Scale Invariant Theory for Bianchi Type II, VIII, and IX Space-Times

Chapter 3
CHAPTER-3

SCALE INVARIANT THEORY FOR BIANCHI TYPE II, VIII AND IX SPACE-TIMES

3.1 Introduction

The role played by SITG concerned with the elementary particle physics and cosmology has been discussed by some authors, viz. Dirac (1971, 1974), Hoyle and Narlikar (1974) and Canuto et al. (1971, 1977a,b). In the later stage Wesson (1981) proposed a simple formulation of SITG incorporating a gauge function $\beta(x^i)$, where $x^i$ are co-ordinates in the four dimensional space-times. In this theory Mohanty and Daud (1994, 1997) have studied the vacuum cosmological models when the space-time is described by homogeneous and anisotropic Bianchi type-I metric with different types of Dirac gauge functions. It has been shown there that the models reduce to Kasner (1921) model when dimensionless cosmological constant $\Lambda_0$ is zero, but for $\Lambda_0 \neq 0$ the models isotropize as in the Einstein theory.

Following chapter-2, in this chapter we have tried to construct the cosmological models of Bianchi type II ($\delta = 0$), VIII ($\delta = 1$) and IX ($\delta = -1$) space-times with a matter field in the form of a perfect fluid in SITG with Dirac gauge function $\beta = \frac{1}{ct}$. Here it is also observed that Bianchi type II metric ($\delta = 0$) and Bianchi type IX metric ($\delta = -1$) are not compatible whereas Bianchi type VIII ($\delta = 1$) is compatible with this theory. The presence of gauge field indicates some distinguishable features in the models. As
before, we have set the field equations of SITG in section 3.2. Then in section 3.3, we have obtained the explicit exact solution for $\delta = 1$ of the field equations with the help of the energy momentum tensor for a perfect fluid where we confine ourselves to the equation of state $P_t = \frac{\rho_t c^2}{3}$. We have also studied the physical behavior described by solution in terms of dimensionless cosmological constant in section 3.4.

### 3.2 Field Equations

Here we consider line elements for homogeneous anisotropic Bianchi type II, VIII and IX in a locally rotationally symmetric system with a gauge function $\beta = \beta(ct)$ as:

$$ds^2 = -c^2 dt^2 + S^2 dx^2 + R^2 \left[ dy^2 + f^2(y) dz^2 \right] - S^2 h(y) \left[ 2 dx - h(y) dz \right] dz \quad (3.1)$$

where $R = R(t), S = S(t)$ and

$$f(y) = \begin{bmatrix} \sin y \\ y \\ \sinh y \end{bmatrix}, \quad h(y) = \begin{bmatrix} \cos y \\ -\frac{y^2}{2} \\ -\cosh y \end{bmatrix} \quad \text{for} \quad \delta = \begin{bmatrix} +1 \\ 0 \\ -1 \end{bmatrix} \quad (3.2)$$

Here we intend to take an attempt to build cosmological models in these space-times with a gravitating perfect fluid having the energy-momentum tensor given by eqns. (2.2) and (2.3).

The non-vanishing components of conventional Einstein’s tensor (1.63) for the metric (3.1) are:

$$G_{11} = S^2 \left[ \frac{\mathcal{R}_{44}}{c^2} + \frac{R^4}{R c^2} + \frac{\delta}{R^2} - \frac{3S^2}{4R^4} \right], \quad (3.3)$$
Hereafter, the suffix 4 after a field variable denotes exact differentiation with respect to time \( t \).

