CHAPTER - 5

UPPER BOUND OF THE NUMBER OF REAL ROOTS
OF A RANDOM ALGEBRAIC POLYNOMIAL
5.1 ON THE UPPER BOUND OF THE NUMBER OF REAL ROOTS OF A RANDOM ALGEBRAIC EQUATION WITH DEPENDENT COEFFICIENT.

5.1.1 Introduction

Let

\[ f_n(z, \omega) = \sum_{k=0}^{n} a_k(\omega) z^k \]

represents a random algebraic equation of degree 'n' whose coefficients are random variables. The study of real roots of \( f_n(z, \omega) = 0 \), its bounds, its distribution etc. have been studied by many authors to a great extent. In all literature hitherto the random coefficients are considered to be independent random variables. Recent studies are those of Sambadham [27], where they considered the \( a_k(\omega) \) to be normally distributed with mean zero and joint density function

\[ |M|^{1/2} (2\pi)^{-n/2} \exp \left[ -\frac{1}{2} \vec{a}' M \vec{a} \right] \]

where \( M^{-1} \) is the moment matrix \( \delta_{ij} = 1, \rho_{ij} = \rho, 0 < \rho < 1, i \neq j, j = 1 \ldots n \) and \( \vec{a}' \) is the transpose of the column vector \( \vec{a} \). They got the same result on upper bound of \( N_n \) as in case of independent case. Here our aim is to study the case of dependent coefficients with symmetric stable distribution which has not been studied till date.

Since consideration of random coefficients of RFS series as stochastic integral gives an opportunity to study RFS series with dependent coefficients, we
have tried to extend this technique to study random algebraic polynomial with
dependent random coefficients.

Infact we have considered the weighted random algebraic polynomial
(5.2) \[ \sum a_k A_k(\omega) z^k = 0 \]
where the random coefficients \( A_k(\omega) \) are as before random Fourier-Stieltjes
coefficients of periodic symmetric stable process \( x(t,\omega) \) and the weight \( a_k \) are
Fourier coefficients of L^p functions. It is obvious that \( A_k(\omega) \)'s are not necessarily
independent except when \( x(t) \) is a Wiener process. To bend the study of bounds
of \( N_n \) for dependent case the most important role is played by the weight \( a_k \).
Consideration of \( a_k \) as Fourier coefficients of L^p function paves the way to find the
measure of exceptional set through the probability measure of stochastic
integral and which ultimately turns to L^p measure of the function whose Fourier
coefficients are the weights \( a_k \). Thus, we have an alternative technique to
calculate the measure of exceptional set which is easy to handle. Here our
motivation is to study the problem of Mishra and Samal [22] when the random
coefficients are dependent random variables.

We have considered the weighted random algebraic polynomial.

(5.3) \[ \sum_{k=0}^{n} a_k A_k(\omega) z^k, \]
where the random coefficients are Fourier-Stieltjes coefficients of symmetric stable process. It can be noticed that $A_k(\omega)$ have symmetric stable distribution and are not necessarily independent except when $x(t)$ is Wiener process. The weight $a_k$ are not mere constant as in [23] but Fourier coefficient of $L^p$ functions.

Consideration of the weight $a_k$ as Fourier coefficient plays a crucial role in the study of random algebraic polynomial for dependent random coefficients. We have used a different technique to find the measure of the exceptional set. In fact the measure can be found from the probability measure of the stochastic integral which is easier to handle. We have adopted the pattern of proof of Samal and Mishra [22] and have shown that

$$N_n \leq \mu (\log n)^2$$

except for a set of measure at most

$$\frac{\mu \sum_0^n |a_k|^\alpha}{n^{\delta - \beta}} + \frac{\mu \left[ \sum_0^n |a_k| \right]^{-1}}{n^{\delta - \beta - \gamma}} \quad 0 < \beta < 1.$$

**Remark**

The measure of exceptional set where weight $a_k = 1$ is

$$\frac{\mu}{n^{\delta - \beta}} + \frac{\mu}{n^{\delta - \beta - \gamma}}$$

which is smaller than that of Samal and Mishra [22]. Thus, ours is an improvement over Samal and Mishra [22]. Also ours is more general in the sense that Samal and Mishra [22] results can be found from ours by taking $a_k = 1$.
5.1.2 Results:

Theorem 5.1:

Let \( x(t) \) be a periodic symmetric stable process with period one almost surely and its increment has characteristic function \( e^{-ct|t|^\alpha} \), \( 1 \leq \alpha \leq 2 \), \( c \) being a positive constant. Let \( N_n \) be the number of real roots of the random algebraic equation.

\[
    f_n (z, \omega) = \sum_{k=0}^{\infty} a_k A_k(\omega) z^k = 0,
\]

where \( A_k(\omega) \)'s are Fourier-Stieltjes coefficients of periodic stable process \( x(t) \) and \( a_k \)'s are the Fourier coefficients of \( f \) in \( L^p[0,1] \) \( p \geq \alpha, 1 \leq \alpha \leq 2 \). Then there exists \( n_0 \) such that for each \( n > n_0 \), the number of real roots of the equation \( f_n (z, \omega) = 0 \) is at most

\[
    N_n \leq \mu (\log n)^2
\]

except for a set of measure at most

\[
    \mu \sum_{k=0}^{n} |a_k|^\alpha / n^{n^\alpha - \beta} + \mu \sum_{k=0}^{n} |a_k|^{-1} / n^{n^\beta - \alpha} \leq 0 < \beta < 1.
\]

For the proof of theorem we require the following lemmas (positive constants are denoted by \( \mu \)):

Lemma 5.1.1: (Jensen's theorem)

The number of zeros of a regular function \( f(z) \) in a circle of centre \( Z_0 \) and radius \( r \) does not exceed.
Log \left[ \frac{M}{|f(Z_0)|} \right] / \log \frac{R}{r},

where M is an upper bound of |f| in a concentric circle of radius R.

**Lemma 5.12**: (cf. Samal and Mishra [22])

If f(t) is any continuous function in [a, b] and x(t) be a symmetric stable process with index \( \alpha, 0 < \alpha \leq 2 \) then for \( \delta > 0 \)

\[
P \left[ \left| \int_{a}^{b} f(t) \, dx(t) \right| > \delta \right] \leq \frac{C^{2\alpha+1}}{\delta^{\alpha+1}} \int_{a}^{b} |f(t)|^\alpha \, dt,
\]

C being a positive constant.

**Lemma 5.13** (cf. Lukacs [9])

The characteristic function of the Stochastic integral \( \int_{a}^{b} f(t) \, dx(t) \) is

\[
h(u) = \exp \left[ \int_{a}^{b} \phi(u, f(t)) \, dt \right],
\]

where \( \phi \) in the logarithm of the characteristic function of the increment of the stochastic process x(t).


