Scale Invariant Theory
for Bianchi Type-III
Space-Time

Chapter 4
4.1 Introduction

The SITG formulated by Wesson (1981) is being considered by Mohanty and Daud (1995). They have shown that the Bianchi type I stiff perfect fluid [Zeldovich (1962)] is compatible in SITG (1981) and constructed Bianchi type I cosmological stiff fluid models. Further, Mohanty and Daud (1997) have constructed cosmological models governed by vacuum field equations when the space-time is described by a homogeneous and anisotropic Bianchi type I metric with different type of gauge functions and studied the properties of the models.

In continuation with the previous chapters, here we have taken an attempt to study the compatibility of Bianchi type III cosmological model with a matter field in the form of a perfect fluid and Dirac gauge function $\beta = \beta(ct)$ in SITG and have found that the theory is not compatible with this space-time. As before, we have set up the field equations of SITG in section 4.2 and some discussions with respective conclusions have been made in section 4.3.

4.2 Field Equations

The line element for an exact spatially homogeneous and anisotropic Bianchi type III space-time with a gauge function $\beta = \beta(ct)$ as:

$$ds^2 = -c^2dt^2 + e^{-A}dx^2 + e^{2(B+z)}dy^2 + e^{2D}dz^2 \quad (4.1)$$
where A, B and D are functions of cosmic time t only.

The introduction of a new time variable T defined by

\[ dT = \frac{1}{e^{A+B+D}} \, dt \]  

transforms the metric (4.1) into

\[ ds^2 = -c^2 e^{2(A+B+D)} \, dT^2 + e^{2A} \, dx^2 + e^{2(B+x)} \, dy^2 + e^{2D} \, dz^2 \]  

which describes the space-time in general theory of relativity.

Here we intend to take an attempt to build cosmological models in this space-time with a perfect fluid having the energy momentum tensor as mentioned in eqns. (2.2) and (2.3).

The non-vanishing components of conventional Einstein’s tensor (1.63) for the metric (4.3) are

\[ G_{11} = \frac{e^{-2(B+D)}}{c^2} \left[ B_{44} + D_{44} - A_4 B_4 - B_4 D_4 - D_4 A_4 \right], \]  

\[ G_{14} = B_4 - A_4, \]  

\[ G_{22} = \frac{e^{-2(A+D-x)}}{c^2} \left[ A_{44} + D_{44} - A_4 B_4 - B_4 D_4 - D_4 A_4 \right], \]  

\[ G_{33} = \frac{e^{-2(A+B)}}{c^2} \left[ A_{44} + B_{44} - A_4 B_4 - B_4 D_4 - D_4 A_4 - c^2 e^{2(B+D)} \right], \]  

and

\[ G_{44} = -\left[ A_4 B_4 + B_4 D_4 + D_4 A_4 - c^2 e^{2(B+D)} \right]. \]
Hereafter, the suffix 4 after a field variable denotes ordinary differentiation with respect to time $T$.

Using the comoving coordinate frame where $U^i = \delta^i_4$, the non vanishing components of the field eqns. (1.66) for the metric (4.3) can be written in the following explicit forms:

\[
G_{11} = -\kappa p_m c^2 e^{2A} - \frac{e^{-2(B+D)}}{c^2} \left[ -2A_4 \frac{B_4}{\beta} + 2 \frac{B_{44}}{\beta} \frac{\beta_4^2}{\beta^2} \right] - \Lambda_0 \beta^2 e^{2A}, \quad (4.9)
\]

\[
G_{14} = 0 \Rightarrow B = A + k_1, \quad k_1 \text{ is an integrating constant,} \quad (4.10)
\]

\[
G_{22} = -\kappa p_m e^{2(B+x)} - \frac{e^{-2(A+D-x)}}{c^2} \left[ -2B_4 \frac{B_4}{\beta} + 2 \frac{B_{44}}{\beta} \frac{\beta_4^2}{\beta^2} \right] - \Lambda_0 \beta^2 e^{2(B+x)}, \quad (4.11)
\]

\[
G_{33} = -\kappa p_m e^{2D} - \frac{e^{-2(A+B)}}{c^2} \left[ -2D_4 \frac{B_4}{\beta} + 2 \frac{B_{44}}{\beta} \frac{\beta_4^2}{\beta^2} \right] - \Lambda_0 \beta^2 e^{2D} \quad (4.12)
\]

and

\[
G_{44} = -\kappa p_m c^4 e^{4(A+B+D)} + \left[ 2(A_4 + B_4 + D_4) \frac{\beta_4}{\beta} + 3 \frac{\beta_4^2}{\beta^2} \right] + \Lambda_0 \beta^2 e^{2(A+B+D)} \quad (4.13)
\]

which in turn give

\[
G_{11} = -\kappa p_m e^{2A} - \frac{e^{-2(A+D+k_2)}}{c^2} \left[ -2A_4 \frac{B_4}{\beta} + 2 \frac{B_{44}}{\beta} \frac{\beta_4^2}{\beta^2} \right] - \Lambda_0 \beta^2 e^{2A}, \quad (4.14)
\]
\begin{align*}
G_{22} &= -\kappa p_m e^{2(A+D+k_i)} - \frac{e^{-2(A+D-x)} - 2A_4 \frac{\beta_4}{\beta} + 2 \frac{\beta_4^2}{\beta^2}}{c^2} - \Lambda_0 \beta^2 e^{2(A+D-x)}, \\
G_{33} &= -\kappa p_m e^{2D} - \frac{e^{-2(2A+D_k_i)} - 2D_4 \frac{\beta_4}{\beta} + 2 \frac{\beta_4^2}{\beta^2}}{c^2} - \Lambda_0 \beta^2 e^{2D} \\
G_{44} &= -\kappa p_m e^{2} e^{2(2A+D+k_i)} + \left[ 2(2A_4 + D_4) \frac{\beta_4}{\beta} + 3 \frac{\beta_4^2}{\beta^2} \right] + \Lambda_0 \beta^2 c^2 e^{2(2A+D+k_i)}
\end{align*}