Using the comoving co-ordinate frame where \( U^i = \delta^i_4 \), the non-vanishing components of the field eqns. (1.66) can be evaluated for the metric (3.1) and can be written in the following explicit form:

\[
G_{11} = -\kappa m S^2 - \frac{S^2}{c^2} \left[ 4 \frac{R_4 \beta^4}{R \beta} + 2 \frac{\beta_{44}}{\beta^2} \frac{\beta^2}{\beta^2} + \Lambda \beta^2 c^2 \right],
\]

\[
G_{13} = \kappa m S^2 h + \frac{S^2 h}{c^2} \left[ 4 \frac{R_4 \beta^4}{R \beta} + 2 \frac{\beta_{44}}{\beta} \frac{\beta^2}{\beta^2} + \Lambda \beta^2 c^2 \right],
\]

\[
G_{22} = -\kappa m R^2 - \frac{R^2}{c^2} \left[ 2 \frac{R_4 \beta^4}{R \beta} + 2 \frac{S_4 \beta^4}{S \beta} + 2 \frac{\beta_{44}}{\beta} \frac{\beta^2}{\beta^2} + \Lambda \beta^2 c^2 \right],
\]
\[
G_{33} = -\kappa p \left( R^2 f^2 + S^2 h^2 \right) - \frac{R^2 f^2}{c^2} \left[ \frac{2 R_4 b_4}{R \beta} + 2 \frac{S_4 b_4}{S \beta} + \frac{2 b_4^2}{\beta^2} - \frac{\Lambda_0 \beta^2 c^2}{\beta^2} \right] \\
- \frac{S^2 h^2}{c^2} \left[ \frac{4 R_4 b_4}{R \beta} + 2 \frac{b_4^2}{\beta^2} - \frac{\beta_4^2}{\beta^2} + \Lambda_0 \beta^2 c^2 \right] 
\] (3.11)

and

\[
G_{44} = -\kappa p_m c^4 + \left[ 4 \frac{R_4 b_4}{R \beta} + 2 \frac{b_4^2}{\beta^2} + \frac{S_4 b_4}{S \beta} + \frac{3 \beta_4^2}{\beta^2} + \Lambda_0 \beta^2 c^2 \right]. 
\] (3.12)

Where eqn.(3.11) is redundant since \(G_{33} = f^2 G_{22} - h G_{13}\).

In the usual way suggested by (Wesson 1981a,b), the quantities \(p_v\) and \(\rho_v\) that involve neither the Einstein tensor of conventional theory nor the properties of conventional matter can be obtained as :

\[
4 \frac{R_4 b_4}{R \beta} + 2 \frac{b_4^2}{\beta^2} + \Lambda_0 \beta^2 c^2 = -\kappa p_v c^2, 
\] (3.13)

\[
2 \frac{R_4 b_4}{R \beta} + 2 \frac{S_4 b_4}{S \beta} + \frac{2 b_4^2}{\beta^2} + \frac{\Lambda_0 \beta^2 c^2}{\beta^2} = -\kappa \rho_v c^4 
\] (3.14)

and

\[
4 \frac{R_4 b_4}{R \beta} + 2 \frac{S_4 b_4}{S \beta} + 3 \frac{\beta_4^2}{\beta} + \Lambda_0 \beta^2 c^2 = \kappa \rho_v c^4 
\] (3.15)

It is evident from the aforesaid equations that the pressure \(p_v\) being isotropic (as in chapter 2) is consistent only when

\[
R = k_1 S, \text{ since } \beta_4 \neq 0 
\] (3.16)
Using the consistency condition (3.16), the pressure and energy density for the vacuum case reduce to

\[
P_v = -\frac{1}{\kappa c^2} \left[ 4 \frac{R_{44}}{R} \beta^2 + 2 \frac{\beta_{4t}}{\beta} \frac{\beta_4^2}{\beta^2} + \Lambda_0 \beta^2 c^2 \right]
\]

(3.17)

\[
\rho_v = \frac{1}{\kappa c^4} \left[ 6 \frac{R_{44}}{R} \beta^2 + \frac{\beta_4^2}{\beta^2} + \Lambda_0 \beta^2 c^2 \right]
\]

(3.18)

As before, here \( p_v \) and \( \rho_v \) refer to the properties of vacuum only in conventional physics. So the total pressure and energy density can be defined as in eqns. (2.20) and (2.21).