If \( f' \) is in \( L^p \), \( 1 \leq p \leq 2 \) and Fourier series of \( f' \) is of type \( \sum_{k=0}^{\infty} c_k e^{ikx} \), then

\[
A \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f'|^p \, dt \right]^\frac{1}{p} \geq \left[ \sum_{k=0}^{n} |c_k|^p (k+1)^{p-2} \right]^\frac{1}{p},
\]

where A is an absolute constant.
Proof of Theorem 5.1

Let $\frac{2}{3}$ be a fixed number greater than $\frac{1}{\log 2}$. Let us denote the circle with centre $x_m = 1 - \frac{1}{2m}$ and of radius $r_m = \frac{1}{2} (1 - x_m)$ as $c_m$ and the circle with centre $x_0 = 1$ and radius $r_0 = \frac{1}{r}$ as $c_0$. Thus, the circles $c_0, c_1, c_2, \ldots, c_k, \ldots$ cover the closed segment $[\frac{1}{2}, 1]$. Let $\Gamma_m$ be the circle concentric with $c_m$ and of double radius. It can be verified that all circles are interior to the circles $|z| = 1 + \frac{2}{n}$. Thus, by lemma (5.11) it follows that the number of zeros of $f_n(z, \omega) = 0$ in the circle $c_m$ is at most

$$
\log \left[ \frac{\max |f_n(\omega)|}{|z| \leq 1 + \frac{2}{n}} \right] \leq \log 2.
$$

Now

$$
f_n(z, \omega) = \sum_{k=0}^{n} a_k A_k z^k
$$

$$
= \sum_{k=0}^{n} a_k z^k \int_{0}^{1} e^{-2\pi k t} dx(t)
$$

$$
= \int_{0}^{1} \sum_{k=0}^{n} a_k z^k e^{-2\pi k t} dx(t).
$$

Denoting $a_k e^{-2\pi k t}$ as $g_k(t)$ and $\sum_{k=0}^{n} g_k(t)$ as $G_n(t)$, we get

$$\Pr \left( \left| \sum_{k=0}^{n} a_k A_k (\omega) \right| > \delta \right)$$
\[ P \left[ \left| \int_0^1 \sum_{k=0}^n a_k e^{-2\pi k z} dt \right| > \delta \right] = \int_0^1 G_n(t) dt > \delta \]

\[ \leq \frac{c_1 t^{1+\alpha}}{(e+1) \delta_2} \int_0^1 |G_n(t)|^\alpha dt \text{ (by Lemma 5.1.2)} \]

\[ \leq \frac{c_2 t^{1+\alpha}}{\delta_2} \int_0^1 \left| \sum_{k=0}^n |a_k|^\alpha dt \right. \]

\[ = \frac{c_2 t^{1+\alpha}}{\delta_2} \left[ \sum_{k=0}^n |a_k|^\alpha \right] . \]

So

\[ P \left[ \left| \int_0^1 \sum_{k=0}^n a_k e^{-2\pi k t} dt \right| \leq \delta \right] \geq 1 - \frac{c_1 t^{1+\alpha}}{(e+1) \delta_2} \left[ \sum_{k=0}^n |a_k|^\alpha \right] . \]

Thus, outside a set of measure atmost \( \frac{c_1 t^{1+\alpha}}{(e+1) \delta_2} \sum_{k=0}^n |a_k|^\alpha \)

\[ \operatorname{Max}_{|z| \leq 1 + \frac{n}{2}} |f_n(z, \omega)| \leq \operatorname{Max}_{|z| \leq 1 + \frac{n}{2}} \left| \sum_{k=0}^n a_k z^k e^{-2\pi k t} dt(t) \right| \]

\[ (5.5) \text{.................................................} \leq \left( 1 + \frac{2}{n} \right)^n \delta . \]

Now considering \( \delta \) as \( (n + 1)^4 \) we get,

\[ \operatorname{Max}_{|z| \leq 1 + \frac{n}{2}} |f_n(z)| \leq \left( 1 + \frac{2}{n} \right)^n (n + 1)^4 \]
outside a set of measure which does not exceed \( \frac{\mu}{(n+1)^{4\alpha}} \sum_{k=0}^{n} |a_k|^\alpha \).

By lemma (5.1.3), the characteristic function of \( \int_{0}^{1} f(t) \, dx(t) \) is

\[
(5.7) \quad \exp \left[ \int_{a}^{b} -c|\alpha| f(t) |\alpha| \, dt \right].
\]

Now,

\[
f_n(z, \omega) = \int_{0}^{1} \sum_{k=0}^{n} a_k e^{-2\pi i t z} z^k \, dx(t)
\]

\[
= \int_{0}^{1} \sum_{k=0}^{n} g_k(t) z^k \, dx(t)
\]

\[
= \int_{0}^{1} G_n(z, t) \, dx(t),
\]

where \( G_n(z, t) = \sum_{k=0}^{n} g_k(t) z^k \).

From (5.7) the characteristic function of \( f_n(z, \omega) \) is

\[
h(u) = \exp \left[ \int_{0}^{1} -c|\alpha| G_n(z, t) |\alpha| \, dt \right].
\]
We know from inversion formula, for small $\varepsilon > 0$

$$P \left[ |x(\omega)| < \varepsilon \right] = \frac{1}{2\pi} \lim_{t \to \infty} \int_{-1}^{1} \frac{e^{i\omega t} - e^{-i\omega t}}{iu} \phi(u) du$$

where $\phi$ is the characteristic function of $x(\omega)$.

Hence,

$$P \left[ |f_n(z)| < \varepsilon \right] = \frac{1}{2\pi} \lim_{t \to \infty} \int_{-1}^{1} \int_{0}^{\infty} e^{-c|u|^\alpha t} h(u) du$$

$$= \frac{2\varepsilon}{\pi} \lim_{t \to \infty} \int_{0}^{\infty} \int_{0}^{\infty} e^{-c|u|^\alpha t} \left( \int_{0}^{1} |G_n(z, t)|^\alpha dt \right) du$$

Since $\int_{0}^{1} |G_n(z, t)|^\alpha dt$ is constant with respect to $u$ denoting it by $k_t$,

after simplification

$$P \left[ |f_n(z)| < \varepsilon \right]$$

$$\leq \frac{2\varepsilon}{\pi} \int_{0}^{\infty} e^{-c|u|^\alpha k_t} du$$

$$= \frac{2\varepsilon}{\pi} \int_{0}^{\infty} \frac{e^{-y}}{ck_t\alpha u^{\alpha-1}} \left( \text{Taking } ck_t u^\alpha = y \right)$$

$$= \frac{2\varepsilon}{\pi k_t^\alpha} \int_{0}^{\infty} e^{-y} y^{1/\alpha-1} dy$$
Substituting the value of \( k_t \)

\[ P \left[ |f_n(z)| < \varepsilon \right] \leq \mu \varepsilon \left[ \int_0^1 \left| \sum_{k=0}^{n} a_k e^{-2\pi k t} z^k \right|^\alpha dt \right]^{\frac{1}{\alpha}}. \]

Thus by lemma (5.1.4) and considering \( a_k e^{-2\pi k t} \) in place of \( c_k \), we get

\[ P \left[ \left| \sum_{k=0}^{n} a_k e^{-2\pi k t} z^k \right|^\alpha dt \right] \geq \mu \left[ \sum_{k=0}^{n} |a_k|^\alpha (k+1)^{\alpha-2} \right]^{\frac{1}{\alpha}}. \]

where \( 1 < \alpha < 2 \)

