \)

After the delay interval from Eqsqs. (5.17)-(5.21), \( F_{0} = -2S, \ \\
F_{1} = -S, \ S = 0.3568 \). During the delay interval equations for \( L(t) \) and \( q(t) \) can be solved analytically. The control laws
and the state are found out to be:

\[ u_0(t) = -0.7301694 x(t) - \int_0^{2/3} \left\{ \frac{(\alpha - 1)}{(a \exp(-2\sqrt{3}s) - 1)} \right\} ds \]

\[ 0.866(\alpha \exp(-2\sqrt{3}s) + 1)/(a \exp(-2\sqrt{3}s - 1)) \cdot 0.5 \]

\[ u_1(t + s - 2/3) \]

\[ u_0(t) = 0.1784 [0.3164 x(t) + \int_0^{2/3} \left\{ \frac{0.866(a \exp(-2\sqrt{3}s) + 1)}{(a \exp(-2\sqrt{3}s - 1) - 0.5)} \right\} (0.910735 \exp(\sqrt{3}s)/(1891.3346 \exp(-2\sqrt{3}s) + 1)) u_1(t + s - 2/3) ds] \]

\[ x(t) = (\alpha \exp(-2\sqrt{3}s) - 1) \exp(\sqrt{3}s) [1/(\alpha - 1)] - \int_0^t \left\{ \frac{\exp(-2\sqrt{3}s)}{(a \exp(-2\sqrt{3}s - 1))^{2/3}} \right\} \left\{ \int_0^{2/3} (0.366 a \exp(-2\sqrt{3}s) + 1.366 \exp(\sqrt{3}\tau) u_1(\tau - 2/3) d\tau \right\} ds - 0.5 \int_0^t \left\{ \exp(-\sqrt{3}s)/(a \exp(-2\sqrt{3}s) - 1) \right\} u_1(s - 2/3) ds \]

where \( a - 1 = 891.3346 \). Solving these equations simultaneously, values of \( x \), \( u_0 \), and \( u_1 \) at various values of \( t \) are obtained and plotted in Figs. 5.11, 5.12, and 5.13. Along with are also plotted results of Soliman and Ray [83], just for a comparison. For the present case there are two laws \( u_0 \) and \( u_1 \) as against one law \( u \) in [83].

5.3. Systems with Delayed Input and Delayed Input Derivative [179]–[181].

5.3.1. Introduction

The mathematical model of certain physical systems naturally contain derivative of inputs. Few such examples are: Optimisation of drum-boiler systems [157], [157A].
EXAMPLE - 5.1

--- SOLIMAN'S
--- PRESENT

FIG. 5.1.1 COMPARISON OF STATE TRAJECTORIES

FIG. 5.1.2 CONTROL FUNCTION $U_1(t)$

FIG. 5.1.3 COMPARISON OF CONTROL FUNCTIONS $U_0(t)$ and $U(t)$
linearisation of nonlinear plants [119; p 275], stabilisation of a synchronous machine through actions of exciter stabiliser with conventional velocity governing [158], the usual nth order systems of linear differential equations with forcing function containing derivative terms [116; p 675], including additional derivative controls for better performance (e.g., [158A] and references therein). Various aspects of such systems have been discussed in literature [159]. However in all these the inherent non-negligible time delays in the controller (e.g., those arising out of saturation, fluid flow etc. which do not allow the input to act instantaneously causing thereby undesirable effects discussed earlier) have not been taken into account. Therefore in a realistic modelling along with the input and its derivative their respective delayed components may also be included in the system equations, and that will be taken up here.

The main aim of this section is to introduce, in a formal way, QOCP for LTI systems with delayed control and delayed control derivative together with their nondelayed components, both for continuous-time and discrete-time representations. Since it is necessary, an associated topic on controllability is also included. The system is first transformed into one without derivative terms in the control, and then standard methods are extended to establish new results.

An interesting way to look at such models is, for example, when a control law \( u = K x(t) \) is administered to system (5.54) below. Then the closed loop system resembles
a neutral DDE system which is a subject of much current investigation. Analyses of such systems are quite complicated compared to a retarded DDE system (e.g., Chs 3, 5, 10 and 12), [3; Chs 1 and 12], particularly their stability and stabilisability aspects ([1], [3]-[6], [65B], [142A]). However, the present investigation does not consider this aspect of the problem explicitly. The control law found out here has a complicated structure, and the overall system might assume a form similar to a neutral FDE system [3; Ch. 12].

This aspect of it is also not accounted for. The investigation here are only purely preliminary in nature, and the closed loop system is being assumed to be asymptotically stable, in whichever form it may be.

5.3.2 System Description

5.3.21 Continuous-time system:

Let the LTI continuous-time system having same constant time delay in the control as well as in the control derivative be modelled by the following system of equation

\[ \dot{x}(t) = A x(t) + B_0 u(t) + B_{01} \dot{u}(t) + B_1 u(t - T) + B_2 \dot{u}(t - T), \quad t \geq 0 \]

with

\[ x(0) = x_0, \quad u(s) = \Theta(s)u(s) = \Psi(s), \quad s \in [-T, 0]. \quad (5.54) \]

Usually from physical reasons, \( \Theta(s) \) and \( \Psi(s) \) are taken to be null functions. Here, the states \( x \in \mathbb{R}^n \), controls
To use the already known results for CDS, system (5.54) needs to be transformed into a form without $u(.)$ terms. Therefore, following an identical procedure suggested by Nicholson [157] for non-delayed systems define the SSET (Strict System Equivalent Transformation) ([119; p 52]; later described in Sec. 5.3.3) of $x(t)$ to a new state variable $z(t) \in \mathbb{R}^n$ through

$$z(t) = Ax(t)$$

where the $(n + 2m)$ vectors,

$$z(t) : = \text{Colm} [z(t), -u(t), -V(t)]$$

$$x(t) : = \text{Colm} [x(t), -u(t), -V(t)]$$

$$V(t) : = u(t - T)$$

and the $(n + 2m)$-square transformation matrix

$$A : = \begin{bmatrix} I_n & B_{01} & B_2 \\ 0 & I_m & 0 \\ 0 & 0 & I_n \end{bmatrix}.$$  

(5.56)

This in fact gives the relation

$$x(t) = z(t) + B_{01} u(t) + B_2 V(t).$$

(5.57)

Then system Eqn. (5.54) is transformed into

$$z(t) = A z(t) + B_a u(t) + B_b V(t)$$

(5.58)
The state Eqn. (5.58) together with an output \( w(t) \in \mathbb{R}^n \) (later in Sec. 5.3.3) will be called the 'Transformed System' (TS) of (5.54) - (5.54a).