Here eqn. (4.15) is redundant since $G_{22} = e^{2(A+D+k_i)} G_{11}$ (i.e., $G_2 = G_1$).

Moreover, in Bianchi type space time this equation is usually treated as a condition for isotropy of pressure $'p_m'$ for constructing exact solution to the field equations and hence the corresponding cosmological model. The analogous condition Kramer (1980) for spherically symmetric space-time is $G_1 = G_3$.

As before, $p_\nu$ and $\rho_\nu$ can be obtained as:

\begin{align*}
e^{-2(2A+D+k_i)} \left( 2A_4 \frac{\beta_4}{\beta} - 2 \frac{\beta_4^2}{\beta^2} \right) - \Lambda_0 \beta^2 c^2 &= -\kappa p_\nu c^2, \tag{4.18} \\
e^{-2(2A+D+k_i)} \left( 2D_4 \frac{\beta_4}{\beta} - 2 \frac{\beta_4^2}{\beta^2} \right) - \Lambda_0 \beta^2 c^2 &= -\kappa \rho_\nu c^2 \tag{4.19}
\end{align*}

and
In this case when there is no matter and gauge function $\beta$ is taken as a constant, one recovers the relation

$$c^2 p_v = c^4 \frac{\lambda_{GR}}{8\pi G} = -p_v$$

i.e. $c^2 p_v + p_v = 0$  \hspace{1cm} (4.21)

which represents the equation of state for false vacuum case. Here $\lambda_{GR} = \lambda_0 \beta^2$ = constant is the cosmological constant in general relativity. Also $p_v$ being dependent on constants $\lambda_{GR}$, $c$ and $G$ is uniform in all directions and hence isotropic in nature.

It is evident from the aforesaid equations that $p_v$, being isotropic, is consistent only when

$$D = \Lambda + k_2, \text{ since } \beta_4 \neq 0$$

where $k_2$ is the integrating constant.

Using the consistency condition (4.22), the pressure and energy density for the false vacuum case reduce to

$$p_v = -\frac{1}{\kappa c^2} \left( e^{-2(3A + D + k_1)} \left[ 2A_4 \frac{\beta_4}{\beta} + 3 \frac{\beta_4^2}{\beta^2} + \Lambda_0 \beta^2 c^2 \right] \right)$$

and

$$\rho_v = -\frac{1}{\kappa c^2} \left( e^{-2(3A + D + k_1)} \left[ 6A_4 \frac{\beta_4}{\beta} + 3 \frac{\beta_4^2}{\beta^2} + \Lambda_0 \beta^2 c^2 \right] \right)$$

(4.23) and

(4.24)
The definition of above quantities [4.23,4.24] is natural as regards to the scale invariant properties of the vacuum. So the total pressure and energy density can be defined as in eqns. (2.20) and (2.21).

So $p_t$ and $\rho_t$ can be expressed explicitly as:

$$e^{-2(3A+k_1+k_2)}\left[2A_{44} - 3A_4^2\right] = -\kappa p_t c^2 , \quad (4.25)$$

$$e^{-2(3A+k_1+k_2)}\left[2A_{44} - 3A_4^2 - c^2 e^{2(2A+k_1+k_2)}\right] = -\kappa \rho_t c^2 \quad (4.26)$$

and

$$e^{-2(3A+k_1+k_2)}\left[3A_4^2 - c^2 e^{2(2A+k_1+k_2)}\right] = -\kappa p_t c^2 \quad (4.27)$$

Now eqns. (4.25) and (4.26) yield

$$c^2 e^{2(2A+k_1+k_2)} = 0 \quad (4.28)$$

Eqn. (4.28) leads to inconsistency for both the cases, i.e. $A$ is negative and very large or $c = 0$. Neither of the cases is acceptable as the solution of the field equations can be obtained in the SITG which is mainly governed by a gauge function involving non-zero velocity of light and hence, the velocity of light plays the most vital role in this theory. Thus, the SITG is not compatible with Bianchi type III space-time.

### 4.3 Discussion

Wesson [1981] assumed that

(a) The metric describing the space time is diagonal and spherically symmetric;

(b) The gauge function $\beta$ depends on time coordinate only;

(c) The energy momentum tensor is that of a perfect fluid in his formulation of SITG.
Because of the physical importance of the Bianchi type cosmological models, we have attempted to study the SITG in a space-time governed by a diagonal Bianchi type III metric with a Dirac gauge function $\beta$ (depending on time only), in view of homogeneity of the space-time. However, it is found that this theory is also not compatible with Bianchi type III space-time.