Thus, considering \( \varepsilon = \frac{1}{n^2} \) we get

\[ (5.8) \quad \ldots \quad P \left[ |f_n(z)| < \frac{1}{n^2} \right] \leq \frac{\mu}{n^2} \left[ \sum_{k=0}^{n} |a_k|^\alpha (k+1)^{\alpha-2} \right]^{\frac{1}{\alpha}}. \]

This being independent of \( m \), holds for all circles \( c_0, c_1, \ldots c_{\log n} \) and gives the measure of the exceptional set in a simpler form. Now by (5.4), (5.6) and (5.8) the number of real zeros of \( f_n(z) = 0 \) in all \( (k+2) \) circles is at most.

\[ \frac{\mu (k+2) \log \left( e^2 (n+1)^4 n^4 \right)}{\log 2} < \mu (\log n)^2 \]

as in [22]
outside a set of measure at most

\[ \log n \mu \frac{\sum_{0}^{n} |a_k|^\alpha}{(n+1)^{\beta \alpha}} + \frac{\mu}{n^2} \left[ \sum_{0}^{n} |a_k|^\alpha (k+1)^{-a-2} \right] \frac{1}{\alpha} (1 + \frac{1}{\epsilon} \log n) \]

\leq \frac{\mu}{n^{\alpha - \beta}} \sum_{0}^{n} |a_k|^\alpha + \frac{\mu}{n^{\beta - p}} \left[ \sum_{0}^{n} |a_k|^\alpha (k+1)^{-a-2} \right] \frac{1}{\alpha}

where \(0 < \beta < 1\)

(5.9) \[= \frac{\mu}{n^{\alpha - \beta}} \sum_{0}^{n} |a_k|^\alpha + \frac{\mu}{n^{\beta - p}} \left( \sum_{0}^{n} \frac{|a_k|^\alpha}{(k+1)^{2-a}} \right) \frac{1}{\alpha}.\]

\[= S_1 + S_2 \text{ (say)}.

Now \[S_2 = \frac{\mu}{n^{\alpha - \beta}} \frac{1}{\left[ \sum_{0}^{n} \frac{|a_k|^\alpha}{(k+1)^{2-a}} \right]^{1/\alpha}}\]

\leq \frac{\mu}{n^{\alpha - \beta}} \frac{(n+1)^{2-a}}{\left[ \sum_{0}^{n} |a_k|^\alpha \right]^{1/\alpha}}

\leq \frac{\mu}{n^{\beta - p}} \frac{1}{n^{1-2/a} \left[ \sum_{0}^{n} |a_k|^\alpha \right]^{1/\alpha}}

= \frac{\mu}{n^{6-\beta - 2/a}} \left[ \sum_{0}^{n} |a_k|^\alpha \right]^{-1/\alpha}.\]
Applying Holder's inequality to $\sum_{k=0}^{n} |a_k b_k|$ we get

$$\sum_{k=0}^{n} |a_k b_k| \leq \left[ \sum_{k=0}^{n} |a_k|^\alpha \right]^\frac{1}{\alpha} \left[ \sum_{k=0}^{n} |b_k|^\alpha' \right]^\frac{1}{\alpha'}$$

$\alpha$ & $\alpha' \geq 1$, $1/\alpha + 1/\alpha' = 1$.

If $b_n = 1$ it gives

$$\sum_{k=0}^{n} |a_k| \leq \left[ \sum_{k=0}^{n} |a_k|^\alpha \right]^{\frac{1}{\alpha}} (n+1)^{1-\frac{1}{\alpha}}$$

$$\Rightarrow \left[ \sum_{k=0}^{n} |a_k|^\alpha \right]^{\frac{1}{\alpha}} \geq \sum_{k=0}^{n} \frac{|a_k|}{(n)^{1-\frac{1}{\alpha}}} \quad (b_0 = 0).$$

Thus, from (5.9) the measure of the exceptional set is atmost

$$\frac{\mu \sum_{k=0}^{n} |a_k|^\alpha}{n^{\alpha-\rho}} + \frac{\mu n^{1-\frac{1}{\alpha}}}{n^{\rho - \rho - \frac{1}{\alpha}} \sum_{k=0}^{n} |a_k|^{-\frac{1}{\alpha}}}$$

$$= \frac{\mu \sum_{k=0}^{n} |a_k|^\alpha}{n^{\alpha-\rho}} + \frac{\mu}{n^{1-\rho - \frac{1}{\alpha}} \sum_{k=0}^{n} |a_k|}.$$

Now considering the range $\left[ 0, \frac{1}{2} \right]$, for which we take the circle with centre 'o' and radius $\frac{1}{2}$, the circle $|z| < \frac{1}{2}$ is interior to the circle $|z| \leq 1$. As before by
Lemma (5.1.1) the number of zeros of any $f_n(z)$ in the circle $|z| \leq \frac{1}{2}$ (say $c'$) does not exceed

$$\log \frac{\max_{|z| \leq 1} |f_n(z)|}{\max_{|z| \leq 1} |f_n(z_0)|} \lesssim \log 2$$

and proceeding in the same pattern from (5.5), we get

$$\max_{|z| \leq 1} |f_n(z)| \leq (n+1)^4$$

$$\leq \mu n^4$$

outside a set of measure at most

$$\frac{\mu}{(n+1)^4} \sum_{|k|=0}^n |a_k|^\alpha$$

and from (5.8)

$$\Pr \left[ |f_n(z_0)| < \frac{1}{n^2} \right] \leq \frac{\mu}{n^2} \left[ \sum_{|k|=0}^n |a_k|^\alpha (k+1)^{\alpha-2} \right]^{\frac{1}{\alpha}}.$$

So the number of zeros in the circle $c'$ does not exceed

$$\log \frac{\mu n^\alpha}{\log 2} < \mu (\log n)$$

$$< \mu (\log n)^2,$$

as in [22].

Thus, total no. of roots in $[0,1]$ of the random polynomial (5.3) is at most $\mu (\log n)^2$ and the measure of the exceptional set is at most

$$\frac{\mu}{n^{\alpha_0-\beta}} \sum_{|k|=0}^n |a_k|^\alpha + \frac{\mu}{n^{s-\beta-\delta}} \left[ \sum_{|k|=0}^n |a_k| \right]^{-1}$$

Remark

In particular for $\alpha = 1$ and $\alpha = 2$, $a_k$ need not be a Fourier coefficient. The results hold good when $a_k$'s are only real constants.
Corollary to theorem 5.1

Let $x(t)$ be a periodic symmetric stable process with period one almost surely and its increment has characteristic function $e^{-ct|t|^\alpha}$, $1 \leq \alpha \leq 2$, $c$ being a positive constant. Let $A_k$ be the Fourier Stieltjes coefficient of $x(t)$. Let $N_n$ be the number of real roots and $a_n \to \infty$ as $k \to \infty$ and $\frac{a_k}{k} < \frac{a_n}{n}$ for $k \geq n$, then for $n > n_0$, $N_n \leq \mu(\log n)^2$ except for a set of measure at most

$$\mu \left( \sum_0^n |a_k|^\alpha \right)^{\frac{1}{\alpha}} + \frac{\mu(\log n)^2}{n^{\beta-1}}$$