5.3.2 Discrete-time system:

It will be observed that in the discrete system, the input derivative terms introduce discontinuities in the state response, which are rendered continuous through an SSET. Discretising system Eqn. (5.58) with a sampling time \( \hat{T} \), a submultiple of \( T \) so that \( T = M \hat{T}, M \) an integer leads to

\[
z(K+1) = \bar{\Phi} z(K) + (D_1 + D_2) u(K) + (\Delta_1 + \Delta_2) V(K)
\]

(5.59)

where \( K = K \hat{T} \) refers to the \( K \)th sampling instant, and

\[
V(K) = u(K - M)
\]
\[
\bar{\Phi} = \bar{\Phi} (\hat{T}) = \exp (A \hat{T})
\]
\[
D_1 = \int_{0}^{\hat{T}} \bar{\Phi} (\hat{T} - \tau) B_0 \, d\tau
\]
\[
D_2 = \int_{0}^{\hat{T}} \bar{\Phi} (\hat{T} - \tau) A B_0 \, d\tau
\]
\[
\Delta_1 = \int_{0}^{\hat{T}} \bar{\Phi} (\hat{T} - \tau) B_1 \, d\tau
\]
\[
\Delta_2 = \int_{0}^{\hat{T}} \bar{\Phi} (\hat{T} - \tau) A B_2 \, d\tau.
\]

(5.60)

Thus, from Eqns (5.54) and (5.59) \( z \)-coordinates can be related to \( x \)-coordinates as follows, when discontinuity in \( x \)-coordinates will be directly introduced:

\[
x(K+1) = [\bar{\Phi} x(K) + D_1 u(K) + \Delta_1 V(K)] + [B_o u(K+1) - u(K)] + B_2 \{V(K+1) - V(K)\}.
\]

(5.61)
The latter two terms of (5.61) define the discontinuity produced in the state at the instants \( kT \) and \((k + 1)T\) respectively. Fig. 5.2 illustrates qualitatively the effect of discontinuities introduced at the sampling instants. It can be observed that the system would be specified in the form of Eqn. (5.61) as

\[
x_2(\textit{K}+1) = \Phi x_2(\textit{K}) + D_1 u(\textit{K}) + A_1 \textit{V}^{(\textit{K})} + [B_01(u(\textit{K}+1) - u(\textit{K}))+ B_2 (\textit{V}(\textit{K}+1) - \textit{V}(\textit{K}))],
\]

\[
x_1(\textit{K} + 1) = \Phi x_1(\textit{K}) + D_1 u(\textit{K}) + A_1 \textit{V}^{(\textit{K})} + [B_01(u(\textit{K}) - u(\textit{K}-1)) + B_2 (\textit{V}(\textit{K}) - \textit{V}(\textit{K}-1))]
\]

(5.62)

where discontinuities in \( x \) are represented as \( x_1(\cdot) \) and \( x_2(\cdot) \) respectively, prior to and following the sampling. The transformation to the 'continuous-coordinate system' \( z(\textit{K}) \) can then be related to the responses \( x_1(\textit{K}) \) and \( x_2(\textit{K}) \) as follows:

\[
z(\textit{K}) = x_1(\textit{K}) - B_2 V(\textit{K}-1) - B_01 u(\textit{K}-1)
\]

(5.63)

Or

\[
z(\textit{K}) = x_2(\textit{K}) - B_2 V(\textit{K}) - B_01 u(\textit{K})
\]

(5.64)

so that, if the first square bracket of Eqn. (5.61) represents \( x_1(\textit{K} + 1) \), the second one represents \((x_2(\textit{K}+1) - x_1(\textit{K}+1))\). Hence system (5.54) when discretised can be modelled without discontinuities by the transformed variable \( z(\textit{K}) \) in the form of (5.59), making use of transformations (5.63)-(5.64).

5.3.3. Controllability [180]

As with the system where control derivatives are absent (i.e., \( B_01 = B_2 = 0 \)) two types of controllabilities can be
described: Relative (or $\mathbb{R}^n$) and absolute-controllability, with their usual meaning (Sec 1.1.31 of Chap. 1). Conditions are now formally derived for the TS (5.58), and then it is shown that controllability characteristics of (5.58) can be related to those of system (5.54). The procedure adopted is based on that of Porter-Bradshaw [160] for non-delayed systems. In the following, the formal Laplace transform of time functions will be represented by corresponding s-argumented capital letters. Taking Laplace transform of (5.54)-(5.54a) to form the 'system matrix' [119; p 43],

$$s \mathbf{X}(s) = \mathbf{A} \mathbf{X}(s) + \mathbf{B}_0 \mathbf{U}(s) + \mathbf{B}_0 s \mathbf{U}(s) + \mathbf{B}_1 \mathbf{U}(s) e^{-Ts} + \mathbf{B}_2 \mathbf{U}(s) s e^{-Ts}$$

and

$$\mathbf{Y}(s) = \mathbf{X}(s).$$

Then one can assemble them into

$$\mathbf{P}(s) \overline{\mathbf{X}}(s) = \overline{\mathbf{Y}}(s)$$

where

$$\overline{\mathbf{B}}_1(s) = (\mathbf{B}_0 + \mathbf{B}_0 s), \overline{\mathbf{B}}_2(s) = (\mathbf{B}_1 + \mathbf{B}_2 s),$$

$$\overline{\mathbf{X}}(s) = \text{Colm} [\mathbf{X}(s), -\mathbf{U}(s), -\mathbf{V}(s)] \text{ is } (n+2m) \text{ vector},$$

$$\overline{\mathbf{Y}}(s) = \text{Colm} [0, -\mathbf{Y}(s)] \text{ is } 2m \text{-vector}, \mathbf{V}(s) = e^{-Ts} \mathbf{U}(s)$$

and the $(2n \times (n + 2m))$ system matrix $\mathbf{P}(s)$ is the polynomial matrix given by

$$\mathbf{P}(s) = \begin{bmatrix}
(S I_n - \mathbf{A}) & \overline{\mathbf{B}}_1(s) \\
-\mathbf{I}_n & 0 & 0
\end{bmatrix}.$$
FIG. 5.2 State discontinuity of each sampling instant
gives rise to the \((2n) \times (n + 2m)\)-dimensional system matrix \(\Pi(s)\) for the TS, defined by (the motivation for the structure will be cleared subsequently)

\[
\Pi(s) \overrightarrow{Z}(s) = \overrightarrow{W}(s)
\]

(5.69)

where \(\overrightarrow{Z}(s) = \text{Coln} [Z(s), -U(s), -V(s)]\), \(\overrightarrow{W}(s) = \text{Colm}[0, -W(s)]\), \(n \times m \times m\).

\[
\Pi(s) = \begin{bmatrix}
(sI_n - A) & B_a & B_2 \\
-I_n & B_{o1} & B_2 \\
\end{bmatrix}
\]

(5.70)

In that case, from (5.69), (5.70) and (5.58) it is evident that the output \(w(t)\) of the TS given by

\[
w(t) = z(t) + B_{o1}u(t) + B_2V(t)
\]

(5.71)

is the state \(x(t)\) of the original system (as in (5.57)).

While \(\Pi(s)\) is in the state space form [119; p 43], \(F(s)\) is not SC (in fact it is in ‘polynomial form’ [119; p 50]). Referring to (5.56) it can be seen that

\[
\Pi(s) A = F(s)
\]

(5.72)

also holds which implies that (since \(A\) is unimodular) \(F(s)\) and \(\Pi(s)\) are ‘strict system equivalent’; i.e., they are of same order and give rise to same transfer function [119; p 52, Thm. 3.1]. So controllability characteristic of \(\Pi(s)\) can be related to that of \(F(s)\). From the foregoing discussion it is concluded that:

1) State of the original system (5.54) - (5.54a) is the output of the TS (5.58) - (5.71).
ii) The original system is state controllable iff the TS is output controllable.

Now, the TS (5.58) can be identified as an LTI system having a single control delay whose Relatively State Controllability (RSC) condition is known (Sec 1.1.31) to be

**Proposition 5.3:** System (5.58) is RSC on \([0, t^1]\) for any \(t^1 > T\) iff rank \(P_s = n\), where

\[
P_s = [B_a, B_b, AB_a, AB_b, \ldots, A^{n-1} B_a, A^{n-1} B_b]. (5.73)
\]

Following identical condition for output controllability of an ODE system (e.g., [116; p 396]) and assuming for the moment that \(u(t)\) and \(v(t)\) are disjoint in (5.71) one may state

**Proposition 5.4:** System (5.54) is RSC on \([0, t^1]\) for any \(t^1 > T\) iff rank \(P_c = n\), where

\[
P_c = [B_a, B_b, AB_a, AB_b, \ldots, A^{n-1} B_a, A^{n-1} B_b, \vartheta_1, B_2].
\]

This is a condition of Relative Output Controllability of system (5.58) with output (5.71).

