Scale Invariant Theory for Bianchi Type-$VI_h$
Space-Time

Chapter 6
6.1 Introduction

It is found from the literature that there are two prominent generalizations of Einstein theory. Firstly, in an attempt to unify electromagnetism with gravitation, Weyl (1922) generalized Riemannian geometry by allowing lengths to change under parallel displacement. The theory, being unphysical, was soon rejected, wherein a mathematical technique known as gauge transformation was introduced. Eddigton (1924) pointed out that the gauge transformation represents a change of units of measurement and hence gives a general scaling of the physical system. Secondly, Dirac (1973) rebuilt the Weyl’s unified theory by introducing the notion of two metrics with an additional gauge function $\beta(x^i)$. A scale invariant variation principle was proposed from which gravitational and electromagnetic field equations can be derived. It is concluded there that an arbitrary gauge function is necessary in all SITG.

Mohanty and Daud (1995) have shown that the Bianchi type I cosmological models governed by stiff fluid distribution is compatible with SITG. They observed that the presence of gauge field does indicate some distinguishable features in the models compared to those developed only with the perfect fluid as the material source.

In this chapter, we have studied the feasibility of SITG proposed by Wesson (1981) in Bianchi type VIh space-time with a time dependent gauge function $\beta = \beta(ct)$ in order to share some symmetry with that of the space-time and a matter field in the form of a...
perfect fluid. We have set up the field equations of SITG in section 6.2. In section 6.3, we have constructed Bianchi type VIₜ(h = 1) radiating cosmological model. Some physical properties of the model have been given in section 6.4 and concluding remarks in section 6.5. It is found that Bianchi type VIₜ(h = 1) space-time is feasible in this theory whereas Bianchi type VIₜ(h = -1) and VIₜ(h = 0) space-times are not feasible. In the feasible case non-singular models for the universe filled with disorder radiation and dust are constructed.

6.2 Field Equations

Here we have considered the line elements for Bianchi type VIₜ in a locally rotationally system with a gauge function β = β(ct) as

\[
ds^2_E = -c^2 dt^2 + A^2 dx^2 + B^2 e^{2x} dy^2 + C^2 e^{2hx} dz^2
\]

where \(A = A(t), B = B(t), C = C(t)\) and \(h = -1,0,1\).

Here we have taken an attempt to build cosmological models in the space-time (6.1) with a perfect fluid having the energy momentum tensor of the form (2.2) and (2.3).

Now, the non-vanishing components of conventional Einstein’s tensor (1.63) for the metric (6.1) are:

\[
G_{11} = \frac{A^2}{c^2} \left[ \frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_{4}C_{4}}{BC} - h \frac{c^2}{A^2} \right], \quad \text{(6.2)}
\]

\[
G_{22} = \frac{B' e^{2x}}{c^2} \left[ \frac{A_{44}}{A} + \frac{C_{44}}{C} + \frac{A_{4}C_{4}}{AC} - h^2 \frac{c^2}{A^2} \right], \quad \text{(6.3)}
\]

\[
G_{33} = \frac{C^2 e^{2hx}}{c^2} \left[ \frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_{4}B_{4}}{AB} - \frac{c^2}{A^2} \right], \quad \text{(6.4)}
\]
and

\[ G_{14} = -(1 + h) \frac{A_4}{A} + \frac{B_4}{B} + h \frac{C_4}{C}. \]  \hspace{1cm} (6.6)

The suffix 4 after a field variable denotes exact differentiation with respect to time \( t \) only.

Using the comoving coordinates \((0,0,0,c)\), the non vanishing components of the field eqns. (1.66) with energy momentum tensor (2.2) and (2.3) can now be evaluated for the metric (6.1) and can be written in the following explicit form:

\[ G_{11} = -\kappa \rho_m A^2 - \frac{A^2}{c^2} \left[ 2 \frac{\beta_4}{\beta} \left( \frac{B_4}{B} + \frac{C_4}{C} \right) - \frac{\beta_4^2}{\beta^2} - 2 \frac{\beta_{44}}{\beta} + \Lambda_0 \beta^2 c^2 \right], \]  \hspace{1cm} (6.7)

\[ G_{22} = -\kappa \rho_m B^2 e^{2x} - \frac{B^2 e^{2x}}{c^2} \left[ 2 \frac{\beta_4}{\beta} \left( \frac{A_4}{A} + \frac{C_4}{C} \right) - \frac{\beta_4^2}{\beta^2} - 2 \frac{\beta_{44}}{\beta} + \Lambda_0 \beta^2 c^2 \right], \]  \hspace{1cm} (6.8)