Proof of corollary to theorem 5.1

We have already shown that the number of roots is at most $\mu(\log n)^2$ and the measure of exceptional set is at most

$$\mu \left( \sum_0^n |a_k| \right) + \frac{\mu(\log n)^2}{n^{\beta-1}}$$

(5.11)

Now

$$\frac{a_k}{k} \leq c \frac{a_n}{n} \quad \text{for } k \geq n$$

$$\Rightarrow \frac{a_{n+1}}{a_k} \leq c \frac{k+1}{k} \leq 2c$$

$$\Rightarrow |a_{k+1}|^\alpha \leq 2^\alpha c^\alpha |a_k|^\alpha.$$
\( \Rightarrow |a_k|^\alpha \geq \frac{|a_{k+1}|^\alpha}{2\alpha c^\alpha} \)

\( \Rightarrow \frac{|a_k|^\alpha}{(k+1)^\delta} \geq \frac{1}{2\alpha c^\alpha} \frac{|a_{k+1}|^\alpha}{(k+1)^\delta}, \quad \delta > 0 \)

\( \Rightarrow \sum_{0}^{n} \frac{|a_k|^\alpha}{(k+1)^\delta} \geq \frac{1}{2\alpha c^\alpha} \sum_{0}^{n} \frac{|a_{k+1}|^\alpha}{k+1} \quad (\text{since } \delta > \alpha) \)

\[ \left[ \sum_{0}^{n} \frac{|a_k|^\alpha}{(k+1)^\delta} \right]^{\frac{1}{\alpha}} \leq \frac{1}{M_\alpha} \left[ \sum_{0}^{n} \frac{|a_{k+1}|^\alpha}{k+1} \right]^{\frac{1}{\alpha}} \]

(where \( M_\alpha \) is a constant depending on \( \alpha \))

\[ \leq \frac{1}{M_\alpha} \left[ (n+1) \left( \frac{a_{n+1}}{n+1} \right) \right]^{\frac{1}{\alpha}} \quad \text{since } \frac{a_k}{k} < \frac{a_n}{n}, \text{ for } k \geq n \]

\[ = \frac{1}{M_\alpha} (a_{n+1})^{\frac{1}{\alpha}}. \]

So from (5.11) the measure of the exceptional set is at most

\[ \frac{\mu \left[ \sum_{0}^{n} |a_k| \right]^{\frac{1}{\alpha}}}{n^{\alpha \alpha - \beta}} + \frac{\mu' \left[ a_{n+1} \right]^{\frac{1}{\alpha}}}{n^{\alpha - \beta}} \]

\[ \leq \frac{\mu (n+1) |a_n|^{\alpha}}{n^{\alpha \alpha - \beta}} + \frac{\mu [a_{n+1}]^{\frac{1}{\alpha}}}{n^{\alpha - \beta}} \]

\( (a_n \text{ being increasing sequence}) \)

\[ \leq \frac{\mu (a_{n+1})^{\alpha}}{n^{\alpha \alpha - \beta}} + \frac{\mu}{n^{\alpha - \beta} (a_{n+1})^{\frac{1}{\alpha}}}, \text{ which proves the corollary.} \]
**Theorem 5.2**

Let \( x(t) \) be a periodic symmetric stable process with period one almost surely with characteristic function \( e^{-|t|^\alpha}, \ 1 \leq \alpha \leq 2 \), \( C \) being a positive constant. Let \( A_k(\omega) \) be Fourier-Stieltjes coefficients of \( x(t, \omega) \) and \( a_k \) be Fourier coefficient of \( f' \) in \( L^p [0, 1] \), \( p > \alpha \) and \( N_n \) be the number of real roots of the random algebraic equation \( \sum_{k=0}^{n} a_k A_k(\omega) x^k = 0 \), then for each \( n > n_0 \), \( N_n \leq \mu (\log \log)^\alpha \log n \) except a set of measure at most

\[
\frac{\mu}{(\log n_0 - \log \log \log n_0)^{\alpha-1}} \left( \frac{\sup |a_k|}{\inf \left| a_k \right|} \right)^\alpha.
\]

For the proof of the theorem we require the following lemmas:

**Lemma 5.15** (cf. [13])

If \( x(\omega) \) is a symmetric stable process with index \( \alpha \), \( 1 \leq \alpha \leq 2 \) and \( f(t) \) be any function in \( L^p [a, b] \), \( p \geq 1 \) then for \( \delta > 0 \)

\[
P \left( \left| \int_a^b f(t) \, dx(t) \right| > \delta \right) \leq \frac{c \cdot 2^{\alpha+1}}{(\alpha+1)^{\delta'}} \int_a^b \left| f(t) \right|^{\alpha} \, dt,
\]

where \( \delta' < \delta \) and \( c \) is a positive constant.

**Lemma 5.16**

If \( x(t) \) has symmetric stable distribution with \( c.f \ \ e^{-|t|^\alpha} \), \( f(t) \) continuous function in \( [a,b] \), then

\[
P \left( \left| \int_a^b f(t) \, dx(t) \right| < \varepsilon \right) \leq \frac{2e^{\sqrt{\frac{1}{\pi \alpha}}} \Gamma\left( \frac{1}{\alpha} \right)}{\pi \alpha} \left( \int_a^b \left| f(t) \right|^{\alpha} \, dt \right)^{-\frac{1}{\alpha}}.
\]
Proof of Lemma 5.1.6

We know from inversion formula for any random variable \( X(\omega) \)

\[
P(|X(\omega)| < \varepsilon) = \frac{1}{2\pi} \lim_{l \to \infty} \int_{-l}^{l} \frac{e^{iue} - e^{-iue}}{iu} \phi(u) du
\]

where \( \phi(u) \) is the characteristic function of \( X(\omega) \). Also we know if \( X(t) \) has characteristic function \( \Psi(t) \), the characteristic function of the stochastic integral

\[
\int_{a}^{b} f(t) dx(t) \text{ is } h(u) = \exp \left[ \int_{a}^{b} \phi(u, f(t)) dt \right]
\]

where \( \phi(t) = \log \psi(t) \). Since \( X(t) \) has characteristic function \( e^{-c|t|^\alpha} \), the characteristic function of the stochastic integral \( \int_{a}^{b} f(t) dx(t) \) is

\[
\exp \left[ \int_{a}^{b} -c |u| f(t)|^\alpha dt \right].
\]

By using the inversion formula of Gnedenko and Kolomogrove [6] we have

\[
P \left( \left| \int_{a}^{b} f(t) dx(t) \right| < \varepsilon \right)
\]

\[
= \frac{1}{2\pi} \lim_{l \to \infty} \int_{-l}^{l} \frac{e^{iue} - e^{-iue}}{iu} e^{-c|u| f(t)|^\alpha} du
\]

\[
\leq \frac{1}{2\pi} \lim_{l \to \infty} \int_{0}^{l} \frac{2 \sin u}{u} e^{-c|u| f(t)|^\alpha} du
\]

\[
\leq \frac{2\pi}{e} \lim_{l \to \infty} \int_{0}^{l} e^{-c|u|^\alpha} |f(t)|^\alpha du
\]