Remarks:

1) The RSC conditions are independent of delay \(T\).

2) Generally it is assumed that \(t^1 > T\). Otherwise relative controllability conditions are same as for the nondelayed case [160].

3) Since the whole component of the state appears in the outputs of both the system and the TS, the RSC condition of Proposition 5.3 expresses only a sufficient condition for the RSC of original system. However in the event of
y(t) = C x(t), y ∈ ℝ^p; p < n output of TS is C w(t); and satisfaction of rank P_s = n may be then neither a necessary nor a sufficient condition for satisfaction of the modified condition rank \( \hat{P}_o = p \), where \( \hat{P}_o \) is accordingly formed (e.g., [116; p 396]). But if rank C = p, the above sufficiency condition will be retained.

4) Let \( P_s := [I_n A] \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \), where

\[
P_1 = [B_o, B_1, A B_o, A^2 B_1, \ldots, A^{n-1} B_o, A^{n-1} B_1]
\]

and \( P_2 = [B_{o1}, B_{o2}, A B_{o1}, A B_{o2}, \ldots, A^{n-1} B_{o1}, A^{n-1} B_{o2}] \).

It can be observed, for example, that rank \( P_s = n \) is implied by either rank \( P_1 = n \) and \( P_2 = 0 \) or vice versa; but it is neither necessary nor sufficient for its satisfaction that either rank \( P_1 = n \) or rank \( P_2 = n \). Thus rank \( P_s = n \) may be satisfied in the absence of either \((u(t), u(t - T))\) or \((\dot{u}(t), \dot{u}(t - T))\) but may not be satisfied in the presence of both.

5) RSC in general does not imply stabilisability and is a weak concept which does not adequately reflect the structure of CDS. Therefore, a stronger concept of Absolute - or Complete - State-controllability will be discussed later.

Identically, the RSC condition for the discrete case can be stated in terms of the TS (5.59) with the output

\[
w(k) = z(k) + B_{o1} u(k) + B_{o} V(k).
\] (5.75)
Using the result of Klamura [44; Thm. 4], let $q < q_1 - T - n$ for some integers $q, q_1$ and $[q, q_1] = \{q, q + 1, \ldots, q_1 - 1, q_1\}$.

**Proposition 5.5:** The discrete system (5.59) is RSC on $[q, q_1]$ iff rank $P_{OD} = n$, where

$$P_{OD} = \begin{bmatrix} D_a, D_b, \Phi D_a, \Phi^2 D_b, \ldots, \Phi^{n-1} D_a, \Phi^{n-1} D_b, B_{01}, B_2 \end{bmatrix},$$

$$D_a = D_1 + D_2, D_b = A_1 + A_2 \quad (5.76)$$

Next, the problem of Absolute State Controllability (ASC) will be taken up for continuous-time system, discrete-time conditions can identically be derived. The concept of ASC has been earlier discussed at length (Sec 1.1.31). On the basis of [30], [39], [45] following result is suggested.

**Proposition 5.6:** System (5.56) is ASC on $[0, t_1) t_1 > T$ iff

$$\text{rank } [E, A E, \ldots, \Phi^{n-1} E] = n \quad (5.77)$$

where $E = B_a + \exp(-AT) B_d \in \mathbb{R}^{n \times m}$.

As discussed for RSC, condition of Proposition 5.6 will serve as a sufficient condition for the ASC of system (5.54), and the Absolute Output Controllability (AOC) is the necessary and sufficient condition for ASC of (5.54). A suggested definition of AOC may be:
Definition 5.1: A system (5.58) - (5.71) is AOC if for any initial input \( u_0 \) applied on \([-\alpha, 0]\) for any \( \alpha > 0 \), there exist a time \( \tau = (0, \infty) \) and a control input \( u \) applied on \((0, T]\) such that the output is identically zero on \((\tau, \infty)\).

In fact this is a function space (null) controllability. The AOC will depend on rank of \( B_{o1} \) and \( B_2 \), and whether the identical condition rank \([E, A E, \ldots, A^{n-1} E, B_{o1}, B_2] = n\) is required to be necessary and sufficient is not clear at the moment. Some conditions may be derived using the ideas in [161]. However it appears that the following unproved result will serve as a better sufficient (if not necessary) condition:

\[ \text{Rank } [E, A E, \ldots, A^{n-1} E, E] = n \]

where \( E = B_{o1} + \exp (-\lambda T) B_2 \).

5.4 QOCP for Continuous-Time Systems With Delayed Input Derivative [179]

Consider the continuous time system (5.54) (Sec. 5.3.2) with derivative- and delayed derivative-controls, reproduced here as

\[
\dot{x}(t) = A_0 x(t) + B_0 u(t) + B_{o1} \dot{u}(t)
+ B_1 u(t-T) + B_2 \dot{u}(t-T), \ t \geq 0
\]

and its transformed (to non-input derivative form) system

\[
\dot{z}(t) = A_0 z(t) + B_a u(t) + B_b u(t-T)
\]

where

\[
B_a = B_0 + A_0 B_{o1}, \quad B_b = B_1 + A_0 B_2
\]
through a SSET that gives

\[ x(t) = z(t) + B_{01} u(t) + B_2 u(t - T). \]  

(5.80)

Any QOCP aiming at minimising

\[ J = \frac{1}{2} \int_{\alpha}^{\beta} (z'Qz + u'R_u) \, dt, \]  

or

\[ J = \frac{1}{2} \int_{\alpha}^{\beta} (z'Qz + u'R_0 u + u'(t\cdot T) R_1 u(t\cdot T)) \, dt \]

subject to (5.79) can be solved using well developed standard techniques pointed out earlier (e.g., [45], [83], [96C], [99], [129]). Thus stabilising optimal or suboptimal laws can be obtained. But then the corresponding OCL for system (5.78) derived from that for (5.79) will have still more complicated structure, as evident from the relationship (5.80). However such a law for z-systems may serve as a suboptimal one for the x-system provided of course, the stability is retained. This may be the case if \( T, B_1, B_2 \) are 'small'. Also, the former OCL may be 'simulated' by simulating the plant delays instead of directly measuring them. The scheme will then be a suboptimal one (if still stabilising), being nonoptimal for all initial conditions.

A simpler approach for the input-derivative-delay-system can be examined now. The model to be used is the delay-separated form of system (5.78), i.e., the non-delayed (as well as its derivative) control and delayed (as well as its derivative) control are disjoint. Then in the RHS of (5.78)-(5.79) the term \( u(t\cdot T) \) will be replaced by \( V(t) \), where \((u_0, \dot{u}_0)\) and \((V_1, \dot{V}_1)\) are respectively the controls
acting exclusively in the nondelayed channels and delayed channels. Thus, let the system (5.78) be restructured as

\[
\dot{x}(t) = A_0 x(t) + B_0 u_0(t) + B_{11} \dot{u}_0(t) + B_1 V_1(t) + B_2 \dot{V}_1(t), \quad t \geq 0
\]  

(5.81)

where \( V_1(t) := u_1(t - T), \quad T > 0 \) and initial data is as in Sec. 5.3.1.

The optimisation procedure adopted is identical to that in Sec. 5.2, except that it is simplified for \( C = I_n \). Therefore the derivations etc. will only be outlined.

The Problem Statement: Minimise the cost functional

\[
J = \frac{1}{2} \int_0^\infty \left[ x'(t) Q_0 x(t) + u_0'(t) G_0 u_0(t) + V_1'(t) G_2 V_1(t) \right] dt
\]  

(5.82)

subject to system constraint (5.81) by finding \( u_0(t) \) and \( u_1(t) \) (\( u_1(t) \) symbolising the delayed control), where the symmetric matrices \( Q_0 \geq 0 \in \mathbb{R}^{n \times n}; \quad G_0 > 0, \quad G_2 > 0 \in \mathbb{R}^{n \times n} \).