\[ G_{33} = -\kappa \rho_m C^2 e^{2hx} - \frac{C^2 e^{2hx}}{c^2} \left[ 2 \frac{\beta_4}{\beta} \left( \frac{A_4}{A} + \frac{B_4}{B} \right) - \frac{\beta_4^2}{\beta^2} - 2 \frac{\beta_{44}}{\beta} + \Lambda_0 \beta^2 c^2 \right], \]  \hspace{1cm} (6.9)

\[ G_{44} = -\kappa \rho_m e^4 + \left[ 2 \frac{\beta_4}{\beta} \left( \frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C} \right) - \frac{\beta_4^2}{\beta^2} - 4 \frac{\beta_{44}}{\beta} + \Lambda_0 \beta^2 c^2 \right] \]  \hspace{1cm} (6.10)

and
\[ G_{14} = 0 \Rightarrow \frac{B_4}{B} + h\frac{C_4}{C} = (1 + h)\frac{A_4}{A} \text{ i.e. } A^{1+h} = k_1 BC^h \] (6.11)

where \( k_1 \) is an integrating constant.

As before, the vacuum pressure \( p_v \) and vacuum density \( \rho_v \) can be obtained as:

\[ 2\frac{\beta_4}{\beta}\left[\left(h + 1\right)\frac{A_4}{A} + \left(\frac{h - 1}{h}\right)\frac{B_4}{B}\right] - \frac{\beta_4^2}{\beta^2} - 2\frac{\beta_{44}}{\beta} + \Lambda_0\beta^2c^2 = \kappa p_v c^2, \quad (6.12) \]

\[ 2\frac{\beta_4}{\beta}\left[\left(2h + 1\right)\frac{A_4}{A} - \left(\frac{1}{h}\right)\frac{B_4}{B}\right] - \frac{\beta_4^2}{\beta^2} - 2\frac{\beta_{44}}{\beta} + \Lambda_0\beta^2c^2 = \kappa \rho_v c^2, \quad (6.13) \]

\[ 2\frac{\beta_4}{\beta}\left[\frac{A_4}{A} + \frac{B_4}{B}\right] - \frac{\beta_4^2}{\beta^2} - 2\frac{\beta_{44}}{\beta} + \Lambda_0\beta^2c^2 = \kappa \rho_v c^2 \quad (6.14) \]

and

\[ 2\frac{\beta_4}{\beta}\left[\left(2h + 1\right)\frac{A_4}{A} + \left(\frac{h - 1}{h}\right)\frac{B_4}{B}\right] + 3\frac{\beta_4^2}{\beta^2} - 4\frac{\beta_{44}}{\beta} + \Lambda_0\beta^2c^2 = -\kappa \rho_v c^4 (6.15) \]

In case where there is no matter and gauge function is a constant function it can be easily obtained from eqns. (6.12)-(6.15) that

\[ c^2 \rho_v = -\frac{c^4\Lambda_{GR}}{8\pi G} = -p_v \text{ i.e. } c^2 \rho_v + p_v = 0 \]

Here \( p_v \) is dependent on constants \( \Lambda_{GR} \), \( G \) and \( c \) is uniform in all directions and hence isotropic in nature. The cosmological model with the equation of state is rare in the literature and is known as \( \rho \)-vacuum or false vacuum or degenerate volume model [Davies (1984), Blome and Prierter (1984), Hogan (1984), Kaiser and Stebbins (1984)]. The corresponding model in static case is the well-known de-Sitter model.
Hence, \( p_v \), being isotropic, is consistent only when

\[
A = k_2 B, \quad \text{since} \quad \beta_4 \neq 0
\]  

(6.16)

where \( k_2 \) is an integrating constant.

Using the consistency condition (6.16), the pressure and energy density for the vacuum case reduce to

\[
p_v = \frac{1}{\kappa c^4} \left[ 2 \frac{\beta_4}{\beta} \left( 2 \frac{A_4}{A} \right) \frac{\beta_4^2}{\beta^2} - 2 \frac{\beta_{44}}{\beta} + \Lambda_0 \beta^2 c^2 \right]
\]

(6.17)

and

\[
\rho_v = -\frac{1}{\kappa c^4} \left[ 2 \frac{\beta_4}{\beta} \left( \frac{3 A_4}{A} \right) + 3 \frac{\beta_4^2}{\beta^2} - 4 \frac{\beta_{44}}{\beta} + \Lambda_0 \beta^2 c^2 \right]
\]

(6.18)

Here \( p_v \) and \( \rho_v \) carry the properties of vacuum in conventional physics. The definition of above quantities is natural as regards to the scale invariant properties of vacuum (Wesson 1981 a,b). So the total pressure and energy density can be obtained as in eqns. (2.20) and (2.21).