The field equations in SITG (3.8)-(3.12) can now be written by using the components of Einstein tensor (3.3)-(3.7) with the definitions mentioned earlier vide eqns. (2.20) and (2.21), in the following explicit form:

\[
2 \frac{R_{4t}}{R} + \frac{R_4^2}{R^2} + \frac{\delta c^2}{R^2} - \frac{3 c^2}{4 R^2} = -\kappa \rho_v c^2,
\]

(3.19)

\[
2 \frac{R_{4t}}{R} + \frac{R_4^2}{R^2} + \frac{c^2}{4 R^2} = -\kappa p_v c^2
\]

(3.20)

and

\[
3 \frac{R_4^2}{R^2} + \frac{\delta c^2}{R^2} - \frac{c^2}{4 R^2} = -\kappa \rho_v c^2
\]

(3.21)

From eqns. (3.19) and (3.20), we obtain

\[
\frac{(\delta - 1) c^2}{R^2} = 0
\]

(3.22)

which yields
for $\delta = 0, -1$.

Eqn. (3.22) implies that either the velocity of light $c=0$ or the metric potential $R$ is infinitely large. Both the cases lead to unphysical situation. Thus, the SITG is not feasible when the space-time is governed by LRS Bianchi type II ($\delta=0$) and IX ($\delta=-1$) whereas it is feasible for $\delta=1$ i.e. for Bianchi type VIII space-time. Hence, in the subsequent section, we have constructed a Bianchi type VIII cosmological model in SITG.

### 3.3 Bianchi Type VIII Cosmological Models

In this case the field equations in SITG eqn. (3.19) is redundant and hence, the system of equations consisting of eqns. (3.20) and (3.21) are two equations in three unknowns, viz. $R$, $p_t$ and $p_i$ for $\delta =1$. For the complete determinacy one extra condition is needed. So we take the equation of state

$$p_i = \frac{p_c c^2}{3}$$ (3.23)

which represents matter radiation.

From eqns. (3.20) and (3.21), we obtain

$$R = \frac{(-c^2 t^2 + 4k_2 t + 4k_3)^{1/2}}{2}$$ (3.24)

where $k_2$ and $k_3$ are integrating constants.

Without loss of generality we can take $k_1=1$ in eqn.(3.16), then we have
Now the total pressure $p_t$ and energy density $\rho_t$ can be obtained as:

$$p_t = \frac{\rho_t c^2}{3} = \frac{4k^2 c^2 + \kappa c^2}{\kappa c^2 (c^2 t^2 - 4k^2 t - 4k^3)}$$

(3.26)

The pressure and energy density corresponding to vacuum case can be calculated as:

$$p_v = \frac{1}{\kappa c^2} \left[ \frac{4(c^2 t^2 - 2k)}{t - c^2 t^2 + 4k^2 t + 4k^3} + \frac{3 + \Lambda_0}{t^2} \right]$$

(3.27)

$$\rho_v = \frac{1}{\kappa c^4} \left[ \frac{6(c^2 t^2 - 2k)}{t - c^2 t^2 + 4k^2 t + 4k^3} + \frac{3 + \Lambda_0}{t^2} \right]$$

(3.28)

Now the matter pressure and density can be obtained as:

$$p_m = \frac{1}{\kappa c^2} \left[ \frac{4k^2 c^2 + \kappa c^2}{(c^2 t^2 + 4k^2 t + k^3)^2} + \frac{4(c^2 t^2 - 2k)}{t - c^2 t^2 + 4k^2 t + k^3} + \frac{3 + \Lambda_0}{t^2} \right]$$