For the infinite-time QOCP to be meaningful the system (5.81) is assumed absolutely controllable in the sense of Sec 5.3.3.

5.4.1. The solution

5.4.11 Only delayed controls:

A system with a single (pure) time delay in the control and control-derivative is given as

\[
\dot{x}(t) = A_0 x(t) + B_1 V_1(t) + B_2 \dot{V}_1(t)
\]  

(5.83)
The control problem is stated as: to choose a suitable \( u_1(t) \) for system (5.83) so as to minimise
\[
J = \frac{1}{2} \int_t^\infty [x'(\tau) Q_0 x(\tau) + u_1'(\tau) G_1 u_1(\tau)] \, d\tau \tag{5.84}
\]
where \( t \) is a general starting time. Since \( V_1(t) = u_1(t-T) \), (5.84) may be put as
\[
J = \frac{1}{2} \int_t^{t+T} x'(\tau) Q_0 x(\tau) \, d\tau + \frac{1}{2} \int_t^{t+T} [x'(\tau) Q_0 x(\tau) + V_1'(\tau) G_1 V_1(\tau)] \, d\tau 
\]
\[ = J_1 + J_2 \tag{5.85} \]
Choice of \( u_1 \) on the interval \((t, \infty)\) cannot affect \( J_1 \) since \( u_1 \) is not effective until \( t + T \). Thus, the problem is to choose \( V_1 \) on \((t + T, \infty)\) to minimise \( J_2 \) for total minimum cost. By the choice of SSE transformation
\[
x(t) = z(t) + B_2 V_1 \tag{5.86}
\]
eqns. (5.83), and \( J_2 \) in (5.85) are respectively transformed into
\[
\dot{z}(t) = A_0 z(t) + B_1 V_1(t) \tag{5.87}
\]
\[
\dot{B}_b = B_1 + A_0 B_2 \\
\hat{J}_2 = \frac{1}{2} \int_t^{t+T} [(z + B_2 V_1)' Q_0 (z + B_2 V_1) + V_1' G_1 V_1] \, d\tau . \tag{5.88}
\]
The problem has a standard solution given below. (The next section gives the specialised solution.)

Proposition 5.7:
Let
\[
\hat{G} = G_1 + B_2' Q_0 B_2 \\
\hat{A} = A_0 - B_b' \hat{G}^{-1} B_2' Q_0
\]
The optimal control minimising $J = J_1 + J_2$ subject to system (5.87) is given by

$$u_1(t) = -\hat{G}^{-1}(B_2^{\prime} \hat{P} + B_2^\prime Q_0) z(t + T)$$

(5.89)

where $\hat{P}$ is the positive definite equilibrium solution of the matrix Riccati equation

$$\hat{A} \hat{P} + \hat{P} \hat{A} - \hat{P} B_0 \hat{G}^{-1} B_0^\prime \hat{P} + Q = 0.$$  

(5.90)

From (5.87)

$$z(t+T) = \Phi(T) z(t) + \int_t^{t+T} \Phi(t+T-\tau) B_0 V_1(\tau) \, d\tau$$

Or

$$= \Phi(T) z(t) + \int_{t-T}^t \Phi(t-\tau) B_0 u_1(\tau) \, d\tau.$$  

(5.91)

Further, substitution of Eqns. (5.91) and (5.86) in (5.89) gives the desired LTI law in terms of $x(t)$. As expected, the law involves a knowledge of entire delay storage $u_1(t)$ between $t-T$ and $t$, and depends on feedback from complete state $\{x(t), u_1(t + \theta), \theta \in [-T, 0]\}$. Also (taking Laplace transformation) its structure is similar to that of the Smith Predictor [144].

5.4.1.2 Both delayed- and nondelayed- controls:

Considering the original problem it is observed that controls $u_1$ and $\dot{u}_1$ are not effective until after a delay $T$ whereas $u_0$ and $\dot{u}_0$ variables have immediate effect. So
after a time $T$ complete freedom of the choice for $u_0$ and $u_1$ is allowed. Referring to Sec. 5.2.2, the cost $J$ in (5.82) is split as in Eqn. (5.13) being (dropping the arguments)

$$J = \frac{1}{2} \int_0^T \left( x'Q_0 x + u_0' G_c u_0 \right) \, dt + \frac{1}{2} \int_T^\infty \left( x'Q_0 x + u_0' G_0 u_0 + V_1' G_1 V_1 \right) \, dt$$

$$: = J_1 + J_2 . \quad (5.92)$$

So

$$\min_{u_0, u_1} J = \min_{u_0(0, T)} \left[ J_1 + \min_{u_0(T, \infty)} J_2 \right]$$

and the optimisation is carried out in two stages:

(i) usual application of LQOCP during $[T, \infty)$ to minimise $J_2$,

(ii) minimisation of $J$ during $[0, T]$ so that the disturbance response due to initial delay storage $V_1(0, T)$ is eliminated at all times: a standard linear tracking problem.

Now using the SSET (5.57) as

$$x(t) = z(t) + B_{01} u_0(t) + B_2 V_1(t) \quad (5.93)$$

Consider the transformed system of (5.81) as

$$\dot{z}(t) = A_0 \ z(t) + B_a u_0(t) + B_b V_1(t) \quad (5.94)$$

where $B_a = B_0 + A_0 B_{01}$, $B_b = B_1 + A_0 B_2$.

Let

$$B_{0t} := [B_a, B_b], \quad B_{olt} := [B_{01}, B_2]$$
Then, the following result can be obtained.

Proposition 5.8: The optimal controls to minimise $J$ in (5.82) over $[T, \infty)$ subject to system (5.81) (and so (5.94) - (5.93)) are given by

$$u_0(t + T)' = -K_0 z(t + T)$$

$$V_1(t + T) = u_1(t) = -K_1 z(t + T)$$

where (m,n) gain matrices

$$K_0 = (D_1 B_a' + D_2 B_b') P + (D_1 B_{0l} + D_2 B_2') Q_0$$

$$K_1 = (D_3 B_a' + D_4 B_b') P + (D_3 B_{0l} + D_4 B_2') Q_0$$

(5.97)
and $P$ is the (n.n) symmetric positive definite equilibrium solution of the matrix Riccati equation

$$FA + A'P - PB_{ot}C^{-1}B_{ot}'P + Q = 0.$$  (5.98)

Also, the complete LTI control laws for a general starting time are

$$u_0(t) = -Kz(t) - \bar{G}_0^{-1}B_{ol}Q_0B_2u_1(t - T)$$
$$- \bar{G}_0^{-1}B_a \int_0^T \Phi'(r, 0) [P_S(r)B_{ls}$$
$$\Phi(r, 0)] P_s(t + r - T) dr,$$  (5.99)

$$u_1(t) = -K_1 \Phi(T, 0)z(t) - K_1 \int_0^T [\Phi(T, 0)\Phi^{-1}(r, 0)$$
$$- \Psi(r, 0)P_S(r)] B_{ls} + Q_0B_{2s}u_1(t + r - T) dr$$  (5.100)

where symmetric p.d. (n.m) matrix $P_S(t)$ satisfies the matrix Riccati equation

$$-\dot{P}_S(t) = P_S(t)A_S + A'_SP_S(t) - P_S(t)B_{a}\bar{G}_0^{-1}B_{a}'P_S(t) + Q_s$$

$$P_S(T) = P,$$  (5.101)

$$K = \bar{G}_0^{-1}[B_a'P_S(0) + B_{ol}'Q_0] \in \mathbb{R}^{m,n};$$  (5.102)

\(\Phi(t, \tau)\) is the solution of the time varying matrix differential equation in the interval \((\tau, T)\) given by

$$\frac{\partial \Phi(t, \tau)}{\partial t} = [A_S - B_a\bar{G}_0^{-1}B_a'P_S(t)]\Phi(t, \tau), \Phi(\tau, \tau) = I$$  (5.103)

and

$$\Phi(t, s) = \int_0^s \Phi(T, \tau)B_a\bar{G}_0^{-1}B_a'\Phi'(s, \tau) d\tau.$$  (5.104)

Further, substitution of (5.93) in the above gives the laws with respect to $x(t)$.  