(denoting \( \int_{a}^{b} |f(t)|^\alpha dt \) as \( k_t \) which is constant with respect to \( u \))
\[
\leq \frac{2\pi}{\alpha} \lim_{t \to \infty} \int_0^\infty e^{-c|u|^\alpha} du
\]

\[
\leq \frac{2\pi}{\alpha} \int_0^\infty e^{-c|u|^\alpha} du
\]

\[
= \frac{2\pi}{\alpha} \int_0^\infty \frac{e^{-y u^{1-\alpha}}}{c_k \alpha} dy
\]

(taking \( y = c_k t u^\alpha \))

\[
= \frac{2\pi}{\alpha} \int_0^\infty \frac{e^{\gamma} y^{\frac{1-\alpha}{\alpha}}}{c_k^{1/\alpha} c_k^{1/\alpha}} dy
\]

\[
= \frac{2\pi e^{\gamma}}{\pi \alpha c_k^{1/\alpha}} \int_0^\infty e^{-\gamma y^{\frac{1}{\alpha}-1}} dy
\]

\[
= \frac{2\pi}{\pi \alpha c_k^{1/\alpha}} \Gamma\left(\frac{1}{\alpha}\right)
\]

\[
= \frac{2\pi e^{\gamma}}{\pi \alpha c_k^{1/\alpha}} \left( \int_a^b |f(t)|^\alpha dt \right)^{\frac{1}{\alpha}}
\]

which proves the lemma.
Lemma 5.1.7 (c.f Zygmund [28] Vol. II p. 110)

(a) If \( f \) is in \( L^p \), \( 1 < p < 2 \) and the Fourier series of \( f \) is of type
\[
F \sim \sum_{k=0}^{\infty} c_k e^{ikx}
\]
then,
\[
\left( \sum_{k=0}^{\infty} |c_k|^p (k+1)^{p-2} \right)^{\frac{1}{p}} \leq A \left( \frac{1}{2\pi} \int_0^{2\pi} |f|^p \, dx \right)^{\frac{1}{p}},
\]
where \( A \) is an absolute constant.

(b) When \( p=1 \) and \( p = 2 \) \( c_k \) need not be Fourier coefficient of \( F \) but for any polynomial
\[
p(z) = \sum_{k=0}^{n} c_k z^k
\]
(i) \( \sum |c_k|^2 = \frac{1}{2\pi} \int_0^{2\pi} |p e^{ix}|^2 \, dx \)

(ii) \( \sum \frac{|s_k|}{k!} \leq \frac{1}{2} \int_0^{2\pi} |p e^{ix}| \, dx \).

For the proof of the theorem we have followed the pattern of Samal & Mishra [24]. Our technique of finding the measure of exceptional set is through the probability measure of stochastic integral.

Proof of the Theorem 5.2

We first cover the interval \([0,1]\) by circle \( C_0, C_1, c_{m_0}, c_{m_0+1}, \ldots, c_m, C_1 \) where \( C_0 \) have centre at \( z=0 \) and is of radius \( \frac{1}{2} \), \( C_c \) has centre \( \frac{3}{4} - \frac{\log \log n_0}{(2n_0)} \) and is of radius \( \frac{1}{4} - \frac{\log \log n_0}{(2n_0)} \), \( C_m \) has centre \( x_m = 1 - \frac{1}{2^m} \) and radius \( r_m = \frac{1}{2}(1-x_m) = \frac{1}{2^{m+1}} \) for \( m = m_0, \ldots, M \) where
\[ m_0 = \left[ \frac{\text{log } n - \text{log } \text{log } n + \text{log } 3}{\text{log } 2} \right] - 1 \]

and

\[ \frac{\text{log } n - \text{log } \text{log } n}{\text{log } 2} - 1 < M < \frac{\text{log } n - \text{log } \text{log } n}{\text{log } 2} \]

\( C_1 \) has centre at \( z = 1 \) and is of radius \( \frac{\text{log } n}{n} \). \( \Gamma_i \) is the circle concentric with \( C_c \) and with twice its radius. So all the \( \Gamma_i \) are interior to the circle

\[ |z| = 1 + \frac{2\text{log } n}{n} \]

Let \( f_n(\omega) = \sum_{k=0}^{n} a_k A_k(\omega) \), where \( A_k(\omega) = \int_{0}^{1} e^{-2\pi k\omega t} dt \).

Thus,

\[ f_n(\omega) = \sum_{k=0}^{n} a_k \int_{0}^{1} e^{-2\pi k\omega t} dt \]

\[ = \int_{0}^{1} \sum_{k=0}^{n} a_k e^{-2\pi k\omega t} dt \]

So by lemma 5.15

\[ P(\{|f_n(z, \omega)| > \delta\}) \]

\[ = P\left(\left| \int_{0}^{1} \sum_{k=0}^{n} a_k e^{-2\pi k\omega t} dt \right| > \delta \right) \]

\[ \leq \frac{2^{1+\alpha}c}{(1+\alpha)\delta^{\alpha}} \int_{0}^{1} \left| \sum_{k=0}^{n} a_k e^{-2\pi k\omega t} \right|^{\alpha} dt \]

Thus,

\[ P\left(\{|f_n(\omega)| > (n+1)^2\} \right) \leq \frac{2^{1+\alpha}c}{(1+\alpha)\delta^{\alpha}} \sum_{k=0}^{n} |a_k|^{\alpha} \]

\[ \frac{\mu}{\delta^{\alpha}} \sum_{k=0}^{n} |a_k|^{\alpha} \]

where \( \delta' = (n+1)^2 \cdot \varepsilon \)
\[ \Rightarrow P\left( |f_n(z, \omega)| < (n+1)^2 \right) > 1 - \frac{\mu}{\delta_n^2} \sum_{0}^{n} |a_k|^{\alpha} . \]

Since \( n \) is large \( (n+1)^2 - \varepsilon \sim (n+1)^2 \).

Thus,

\[ |f_n(z, \omega)| < (n+1)^2 \left( \frac{(1+2\log \log n)^2}{n} \right) . \]

outside a set of measure at most

\[ \frac{\mu}{(n+1)^2} \sum_{0}^{n} |a_k|^{\alpha} . \]

Thus,

\[ f_n(x, \omega) < (n+1)^2 e^{2\log \log n} \]

outside a set of measure at most

\[ (5.13) \quad \sum_{0}^{n} |a_k|^{\alpha} . \]

By using lemma (5.1.6), we get

\[ P\left( |a_0 A_0 (\omega)| < \frac{1}{(n+1)^2} \right) \]

\[ = P\left( \left| \int_{0}^{1} a_0 dx(t) \right| < \frac{1}{(n+1)^2} \right) \]

\[ \leq \frac{2}{\pi} \Gamma(\frac{1}{\alpha}) \int_{0}^{1} |a_0|^{\alpha} dt \]

\[ \leq \frac{\mu}{(n+1)^2} \left( \int_{0}^{1} |a_0|^{\alpha} dt \right)^{\frac{-1}{\alpha}} \]
Thus, outside a set of measure $\frac{\mu}{(n+1)^2} |a_0|^{-1}$.