(3.29)

$$\rho_m = \frac{1}{\kappa c^4} \left[ \frac{12k^2 c^2 + \kappa c^2}{(c^2 t^2 + 4k^2 t + k^3)^2} + \frac{6(c^2 t^2 - 2k)}{t - c^2 t^2 + 4k^2 t + k^3} - \frac{3 + \Lambda_0}{t^2} \right]$$

(3.30)

Thus, the radiating LRS Bianchi type VIII model in SITG is given by the eqns. (3.25) and (3.26) and the model in this case becomes

$$ds_w^2 = \frac{1}{c^2 t^2} \left[ -c^2 dt^2 + \frac{1}{4} (c^2 t^2 + 4k^2 t + 4k^3) \left\{ (dx^2 + dy^2 + \sin^2 ydz^2) - \cos y(2dx - \cos y dz) dz \right\} \right]$$

(3.31)
3.4 Discussion

In this chapter, we have taken an attempt to formulate the SITG Wesson (1981) replacing diagonal spherical symmetric metric by non-diagonal Bianchi type II (δ=0), VIII (δ=1) and IX (δ=-1) metrics, because of the physical importance of Bianchi cosmologies. However, this theory is not feasible for Bianchi type II (δ=0) as studied earlier exclusively in chapter 2 and so also for Bianchi type IX.

For δ = 1, the scalar expansion Θ = U^1 calculated as

\[ \Theta = \frac{3(c^2 t - k_2)}{c(c^2 t^2 - 4k_1 t - 4k_3)} \]

from which it is evident that

\[ \Theta \to 0 \quad \text{as} \quad t \to \frac{k_2}{c^2} \]

and \[ \Theta \to \pm \infty \quad \text{(according as} \quad t > \frac{k_2}{c^2} \text{or} \quad t < \frac{k_2}{c^2} \text{) as} \quad t \to \frac{2}{c^2} \left[ k_2 \pm \sqrt{k_2 + k_3 c^2} \right]. \]

For \[ t > \frac{k_2}{c^2} (k_2 > 0) \], the model is expanding till \[ t > \frac{k_2 \pm \sqrt{5k_2^2 - 4k_3 c^2}}{c^2} \] and subsequently the model is contracting till infinite future and this behaviour is just opposite for \[ t < \frac{k_2}{c^2} \]. In both the cases the rate of expansion/contraction is uniform.

For \[ k_2 = 0 \],

\[ \Theta \to 0 \quad \text{as} \quad t \to 0 \quad \text{and} \quad \Theta \to \infty \quad \text{as} \quad t \to \frac{2\sqrt{k_3}}{c}, k_3 > 0. \]
For $k_3 = 0$,

$$\Theta \to 0 \text{ as } t \to \frac{k_2}{c^2}, k_2 > 0 \text{ and } \Theta \to \infty \text{ as } t \to 0.$$ 

In both the aforesaid cases the universe is ever expanding where the parameters $k_2$ and $k_3$ involved are positive.

It has also been observed that

$$\frac{\rho_m}{\Theta^2} = \frac{4(k_2^2 + k_1c^2)}{3kc^2(k_2 - c^2t)} \to \infty \text{ as } t \to \frac{k_2}{c^2}$$

and

$$\frac{\rho_m}{\Theta^2} \to 0 \text{ as } t \to \infty.$$ 

which confirms the homogeneity nature of the space-time. The shear scalar 'σ' found to be zero indicating that the shape of the universe remains unchanged during evolution. It is clear that $\frac{\sigma^2}{\Theta^2} = 0$. Thus, the space-time is isotropized during evolution in SITG. Here we found that the vorticity 'w' vanishes which indicates that $U^i$ is hypersurface orthogonal. Since the acceleration is $\ddot{U}^i$ found to be zero, the matter particles follow geodesics in this theory. Further, we have

$$\rho_m \to -\infty \text{ as } t \to 0 \text{ and } \rho_m \to 0 \text{ as } t \to \infty.$$ 

This indicates that there is a big bang like singularity at the initial epoch.