Using the components of Einstein tensor (6.2) – (6.6) and the results obtained in eqns. (6.16) – (6.18), the field eqns. (6.7) – (6.11) with the definitions of \( p_t \) and \( \rho_t \) given in eqns. (2.23) and (2.24) can now be written in the following explicit form:

\[
2 \frac{A_{44}}{A} + \frac{A_4^2}{A^2} - \hbar \frac{c^2}{A^2} = -\kappa p_t c^2,
\]

(6.19)

\[
2 \frac{A_{44}}{A} + \frac{A_4^2}{A^2} - \hbar^2 \frac{c^2}{A^2} = -\kappa p_t c^2,
\]

(6.20)
As the total pressure is isotropic in nature we obtain from eqns. (6.19) - (6.22) that

\[ \frac{2 A_{44}}{A} + \frac{A_{4}^{2}}{A^{2}} - \frac{c^{2}}{A^{2}} = -\kappa p_{1}c^{2} \]  

(6.21)

and

\[ 3 \frac{A_{4}^{2}}{A^{2}} - (1 + h + h^{2}) \frac{c^{2}}{A^{2}} = \kappa \rho_{1}c^{4} \]  

(6.22)

As the total pressure is isotropic in nature we obtain from eqns. (6.19) - (6.22) that

\[ \frac{h(h - 1)c^{2}}{A^{2}} = 0 \quad \text{and} \quad \frac{(h^{2} - 1)c^{2}}{A^{2}} = 0 \]  

(6.23)

These equations simultaneously hold good for \( h = 1 \). Otherwise it leads to unphysical situation, i.e. either the velocity of light vanishes or the metric potential \( A \) is infinitely large. So this theory is not feasible when the space-time is governed by LRS Bianchi type \( \text{VI}_{h}(h = -1) \) and \( \text{VI}_{h}(h = 0) \) metrics whereas it is feasible only for Bianchi type \( \text{VI}_{h}(h = 1) \) metric.

6.3 Bianchi Type \( \text{VI}_{(h=1)} \) Radiating Cosmological Model

Now, we have two field equations with three unknowns \( p_{1}, \rho_{1}, \) and \( A \) for \( h = 1 \). For the complete determinacy one extra condition is needed. We, therefore, consider the equation of state

\[ p_{t} = \frac{\rho_{1}c^{2}}{3} \]  

(6.24)

Hence, from eqns. (6.19) - (6.22), for \( h = 1 \), we obtain

\[ A = \left(c^{2}t^{2} + d_{1}t + d_{2}\right)^{\frac{1}{2}} \]  

(6.25)
where $d_1$ and $d_2$ are two real constants of integration.

Without loss of generality, we take $k_1 = k_2 = 1$ in eqns. (6.11) and (6.16) respectively.

Subsequently, we have

$$A = B = C = (c^2 t^2 + d_1 t + d_2)^{1/2}$$

(6.26)

Now the total pressure $p_t$ and energy density $\rho_t$ can be calculated as:

$$p_t = \rho_t c^2 = \frac{1}{4k\epsilon^2} \left[ \frac{(d_1^2 - 4d_2 c^2)}{(c^2 t^2 + d_1 t + d_2)^2} \right]$$

(6.27)

Now considering Dirac gauge function in the form $\beta = \frac{1}{\epsilon t}$ the pressure and energy density corresponding to vacuum can be calculated as:

$$p_v = -\frac{1}{k\epsilon^2} \left[ \frac{2(2c^2 t + d_1)}{t(c^2 t^2 + d_1 t + d_2)} - \frac{\Lambda_0 - 5}{t^2} \right]$$

(6.28)

and

$$\rho_v = \frac{1}{k\epsilon^4} \left[ \frac{3(2c^2 t + d_1)}{t(c^2 t^2 + d_1 t + d_2)} - \frac{\Lambda_0 - 5}{t^2} \right]$$

(6.29)

Subsequently, the matter pressure and density can be obtained as:

$$p_m = \frac{1}{k\epsilon^2} \left[ \frac{(d_1^2 - 4d_2 c^2)}{4(c^2 t^2 + d_1 t + d_2)^2} + \frac{2(2c^2 t + d_1)}{t(c^2 t^2 + d_1 t + d_2)} - \frac{\Lambda_0 - 5}{t^2} \right]$$