(5.14) $\frac{\mu}{(n+1)^2} |a_0|^{-1}$.

(5.15) $f_n(z, \omega) \geq \frac{1}{(n+1)^2}$.

If $N_0$ denotes the number of real roots of $f_n(z, t)$ in $C_0$ circle then by Jensen's theorem

$$N_0 \leq \frac{\log \left( \frac{\mu}{|f_n(z, \omega)|} \right)}{\log 2},$$

where $M$ is the upper bound of $f_n(z, \omega)$ in $\Gamma_0$.

So

$$N_0 \leq \log \left( \frac{(n+1)^2 e^{2 \log \log n}}{\frac{1}{(n+1)^2}} \right) / \log 2$$

$$= \log \left\{ (n+1)^4 e^{2 \log \log n} \right\} / \log 2$$

$$= 4 \log (n+1) + 2 \log \log n / \log 2,$$

outside a set of measure at most

$$\frac{\mu}{(n+1)^2} \sum_{k=0}^{n} |a_k|^{15} + \frac{\mu |a_0|^{-1}}{(n+1)^2}.$$

So for all $n \geq n_0$, the measure of the exceptional set will be
\[
\sum_{n=n_0+1}^{\infty} \frac{\mu}{(n+1)^{2\alpha}} \sum_{k=0}^{n} |a_k|^\alpha + \mu \sum_{n=n_0+1}^{\infty} \frac{|a_0|^{-1}}{(n+1)^{2\alpha}}
\]

(5.16) \[\leq \frac{\mu}{n^{2\alpha-1}} \sum_{k=0}^{n} |a_k|^\alpha + \frac{\mu}{n_0} |a_0|^{-1} .\]

Let \( N_0 \) be the number of real roots in \( C_0 \) with centre \( \frac{3}{4} - \frac{(\log \log n_0)}{2n_0} \) and radius \( \frac{1}{4} - \frac{\log \log n_0}{2n_0} \).

By Jenson's theorem
\[
N_0 \leq \log \frac{M'}{f_n\left(\frac{3}{4} - \frac{\log \log n_0}{2n_0}, \omega\right)} ,
\]

where \( M' \) is the maximum of \( f(x, \omega) \) in \( \Gamma_\omega \). Since all circles are interior to the circle \( |z| = 1 + \frac{2\log \log n}{n} \) we get as in (5.12)
\[
|f_n(x, \omega)| < (n+1)^2 e^{2\log \log n}
\]
outside a set of measure at most
\[
\frac{\mu}{(n+1)^{2\alpha}} .
\]

Writing \( \frac{3}{4} - \frac{\log \log n_0}{2n_0} \) as \( x_0 \),
\[
P(|f_n(x_0, \omega)| < \epsilon)
\]
\[
= P\left( \left| \sum_{k=0}^{n} a_k A_k(\omega) x_0^k \right| < \epsilon \right)
\]
\[ P \left( \int_0^1 \left| \sum a_k e^{-2n\text{it}} \left( \frac{3}{4} - \frac{\log \log n}{2n} \right)^k \right| \, dt < \varepsilon \right) \]

\[ \leq \frac{2e^{\Gamma(\frac{1}{\alpha})}}{\pi \cdot c \cdot \alpha} \left( \int_0^1 \left| \sum a_k \left( \frac{3}{4} - \frac{\log \log n}{2n} \right)^k \right| \alpha \, dt \right)^{-\frac{1}{\alpha}} \]

(by Lemma 5.15)

\[ \leq \frac{\mu}{(n+1)^2} \left( \sum_{k=0}^{n} |a_k|^\alpha \left( \frac{3}{4} - \frac{\log \log n}{2n} \right)^k \right)^{-\frac{1}{\alpha}} \quad \varepsilon = \frac{1}{(n+1)^2} \]

(5.17)

\[ \leq \frac{\mu}{(n+1)^2} \sum_{k=0}^{n} |a_k|^\alpha . \]

As in Samal and Mishra [24] taking \( \sum_{k=0}^{n} \left( \frac{3}{4} - \frac{\log \log n}{2n} \right)^k \) as \( 0 < \beta < 1 \) we get

\[ N_c < \log \left( \frac{(n+1)^2 e^{2\log \log n}}{\log 2} \right) / \log 2 \]

\[ = \frac{4 \log (n+1) + 2 \log \log n}{\log 2} \]

outside a measure of at most

(5.18)

\[ \frac{\mu}{(n+1)^{2\alpha - 1}} + \frac{\mu}{(n+1)^2} \sum_{k=0}^{n} |a_k|^\alpha . \]

To find the number of real roots of \( f_n(z, \omega) \) in circle \( C_m \) we have followed exactly the method of Samal and Mishra [24] except that to find the measure of \( E_1, F_1 \) and \( G \), we use lemma 5.15 and 5.16 in place of lemma 1 and 2 of [24].
Lemma 3 and 4 of Samal and Mishra [24] can be modified with the only difference that in our case for lemma - 3, we consider
\[ \delta^\alpha = \sum_{k=0}^{\infty} |a_k|^\alpha \left[ |x_k|^\alpha + |y_k|^\alpha \right] \]
and for lemma - 4 we consider
\[ \delta^\alpha = \sum_{k=0}^{\infty} |a_k|^\alpha |z_k|^\alpha , \]
where \( z_k \) being real and
\[ G = \left\{ \omega : \left| \sum_{k=0}^{\infty} a_k A_k(\omega) z_k \right| \leq \varepsilon \right\} . \]

Thus, the results of lemma 3 and 4 of [24] remains same in our case which we have elaborated in brief (the symbol \( \sigma \) in Samal and Mishra [24] has been modified to \( \delta \) in our work).

Lemma 5.18.

Let \( E \) be an arbitrary set. Then for complex number \( z_k \) we have
\[
\int_{z} \log \left| \sum_{k=0}^{n} a_k A_k(\omega) z_k \right| d\omega \leq m(E) \log \delta + \mu m(E) \log \frac{1}{m(E)}
\]
where
\[ \delta = \left\{ \sum_{k=0}^{\infty} |a_k|^\alpha \left( |x_k|^\alpha + |y_k|^\alpha \right) \right\}^{\frac{1}{\alpha}} . \]

\( z_k = x_k + iy_k \).