---
Derivation (outline):

Derivation of these results are quite straightforward and are initially obtained for the system \((5.94)\), and the corresponding cost function obtained from \((5.82)\) and \((5.93)\). As discussed in Sec. 5.2.2, after time \(T\) (i.e., between \([T, \infty)\)) the laws are given by \((5.96)\). To compute \(K_0\) and \(K_1\) the scalar Hamiltonian is formed with the Lagrangian \(\lambda(t) \in \mathbb{R}^n\) as

\[
H(z, \lambda, u_0, V_1, t) = -\frac{1}{2} [(z + B_{o1} u_0 + B_2 V_1)' Q_o \nonumber \\
(z + B_{o1} u_0 + B_2 V_1) + u_0' G_o u_o + V_1' G_1 V_1] 
onumber \\
+ \lambda' [A_o z + B_a u_o + B_b V_1].
\]  

(5.105)

Evaluating the necessary conditions for optimality \([103]\)

\[
\dot{\lambda} = -\frac{3}{\lambda} \frac{\partial H}{\partial z}, \quad \frac{3}{\lambda} \frac{\partial H}{\partial u_0} = 0 \quad \text{and} \quad \frac{\partial H}{\partial V_1} = 0,
\]

and with usual manipulations, following conditions are obtained (utilising designation of terms in \((5.95)\))

\[
[G] \begin{bmatrix} u_0(t) \\ V_1(t) \end{bmatrix} = \begin{bmatrix} B_{a}', -B_{o1} \cdot Q_o \\ B_{b}', -B_2 \cdot Q_o \end{bmatrix} \begin{bmatrix} \lambda(t) \\ z(t) \end{bmatrix}
\]

or

\[
\begin{bmatrix} u_0(t) \\ V_1(t) \end{bmatrix} = G^{-1} \begin{bmatrix} B_{a}' \\ B_{b}' \end{bmatrix} \lambda(t) - G^{-1} \begin{bmatrix} B_{o1} \cdot Q_o \\ B_2 \cdot Q_o \end{bmatrix} z(t),
\]

(5.106)

\[
\begin{bmatrix} z(t) \\ \cdot \lambda(t) \end{bmatrix} = \begin{bmatrix} A & B_{o1} \cdot G^{-1} & B_{o1}' \end{bmatrix} \begin{bmatrix} z(t) \\ \cdot \lambda(t) \end{bmatrix} = [M] \begin{bmatrix} z(t) \\ \cdot \lambda(t) \end{bmatrix}.
\]

(5.107)

The time solution of \((5.107)\) can be defined in terms of the assumed distinct eigenvalues and eigenvector components of

\(M\) as
is the (2n×2n) modal matrix of \( M \). As is known (e.g., [100; Sec. 15.2]), \( M \) has \( \{ s_i \} \), \( i = 1, 2, \ldots, n \) stable, and \( \{ s_j \}, \ j = n + 1, \ldots, 2n \) unstable-eigenvalues. Identifying \( A^1 = \text{diag} \[ s_1, \ldots, s_n \] \) and \( A^2 = \text{diag} \[ s_{n+1}, \ldots, s_{2n} \] \), it is possible that \( U^\top M U = \Lambda = \text{Block diag} \[ \Lambda_1, \Lambda_2 \] \).

Thus, eliminating the divergent modes corresponding to the unstable eigenvalues in \( \Lambda_2 \) so that the condition of asymptotic stability is satisfied, the eigenvector solutions are given by

\[
\begin{align*}
\mathbf{z}(t) &= U_{11} \exp (A_1 t) \cdot U_{11}^\top \mathbf{z}(t_0) \\
\lambda(t) &= U_{21} U_{11}^\top \mathbf{z}(t).
\end{align*}
\] (5.108)

Using (5.108) in (5.106), the OCL in terms of \( \mathbf{z}(t) \) (from (5.95)) at all \( t \) is found to be

\[
\begin{align*}
\mathbf{u}(t) &= \text{Colm} \ [\mathbf{u}_0(t), \mathbf{V}_1(t)] \\
&= G^\top \left( B_{ot} U_{21} U_{11}^\top - B_{ot} \mathbf{Q}_0 \right) \mathbf{z}(t).
\end{align*}
\] (5.109)

Further expanding \( MU = U \cdot \Lambda \) for stable modes only, one obtains

\[
\begin{align*}
A^\top U_{11} + B_{ot} G^\top B_{ot} U_{21} &= U_{11} \Lambda_1,
\end{align*}
\]

and

\[
Q U_{11} - A^\top U_{21} = U_{21} \cdot \Lambda_1.
\]

Eliminating \( \Lambda_1 \) from above relations gives, the algebraic matrix Riccati Eqn.(5.98), where the (n×n) matrix \( \mathbf{P} = -U_{11} U_{11}^\top \)
is its p.d. solution. This, along with (5.109) and expansion of \( G^{-1} \) from (5.95) leads to the values of \( K_0 \) and \( K_1 \) in (5.97). The optimal trajectory is then determined from (5.93) as

\[
x(t) = \left[ I - B_0 t \ G^{-1} (E_0' P + B_0' Q_0) \right] z(t) \quad (5.110)
\]

Or,

\[
= \left[ I - (B_0 K_0 + B_2 K_1) \right] z(t) .
\]

Also, the minimal value of \( J_z \) can be obtained using the fundamental principle of continuous dynamic programming [112A; p 75] to be

\[
J [x(T), T] = J [z(T), T] = \frac{1}{2} z(T)' P z(T) . \quad (5.111)
\]

Once \( K_0 \) and \( K_1 \) are known, it remains to determine \( z(t + T) \) which requires solution of a linear quadratic tracking problem in \([0,T] \) using a standard method (e.g., [112A; p 96]). Let \( z(T) \) and \( x(T) \) be the desired states at \( t = T \) due to application of \( u_0(t) \); \( z_0(t) \) (and \( x_0(t) \)) be the responses due to disturbance \( V_1(0,T) \). Reformulating the cost function

\[
\hat{J}_1 = \frac{1}{2} \left[ z(T) - z_0(T) \right]' H [z(T) - z_0(T)]
\]

\[
+ \frac{1}{2} \int_0^T \left[ (x(t) - x_0(t))' Q_0 (x(t) - x_0(t))
\]

\[
+ u_0'(t) G_0 u_0(t) \right] dt ,
\]

and assuming the costate to be of the form

\[
\lambda(t) = P_S(t) z(t) + w(t)
\]

and finally constraining the disturbance response to zero (i.e., \( x_0(t) = 0, x_0(T) = 0 \), \( z_0(T) = 0 \) for all time) the
control law has the form
\[ u_0(t) = \Phi_0^{-1} \left[ \left( B_a' P_s(t) + B_{01}' Q_0 z(t) \right) 
+ B_a' w(t) + B_{01}' Q_0 B_2 V_1(t) \right]. \] (5.112)

Here \( P_s(t) \) is given as in (5.101) and the \( n \)-vector \( w(t) \) satisfies,
\[ -\dot{w}(t) = [\dot{A}_S - B_a \Phi_0^{-1} B_a' P_s(t)]' w(t) 
+ [P_s(t) B_1 S + Q_0 B_2 S] V_1(t). \] (5.113)
with \( w(T) = 0 \).