Proof

Let
\[ F = \left\{ \omega : \left| \sum_{k=0}^{n} a_k A_k(\omega) z_k \right| \geq \lambda \delta \right\} . \]
\[ m(F) = P \left( \left| \sum_{k=0}^{n} a_k A_k(\omega) x_k + i \sum_{k=0}^{n} a_k A_k(\omega) y_k \right| \geq \lambda \delta \right) \]

\[ \leq P \left( \left| \sum_{k=0}^{n} a_k A_k(\omega) x_k \right| \geq \frac{\lambda \delta}{2} \right) + P \left( \left| \sum_{k=0}^{n} a_k A_k(\omega) y_k \right| \geq \frac{1}{2} \lambda \delta \right) \]

\[ \leq P \left( \left| \sum_{k=0}^{n} a_k \int_{0}^{1} e^{-2\pi k t} dx(t) x_k \right| \geq \frac{\lambda \delta}{2} \right) + P \left( \left| \sum_{k=0}^{n} a_k \int_{0}^{1} e^{-2\pi k t} dx(t) y_k \right| \geq \frac{\lambda \delta}{2} \right) \]

\[ \leq \frac{\mu}{(\lambda \delta)^a} \int_{0}^{1} \left| \sum_{k=0}^{n} x_k a_k e^{-2\pi k t} \right|^\alpha dt + \int_{0}^{1} \left| \sum_{k=0}^{n} y_k a_k e^{-2\pi k t} \right|^\alpha dt \]

\[ \leq \frac{\mu}{\lambda^a \delta^a} \sum_{k=0}^{n} \left\{ |x_k|^\alpha |a_k|^\alpha + |y|^\alpha |a_k|^\alpha \right\} \]

\[ \leq \frac{\mu}{\lambda^a \delta^a} \left\{ \sum_{k=0}^{n} |a_k|^\alpha (|x_k|^\alpha + |y|^\alpha) \right\} \]

(5.19) \[ \leq \frac{\mu \delta_n^\alpha}{\lambda^a \delta^a} , \quad \text{where } \delta_n^\alpha = \sum_{k=0}^{n} |a_k|^\alpha \{|x_k|^\alpha + |y|^\alpha\}. \]

Now,

\[ \int_{E} \log \left| \sum_{k=0}^{n} a_k A_k(\omega) z_k \right| dt = \int_{E/F} + \int_{E \setminus F} \]

\[ = I_1 + I_2 \text{ (say).} \]

Now \[ I_1 \leq m(E/F) \lambda \delta \text{ (as in set } E/F \left| \sum_{k=0}^{n} a_k A_k(\omega) z_k \right| < \lambda \delta ). \]
For I₂, describe

\[ F_i := \left\{ \omega : i \delta \leq \left| \sum_{0}^{n} a_k A_k(\omega)z_k \right| < (i+1)\delta \right\} \text{ and } i_o = \lambda. \]

I₂ is integral over \( E \cap F = E \cap (\cup F_i) \)

So

\[ I_2 < \sum_{i=i_0}^{\infty} \log \{(i+1)\delta\} m(E \cap F_i). \]

But

\[ \sum_{i=i_0}^{\infty} \log (i+1) M(F_i) \]

\[ = \sum_{i=i_0}^{\infty} \log (i+1) P\left( \omega : \left| \sum_{0}^{n} a_k A_k(\omega)z_k \right| \leq (i+1)\delta \right) \]

\[ \leq \sum_{i_0}^{\infty} \log (i+1) P\left( \int_{0}^{1} \left| \sum a_k e^{-2\pi i k t} z_k dx(t) \right| \geq i\delta \right) \]

\[ \leq \sum_{i_0}^{\infty} \log (i+1) \frac{a_b a}{i^{a_0}} \]

\[ \leq \mu \sum_{i_0}^{\infty} \log (i+1) \frac{a_b a}{i^{a_0}} \]

\[ < \mu(\log i_o) \]
Thus, \[ I_2 < \sum_{i=0}^{\infty} \log \{(i+1)\delta\} m(E \cap F_i) \]
\[ = \sum_{i=0}^{\infty} \log \delta m(E \cap F_i) + \sum_{i=0}^{\infty} \log (1+i)m(E \cap F_i) \]
\[ = \log \delta m(E \cap F_i) + \mu \log \left( \frac{1}{m(E)} \right). \]

So
\[ \int_E \log \left| \sum_0^\infty a_k A_k(\omega) z_k \right| \, d\omega < m(E) \log \delta + \mu m(E) \frac{1}{m(E)}. \]

**Lemma 5.1.9**

If \( z_k \) are real, and if
\[ G = \left\{ \omega : \left| \sum_0^\infty a_k A_k(\omega) z_k \right| \leq \delta \right\}, \]
where
\[ \delta^\alpha = \sum_{k=0}^{\infty} |a_k|^{\alpha} |z_k|^{\alpha} \]
and if \( E \) is any set having no point in common with \( G \), then \( m(G) < \mu \in Q \) and
\[ \int_E \log \left| \sum_0^\infty a_k A_k(\omega) z_k \right| \, dt > m(E) \log \delta - \mu \phi m(E) \log \left( \frac{1}{m(E)} \right). \]
where
\[ \delta_n^\alpha = \sum_{k=0}^n |a_k|^{\alpha} |z_k|^{\alpha} \text{ and } \phi = \frac{\delta}{\delta_n}. \]

This result we can achieve exactly as in Samal & Mishra [24] for independent coefficients. Only difference is to find measure \( m(G) \) and \( m(E) \), in place of the inequality
\[
(5.19) \quad p\left(|x(\omega)| < \varepsilon \right) \leq \frac{2r^{\frac{1}{2}}}{\pi \alpha \cdot \varepsilon^{\alpha}} \in \quad \text{for } \alpha > 0,
\]
we use our lemma (5.1.6) for dependent coefficients \( A_k(\omega) \). It can be noted that since in equality (5.19) refers to one single random variable to find the measure \( (|f_n(x, \omega)| < \varepsilon) \), there was the need of independency. But in our case since we find \( p(|f_n(x, \omega)| < \varepsilon) \) through the probability measure of stochastic integral
\[
P\left(\left| \int_0^1 \sum a_k e^{-2\pi it^2} dx(t) \right| < \varepsilon \right)
\]
we can avoid dependency of the random coefficients \( A_k(\omega) \) of the random algebraic polynomial \( f_n(x, \omega) \).

Taking
\[ x_m + \left( \frac{5}{2^m+3} \right) \cos \theta = R \cos \phi \]
and
\[ \left( \frac{5}{2^m+3} \right) \sin \theta = R \sin \phi, \]
we shall have
\[
f_n \left( x_m + \frac{5}{2^m+3} e^{i\theta}, \omega \right)
\]
\[ = \sum_{k=0}^n a_k A_k(\omega) \left( x_m + \frac{5}{2^m+3} e^{i\theta} \right)^k \]
\[
= \int_0^1 \sum_{k=0}^n a_k e^{-2\pi i k t} \left( x_m + \frac{5}{2\pi^3} e^{i\theta} \right)^k dt \\
= \sum_{k=0}^n R_k \int_0^1 a_k (\cos K\theta + i \sin k\theta) dt.
\]

Hence, by using lemma (5.1.8) we have

\[
\int_E \log |f_n \left( x_m + \frac{5}{2\pi^3} e^{i\theta}, \omega \right) | d\omega \\
\leq m(E) \log \left( \sum_{k=0}^\infty R^{k\alpha} |a_k|^\alpha \right) + \mu m(E) \log \left( \frac{1}{m(E)} \right) \\
\leq \mu m(E) \log \sum_{k=0}^\infty R^{k\alpha} |a_k|^\alpha + \mu m(E) \log \left( \frac{1}{m(E)} \right).
\]

Again by using lemma (5.1.9) we have

\[
\int_E \log |f_n (x_m, \omega)| d\omega > m(E) \log \left( \sum_{k=0}^\infty |a_k|^\alpha x_m^{k\alpha} \right)^{\frac{1}{\sigma}} - \mu \theta_m m(E) \log \left( \frac{1}{m(E)} \right),
\]

where

\[
\theta_m = \left\{ \frac{\sum_{k=0}^n |a_k|^\alpha x_m^{k\alpha}}{\sum_{k=0}^n |a_k|^\alpha x_m^{k\alpha}} \right\}.
\]

If \( \phi_m(\omega) \) denotes the number of zeros of \( f_n(z, \omega) \) in the circle with centre \( x_m \) and radius \( \frac{1}{2\pi^3} \) from Jensen's theorem as in Samal and Mishra [24] we get
\[ \int_\mathbb{E} \phi_m(\omega) \, d\omega < \mu m(E) \int_0^{2\pi} \log \left( \frac{\sum_{k=0}^\infty |a_k|^\alpha R^{k\alpha}}{\sum_{k=0}^\infty |a_k|^\alpha x_m^{k\alpha}} \right) \, d\theta + \mu \phi_m m(E) \log \left( \frac{1}{m(E)} \right). \]

Let
\[ \beta_m = \sum_{k=0}^\infty |a_k|^\alpha R^{k\alpha} \]
\[ \leq \frac{\sup |a_k|^\alpha \sum_{k=0}^\infty R^{k\alpha}}{\inf |a_k|^\alpha \sum_{k=0}^\infty x_m^{k\alpha}}. \]

But the proof for the value of \( \sum_{k=0}^\infty R^{k\alpha} \sum_{k=0}^\infty x_m^{k\alpha} \) follows the same pattern of Samal & Mishra [24](P. 119) and lastly we get \( \beta_m \leq \left( \frac{\sup |a_k|}{\inf a_k} \right)^\alpha \). Therefore,
\[ \int_\mathbb{E} \phi_m(t) \, dt \leq \mu m(E) \int_0^{2\pi} \log \left( \frac{\sup |a_k|}{\inf |a_k|} \right)^\alpha + \mu \theta_m m(E) \log \frac{1}{m(E)} \]
\[ \leq \mu \theta_m m(E) \log \frac{1}{m(E)}. \]

The rest of the proof in [3 ;art 1.4] applies also to our case with following changes
\[ \theta_m^{\alpha} = \frac{\sum_{k=0}^\infty |a_k|^\alpha x_m^{k\alpha}}{\sum_{k=0}^\infty |a_k|^\alpha x_m^{k\alpha}}. \]
\[
\sup |a_k|^\alpha \sum_{k=0}^{\infty} x_k^n
\leq \frac{\sup |a_k|^{\alpha}}{\inf |a_k|^{\alpha}} \frac{1}{1-x_m^{\alpha(n+1)}}
\]

\[
\leq \frac{1}{1-x_m^{\alpha(n+1)}} \left( \frac{\sup |a_k|}{\inf |a_k|} \right)^\alpha
\]

\[
\leq \left\{ 1 - \left( 1 - \frac{1}{2^n} \right)^{\alpha(n+1)} \right\}^{-\frac{1}{\alpha}} \left( \frac{\sup |a_k|}{\inf |a_k|} \right)^\alpha
\]

So that

\[
p_k \leq \left\{ 1 - \left( 1 - \frac{1}{2^{\alpha(n+1)}} \right)^{\alpha(n+1)} \right\}^{-\frac{1}{\alpha}} \left( \frac{\sup |a_k|}{\inf |a_k|} \right)^\alpha
\]

\[
\leq \left\{ 1 - \left( 1 - \frac{1}{2^{\log n / \log 2}} \right)^{\alpha n} \right\}^{-\frac{1}{\alpha}} \left( \frac{\sup |a_k|}{\inf |a_k|} \right)^\alpha
\]

\[
\leq \left\{ 1 - \left( 1 - \frac{1}{n} \right)^{\alpha n} \right\}^{-\frac{1}{\alpha}} \left( \frac{\sup |a_k|}{\inf |a_k|} \right)^\alpha
\]

\[
\leq (1 - e^{-\alpha})^{-\frac{1}{\alpha}} \left( \frac{\sup |a_k|}{\inf |a_k|} \right)^\alpha
\]
Thus, we shall have

$$\sum_{m=m_0}^{M(t)} \phi_m(t) \leq \mu (\log \log n) \log \left( \sup_{n \geq |a_k|} \right)^\alpha$$
outside a set of measure atmost

$$\frac{\mu}{m_0} + \exp \left( \frac{-\mu}{\mu_1} \right) \left( \sup_{n \geq |a_k|} \right)^\alpha.$$

Taking $\mu = \mu_1 (\log \log n_0)^{a-1}$, we have for all $n > n_0$

$$\sum_{m=m_0}^{M(t)} \phi_m(t) < \mu_1 (\log \log n)^{a} \log n \left( \sup_{n \geq |a_k|} \right)^\alpha$$
outside a set of measure atmost

5.20 \[ \frac{\mu}{(\log n_0 - \log \log n_0)^{a-1}} \left( \sup_{n \geq |a_k|} \right)^\alpha. \]

Let $N_1$ be the number of real zeros of $f_n(x, \omega)$ in the circle $c_1$

$$P \left\{ \left| \sum_{k=0}^{n} a_k A_k(\omega) \right| < \frac{1}{(n+1)^2} \right\} < \frac{\mu}{(n+1)^{2+\frac{1}{a}}} |a_0|. $$

Hence for $n > n_0$ we shall have $\frac{4 \log (n+1) + 2 \log \log n}{\log 2}$ outside a set of measure atmost

5.21 \[ \frac{\mu}{n_0} \sum_{k=0}^{n} |a_k|^\alpha + \frac{\mu}{n_0^{1+\frac{1}{a}}} |a_0|^{-1}. \]

If $N_n$ denotes the number of zeros $f_n(x, \omega)$ in the interval $[0,1]$ we have for all $n > n_0$

$$N_n < \mu (\log \log n)^{\alpha} \log n.$$
The exceptional set is of measure atmost

\[ \frac{\mu}{n_{0}^{2\alpha-1}} \sum_{0}^{n_{0}} |a_k|^\alpha + \frac{\mu}{n_{0}} |a_0|^{-1} + \frac{\mu}{n_{0}^{4\alpha}} |a_0|^{-1} + \]

\[ \frac{\mu}{(\log \log \log \log n_{0}^{-1})} \left( \sup |a_k| \right)^\alpha \left( \inf |a_k| \right) \]

\[ < \frac{\mu}{(\log n_{0} \log \log \log n_{0})^{-1}} \left( \sup |a_k| \right)^\alpha \left( \inf |a_k| \right) , \]

which proves the theorem.