It is well known that the OCL (5.112) is often computationally difficult to realise since it involves \( w(t) \) which must be solved backward in time. Thus a knowledge of \( V_1(0,T) \) in the delay interval is necessary which is quite often unknown initially (in fact, without the null value assumptions of \( z_d \) and \( x_d \) they also appear in (5.112) and (5.113), and must be known a priori). However, using the transition matrix \( \Phi(t,T) \) for (5.113) as in (5.103), and explicit solution \( w(t) \) can be obtained (Sec. 5.2.2). Then following similar steps as in Secs. 5.2.22 and 5.2.23, the control laws for \( u_0(t) \) and \( u_1(t) \) given in Eqns. (5.99) and (5.100) respectively are obtained.

5.5 QOCO for Discrete-Time Systems with Delayed Input Derivative [181]

The discrete version of system (5.78) with control derivative and delayed control derivative is given by
Eqn. (5.61) (Sec. 5.3.22) reproduced below

\[ x(K + 1) = \left[ \Phi x(K) + D_1 u(K) + \Delta_1 V(K) \right] \]
\[ + \left[ B_{01} \{ u(K + 1) - u(K) \} + B_2 \{ V(K) + V(K) \} \right], \quad (5.114) \]

with terms defined in (5.60). Its transformation into the continuous coordinate system through the SSET (5.63) - (5.64) is given in (5.59) as

\[ z(K + 1) = \Phi z(K) + d u(K) + \Delta V(K), \text{ Or} \]
\[ z(K + 1) = \Phi z(K) + D \tilde{V}(K) \quad (5.115) \]

where \( d = (D_1 + D_2), \Delta = (\Delta_1 + \Delta_2), D = [d, \Delta], \tilde{V}(K) = \text{Colm} [u(K), V(K)]. \) Further, let the relationships (5.63) and (5.64) be represented respectively through

\[ x_1(K + 1) = z(K + 1) + B_2 V(K) + B_{01} u(K) \]
\[ = z(K + 1) + B \tilde{V}(K) \quad (5.116) \]
\[ x_2(K) = z(K) + B_2 V(K) + B_{01} u(K) \]
\[ = z(K) + B \tilde{V}(K) \quad (5.117) \]

where \( B = [B_{01}, B_2]. \)

For the optimisation of system (5.115) (i.e., (5.59)), a logical and unambiguous scalar quadratic performance index is suggested as in [157A], and is given in terms of the original states showing discontinuities at sampling instants (Sec. 5.3.22) as
\[ J_N = \sum_{K=0}^{N} \left[ x_1'(K + 1) Q_1 x_1(K + 1) + x_2'(K) Q_2 x_2(K) \right. \\
\left. + u'(K) G_0 u(K) + V'(K) G_1 V(K) \right] \] (5.118a)

which through (5.116) and (5.117) becomes

\[ J_N = \sum_{K=0}^{N} \left[ z(K) Q z(K) + z'(K) W \tilde{V}(K) \right. \\
\left. + \tilde{V}'(K) W' z(K) + \tilde{V}(K) R V(K) \right] \] (5.118b)

where

\[ q = \Phi' q_1 \Phi + q_2, \]
\[ w = \Phi' q_1 (D + B) + q_2 B, \]
\[ R = (D + B)' q_1 (D + B) + B' q_2 B + G, \]
\[ G = \begin{bmatrix} G_0 & 0 \\ 0 & G_1 \end{bmatrix}; \ q_1, q_2 > 0 \text{ and } G_0, G_1 > 0. \]

The system is assumed to be absolutely controllable in the sense of Sec. 5.3.3. Then, the discrete minimum principle \[ [102C] \] may be applied. The conditions of its valid application are satisfied by the linearity of the system, the convexity of the cost functional and the nonsingularity of \( \Phi \). Since (5.118b) can be put as

\[ J_N = \sum_{K=0}^{N} \left[ z'(K), \tilde{V}'(K) \right] \tilde{Q} \left[ \begin{array}{c} Z(K) \\ \tilde{V}(K) \end{array} \right], \quad \tilde{Q} = \begin{bmatrix} Q & W \\ W' & R \end{bmatrix} \]

its convexity requires that the matrix \( \tilde{Q} \) be positive definite. Remembering that (Sec. 3.3.22) the delay time \( T \) and the sampling time \( \hat{T} \) are related by \( T = M \hat{T}, M \) an integer, the cost (5.118a) can be written as
\[ J_N = \sum_{K=0}^{M-1} L_K (z(K), u(K), V(0, M)) \]
\[ + \sum_{K=M}^{N-1} L_K (z(K), u(K), V(K)) + z'(N) S z(N) . \quad (5.119) \]

Choice of \( V(K) \) on \([M, N-1]\) cannot affect the first term of (5.119), since \( V(K) = u(K - M) \) and is not effective until \( K = M \). Thus, the problem is to choose both \( u(K) \) and \( V(K) \) on the interval \((M, N-1)\) to minimise the second term of (5.119). The third term represents the terminal cost in which \( S_N = 0 \) as \( N \to \infty \), and \( V(0, M) \) symbolises the initial delay storage entering as a known plant disturbance. The functional \( L_K(\ldots, \ldots) \) is clear from (5.118b). In what follows these two control laws, one having the immediate effect and the other having the delayed effect, will be respectively designated for convenience as \( u_0(\cdot) \) and \( u_1(\cdot) \) (i.e., \( u_1(i) = V_1(i + M) \)). Following results state these laws, analogous to the continuous-time case. So as in (5.92) minimisation of \( J_N \) is splitted up as

\[
\min J_N = \min_{u_0, u_1} \left[ \min_{u_0(0, M)} \min_{u_1(M, \infty)} J_1 \right] . \quad (5.120)
\]

Let \( W_1 : = \phi' q_1 (d + B_0) + Q_2 B_0 \in \mathbb{R}^{n \times m} \)
\( W_2 : = \phi' q_1 (\Delta + B_2) + Q_2 B_2 \in \mathbb{R}^{n \times m} \)
\( R_1 : = (d + B_0) q_1 (d + B_0) + B_0 \, Q_2 B_0 + G_0 \in \mathbb{R}^{m \times m} \)
\( R_2 : = (\Delta + B_2) q_1 (\Delta + B_2) + B_2 \, Q_2 B_2 + G_1 \in \mathbb{R}^{m \times m} \)
\( S : = (d + B_0) q_1 (\Delta + B_2) + B_0 \, Q_2 B_0 \in \mathbb{R}^{m \times m} \)
Proposition 5.9: The optimal controls to minimise $J_N$ in 
(5.118b) over $[M \infty)$ subject to discrete-system (5.115) are
given by

\[ u_0(i) = F_0(i) z(i) \]

\[ V_{i}(i + M) = u_{1}(i) = F_{1}(i) z(i + M) \quad (5.122) \]

where (m.n) gain matrices

\[ F_0(i) = -[R_{11} W_1' + R_{12} W_2' - (R_{11} d' + R_{12} \Delta_1) \Phi_t^{-1}(\widetilde{K}(i) + q_t)] \]

\[ F_1(i) = -[R_{21} W_1' + R_{22} W_2' - (R_{21} d' + R_{22} \Delta_1) \Phi_t^{-1}(\widetilde{K}(i) + q_t)](5.123) \]

and $\widetilde{K}(i)$ is (m.n) p.d. matrix satisfying the discrete matrix Riccati equation

\[ -\widetilde{K}(i) = q_t + \Phi_t^{'} \widetilde{K}(i+1) [I + D_t \widetilde{K}(i + 1)]^{-1} \Phi_t \quad (5.124) \]

with $\widetilde{K}(N) = S_N = 0$ as $N \rightarrow \infty$.

Also the complete LTI control laws for general starting time
are given by
\[ u_0(i) = \left[ I - H(0) B_{ol} \right]^{-1} \left[ -H(0) x(i) + (H(0) B_2 - \Delta) u(i-M) + G_B \right] \]

\[ + G_A \sum_{p=1}^{M-2} \sum_{K=1}^{M-K-p} E_{M-p} u_1(i-p) \quad (5.125) \]

\[ u_1(i) + F_1(i) \phi [B_2 + B_{ol} H(0) B_2] u_1(i-M) + B_{ol} G(i) \]

\[ - F_1(i) \sum_{p=0}^{M-1} \phi [d G(i+p) + (dH(0) B_2 + \Delta) u_1(i-M+p)] \]

\[ = F_1(i) \phi [I - H(0) B_{ol}]^{-1} H(0) x(i) \]

\[ - F_1(i) \sum_{p=0}^{M-1} \phi [d G(i+p) + (dH(0) B_2 + \Delta) u_1(i-M+p)] \quad (5.126) \]

where

\[ H(0) = R_0 + d' L(1) d^{-1} d' L(1) \phi \]

\[ G_A = \left[ (R_0 + d' L(1) d^{-1} d' L(1) d R_0^{-1} d' - R_0^{-1} d) \right] \]

\[ G_B = (R_0 + d' L(1) d) d' L(1) d \quad (5.127) \]

\[ G(i) = G_A \sum_{p=1}^{M-2} \sum_{K=1}^{M-K-p} E_{M-p} u_1(i-p) \]

\[ + G_A \Lambda_1 u_1(i-M+1) - G_B u_1(i-M) \quad (5.128) \]

\[ \Lambda_1 = \phi' [L(i+1) - L(i+1) d (R_0 + d' L(i+1) d)^{-1} d' \]

\[ \cdot d' L(i+1)] d R_0^{-1} d' - \phi' \quad (5.129) \]

\[ E_1 = - \phi' [L(i+1) - L(i+1) d (R_0 + d' L(i+1) d)^{-1} d' \]

\[ \cdot d' L(i+1)] \Delta \quad (5.130) \]

and \( L(i) \) satisfies

\[ -L(i) = Q + \phi' (L(i+1) - L(i+1) d) \]

\[ \cdot (R_0 + d' L(i+1) d)^{-1} d' L(i+1) \phi \quad (5.131) \]

with \( L(M) = \tilde{K} \)
and $\hat{K}$ is the steady state solution of (5.124) as $N \to \infty$.

Derivation (outline):

The results are derived in two stages: Solution of (A) an infinite-time standard linear quadratic regulator problem in the interval $[M, \infty)$ giving law (5.112), and (B) a discrete linear tracking problem in the interval $[0, M]$ leading to the laws (5.125) and (5.126).

5.5.1 Linear regulator problem:

For this, the cost functional is simplified to

$$J_1 = \sum_{K=M}^{N-1} L_K(z(K), u(K), V(K)) + z'(N) S_N z(N)$$

where from (5.118b) and (5.121) (dropping argument $K$)

$$L_K(z, u, V) = z'Qz + z'W_1 u + z'W_2 V + u'W_1' z + V'W_2' z + u'R_1 u + u'SV + V'S' u + V'R_2 V.$$  (5.132)

Following standard procedure of Dorato and Levis [162] the closed loop OCL are obtained as (5.122)–(5.123). These laws will need determination of $z(i+M)$. As in the continuous time case, from Eqns. (5.59), (5.63) and (5.64) expressing $z$ in terms of $x_2$ (which is taken as the $x$-variable)

$$z(i+M) = \Phi^M[x(i)-E_2u_1(i-M)-B_0u_0(i)] + \sum_{K=0}^{M-1} \Phi^{M-1-K} \cdot [d u_0(i + K) + \Delta u_1(i + K - M)].$$  (5.133)

Evaluation of $z(i+M)$ depends on both $u_0$ and $u_1$ (as initial delay storage), and $u_0$ is as yet undetermined on
Using fundamental principle of dynamic programming, the second part of Eqn. (5.120) gives over $\left[ M, \infty \right)$

$$
\min J_1 = \frac{1}{2} z'(M) \tilde{K} z(M)
$$

(5.134)

where, as $N \to \infty$ the steady state solution of (5.124) $\tilde{K}(i) \to \tilde{K}$. Then, a minimisation of (5.120) is performed on $[0, M]$ taking into account that the initial delay storage $V(0,M)$ enters as a plant disturbance into the system that has input $u_0(i)$. This leads to solving the linear tracking problem so that the disturbance response due to $V(0,M)$ is eliminated at all times.

5.5.2 Linear tracking problem:

Let $z(M)$ be the desired response at $i = M$ due to application of $u_0(i)$, and $z_d(i)$ be the response due to disturbance. Similar argument as in continuous-time case (in Sec. 5.4.12) is followed to minimise

$$
\hat{J}_0 = \frac{1}{2} \left[ z(M) - z_d(M) \right]' \tilde{K} \left[ z(M) - z_d(M) \right]
$$

$$
+ \frac{1}{2} \sum_{i=0}^{M-1} \left[ (z(i) - z_d(i))' Q (z(i) - z_d(i)) \right]
$$

$$
+ u_0'(i) R_0 u_0(i)
$$

(5.135)

for the system (5.59) (with $K = i$). The solution of the discrete linear tracking control under disturbance $V(i) = u_1(i - M)$, following Findyck [163] is
\[ u_0(i) = -[R_0 + d'L(i + 1) \cdot dL(i+1) \cdot \phi z(i)] \]
\[ + [R_0 + d'L(i + 1) \cdot dL(i+1)] \cdot dR' \cdot d'g(i + 1) - R_0 \cdot d'g(i+1) - [R_0 + d'L(i+1)] \cdot dL(i+1) \cdot V(i) \]
\[ : = -H(i) z(i) + G(i) \quad (5.136) \]

where \( L(i) \) satisfies (5.131) with final condition \( L(M) = K \), and \( g(i) \) is obtained from

\[ -g(i) = - A_i g(i+1) E_i V(i) - q z_d(i) \quad (5.137) \]

with \( g(M) = -S z_d(M) = -K z_d(M) \) and \( A_i, E_i \) are as in (5.129), (5.130).

For reasons stated in continuous time case it is assumed that \( z_d(i) = 0 \) throughout the delay interval and so in (5.137) \( g(M) = z_d(M) = z_d(i) = 0 \). Thus the initial control is

\[ u_0(0) = -H(0) z(0) + G(0) \]
\[ G(0) = G_A g(1) - G_B V(0) \quad (5.138) \]

where \( H(0), G_A, G_B \) are as defined in (5.127)-(5.136).

5.5.3 Solution of the tracking and Riccati equations:

The discrete matrix Riccati equation can be solved backwards, starting with the index \( (i+1) = M \), \( L(M) = K \), and the values \( L(M-1), L(M-2), \ldots, L(1), L(0) \) are successively computed from (5.131) and stored. The discrete tracking Eqn. (5.137) can be solved backward in time starting with index \( (i+1) = M \), \( g(i+1) = g(M) = 0 \) to give
When \( j = M - 1 \) and \( M \) respectively, from (5.139) the values of \( g(1) \) and \( g(0) \) are obtained. For the above expression \( \Phi(i) \) and \( \Delta(i) \) are constants and their values depend on \( L(i+1) \) obtained by the solution of the Riccati equation, which are precomputed and stored.

5.5.4 Expressions for control inputs:

Since zero time is completely arbitrary, generalising for any sampling index \( i \) gives

\[
\begin{align*}
u_0(i) &= -H(0) z(i) + G(i), \text{ or} \\
u_0(i) &= -H(0) x(i) + H(0) B_2 V_1(i) + H(0) B_0 u_0(i) + G(i)
\end{align*}
\]

where \( G(i) \) depends on \( g(l + i) \) and past values of \( V_1(i) \); i.e., \( V_1(i-0), V_1(i-1), \ldots, V_1(i-(M-p)) \) etc. On defining \( V_1(i-(M-p)): = u_1(i-M-M-p) = u_1(i-p) \) and using (5.139), Eqn. (5.128) is obtained. This when substituted in (5.140) obtains the law \( u_0(i) \) in (5.125). Having found \( u_0(i) \) in the interval \([0, M]\), \( u_1(i) \) in Eqn. (5.126) may be obtained from expression (5.120) using Eqn. (5.125).

5.6 Examples

To illustrate the procedure, a continuous time scalar system with input derivative is taken. Detailed calculations
are omitted and only end results are given. Effects of varying system parameters as well as weightages in the cost functions are mainly studied. Finally the results are commented upon.

Example 5.2:

Minimise \( J = \frac{1}{2} \int_0^\infty (x^2(t) + u^2(t)) \, dt \)

subject to \( \dot{x}(t) = 3x(t) + u(t - \pi/12) + \dot{u}(t - \pi/12) \),

and \( x(0) = 1, \ u(s) = 0, \ \dot{u}(s) = 0, \ s \in [0, \pi/12) \).

This open loop-unstable system has \( a_0 = 3, \ b_0 = 0, \ b_01 = 0, \ b_1 = 1, \ b_2 = 1, \ T = \pi/12, \ q_0 = 1, \ g_0 = 1 \) and \( g_1 = 0 \). There is no immediate control path and so only \( u_1 \) is to be found out. The transformation (5.80) is \( x(t) = z(t) + V(t) \). Following the procedure of Sec. 5.4 the law and the transformed-state response are found out to be

\[
\begin{align*}
    u_1(t) &= -2.87104 \left[ z(t) + 4 \int_0^{\pi/12} e^{-3s} u_1(t+s-\pi/12) \, ds \right], \\
    z(t) &= e^{3t} \left[ 1 + 4 \int_0^t e^{-3s} u_1(t-s) \, ds \right].
\end{align*}
\]

These values of \( u_1(t), z(t) \) as well as actual state \( x(t) \) are plotted in Fig. 5.3 as PLOTS-I (\( z_1, x_1, U_1 \)).

Example 5.3:

Same system as in Example 5.2 but now weightage on control is reduced to \( g_0 = 0.5 \). Then

\[
\begin{align*}
    u_1(t) &= -4.3866 \left[ z(t) + 4 \int_0^{\pi/12} e^{-3s} u_1(t+s-\pi/12) \, ds \right].
\end{align*}
\]
The \( z(t) \) is similar in nature to that in Example 5.2. These values and corresponding \( x(t) \) are plotted in Fig. 5.3 as PLOTS-II \((z_2, x_2, U_2)\).

**Example 5.4:**

Same system and cost function as in Example 5.2 but now \( a_0 = -0.5 \). Then

\[
\begin{align*}
    u_1(t) &= -0.59398[z(t) + 0.5 \int_0^{\pi/12} e^{0.5s} u_1(t+s-\pi/12)ds], \\
    z(t) &= e^{-0.5t}[1 + 0.5 \int_0^t e^{0.5s} u_1(t-s)ds].
\end{align*}
\]

These values and corresponding \( x(t) \) are plotted in Fig. 5.4 as PLOTS-III \((z_3, x_3, U_3)\).

**Example 5.5:**

Same system and cost function as in Example 5.2 but now \( a_0 = -3 \). Then

\[
\begin{align*}
    u_1(t) &= -0.08592 [z(t) - 2 \int_0^{\pi/12} e^{3s} u_1(t+s-\pi/12)ds], \\
    z(t) &= e^{-3t} [1-2 \int_0^t e^{3s} u_1(t-s)ds].
\end{align*}
\]

These values and corresponding \( x(t) \) are plotted in Fig. 5.4 as PLOTS-IV \((z_4, x_4, U_4)\).

Following observations can be made from the plots in above examples. PLOTS-I for the unstable system show a growing oscillatory tendency of \( z(t) \) until the end of first delay interval when the control 'acts' and checks the growth. But there is a steady state error of around 0.8 in \( z(t) \) and control attains a steady value of -0.8, although the actual
FIG. 5.3 Response of optimal system

PLOTS—— I, II
X • STATE  U • CONTROL
TIME = 1 DIVN = \pi/96 Sec.

(1) \( Z_1, X_1, U_1 \)—— WITH \( \theta_0 = 1, a = 3 \)
(II) \( Z_2, X_2, U_2 \)—— WITH \( \theta_0 = 0.5, a = 3 \)
FIG. 5.4 Response of optimal system

Plots: III, IV

X = State, U = Control

Time: \(1 \text{ DIVN} = \pi / 96\) Sec.

(III) \(Z_3, X_3, U_3\) --- with \(\theta_0 = 1, \sigma = -5\)

(IV) \(Z_4, X_4, U_4\) --- with \(\theta_0 = 1, \sigma = -3\)
state $x(t)$ comes to zero. This may be attributed to the value of control weightage $g_0 = 1$ which is appreciable. In PLOTS-II for reduced $g_0 = 0.5$, these errors reduce but not significantly. However $x(t)$ attains zero value earlier. PLOTS-III are of a stable but lightly damped system with same $g_0 = 1$. Control goes to zero along with the states. PLOTS-IV are for a stabler system with same $g_0 = 1$. The control and states almost simultaneously approach zero. In all these cases the discontinuities in $x(t)$ at $t = \pi/12 = T$ are clearly noticed (Eqn. (5.80)). It is expected that further reduction of $g_0$ (and if necessary, penalising the state more) may give better results for first two cases. Further, presence of $u_0(t)$ terms may also give improved results.

It may be noted that the scalar system $\dot{x}(t) = ax(t) + u(t - T)$ with feedback law $u(t) = -Kx(t)$ will be stable ([1; pp 45, 444, 450]) iff (i) $a < 1/T$ and (ii) $a < K < ((w_1^2/T^2) + a^2)^{1/2}$ where $w_1$ is the root of $w = aT \tan w$ between 0 and $\pi$, and if $aT = 0$, $w_1 = \pi/2$. Thus for $a = 3$ the stability bounds are defined as: $T < 1/3$ and so $3 < K < 3\sqrt{2}$ for $T = \pi/12$. Now with $K = 4$ the closed loop system can be seen to be lightly damped (from root location). However with delayed derivative of the control present the state can be brought to zero earlier at the cost of applying a complicated law. Additional presence of a nondelayed control component can further improve the result.

On the other hand, the following characteristics pertain to the system of Example 5.2 with control $u(t) = p x(t)$.
The closed loop system is $\dot{x}(t) - px(t - \pi/12) = 3x + px(t - \pi/12)$. This is a neutral DDE if $p \neq 0$, and so there exists a possibility of an infinite number of eigenvalues on a vertical strip of the complex plane. From [1; Ch. 12] it is seen that there is one root chain which is given by [1; Thm. 12.10]

$$\lambda_m = \left(\frac{12}{\pi}\right) \log |p| + j(2\pi m + \text{arg}(p)) + O(1),$$

$m = 0, \pm 1, \pm 2, \ldots \ldots$.

So if the root chain is to lie in the open LHP $\text{Re} \lambda < 0$, and is uniformly bounded away from the imaginary axis, then $|p| < 1$. Then from the CQP a rough computation shows that whenever $p < 0$ and $|p| < 1$ there occurs at least one real eigenvalue $\lambda > 0$. Thus let $p > 0$ and $|p| < 1$ (which guarantees a real $\lambda < 0$). But then it is further found out that for every value of $p$ there occurs at least one imaginary zero of the CQP. The system thus cannot be stabilised by a simple proportional feedback of $x(t)$. 