(6.30)

and

$$\rho_m = \frac{1}{k\epsilon^4} \left[ \frac{3(d_1^2 - 4d_2 c^2)}{4(c^2 t^2 + d_1 t + d_2)^2} - \frac{3(2c^2 t + d_1)}{t(c^2 t^2 + d_1 t + d_2)} + \frac{\Lambda_0 - 5}{t^2} \right]$$

(6.31)
Thus, the Bianchi type VI(h=1) model in SITG is governed by the eqns. (6.27) and (6.28) and the metric in this case is

$$\text{ds}_w^2 = \frac{1}{c^2 t^2} \left[ -c^2 dt^2 + \left(c^2 t^2 + d_1 t + d_2 \right) \right] dx^2 + e^{2x} \left(dy^2 + dz^2 \right)$$

(6.32)

With the help of time coordinate transformation i.e. $t = e^T$ the model can be put in the following form

$$\text{ds}_w^2 = -dT^2 + Q^2(T) \left[ dx^2 + e^{2x} \left(dy^2 + dz^2 \right) \right]$$

(6.33)

where

$$Q(T) = \left(1 + \frac{d_1}{c^2} e^{-T} + \frac{d_2}{c^2} e^{-2T} \right)^\frac{1}{2}$$

6.4 Some Physical Properties of the Model

In this section we intend to study some of the physical properties of the model universe obtained in the last section in the transformed coordinate system wherein the behaviour of the physical quantities remain alike even though there is a shift, i.e. $t(0,1,\infty) \rightarrow T(-\infty,0,\infty)$. The new time coordinate being stretched covers the time region from past to future completely. Thus, one gets clear picture of the model in the new time coordinate $T$.

6.4.1 Scalar Expansion of the Model

The scalar expansion of the model

$$\Theta = U_i^{\dot{i}} = 3 \frac{Q_T}{Q}$$
which gives the rate of expansion or contraction of the model found to be

\[ \Theta(T) = \frac{3}{2} \left( \frac{d_1 e^{-T} + 2d_2 e^{-2T}}{c^2 + d_1 e^{-T} + d_2 e^{-2T}} \right) \]

Thus, we have

\[ \Theta(0) = -\frac{3}{2} \left( \frac{d_1 + 2d_2}{c^2 + d_1 + d_2} \right) \quad \text{and} \quad \Theta \to 0 \quad \text{as} \quad T \to \infty. \]

Moreover, on mathematical ground if \(d_1\) and \(d_2\) are both non-zero then they are of opposite sign and \(\Theta(T = \ln z + \ln \left( -\frac{d_2}{d_1} \right)) = 0\). Thus the model is expanding without admitting any singularity during evolution.

The shear scalar \(\sigma\) for the model vanishes which indicates that the shape of the universe is unchanged during the evolution. Since \(\frac{\sigma^2}{\Theta^2} = 0\), the space-time is isotropized during evolution in this theory. The vorticity \(w\) of the radiating fluid of the model also vanishes. Thus, \(U^i\) is hypersurface orthogonal. The acceleration \(\dot{U}^i = 0\) confirms that the matter particles follow geodesics in this theory.

From eqn. (6.30), we have

\[ \rho_m(0) = \text{positive constant with proper choice of the parameters } \{d_1, d_2, \lambda_0\} \quad \text{and} \quad \rho_m \to 0 \quad \text{as} \quad T \to \infty. \] Thus the universe starts evolving with constant matter density at initial epoch.

Further, it has also been observed that
\[
\frac{\rho_m}{\Theta^2_{T=0}} = \text{constant and} \ \frac{\rho_m}{\Theta^2_{T=\infty}} = 0
\]

which confirms the homogeneity nature of the space-time during the evolution.

The spatial volume of the model

\[
V(T, x) = \left(1 + \frac{d_1}{c^2}e^{-T} + \frac{d_2}{c^2}e^{-2T}\right)^{\frac{3}{2}}e^{2x}
\]

yields \(V(0, 0) \rightarrow (1 + \frac{d_1}{c^2} + \frac{d_2}{c^2})^3\)

Further, we find \(V \rightarrow e^{2x}\) as \(T \rightarrow \infty\) and subsequently \(V \rightarrow \infty\) as \(x \rightarrow \infty\) and also \(V \rightarrow \infty\) as \((T, x) \rightarrow (\infty, \infty)\).

Thus, the model is spatially open and temporarily closed and expands uniformly till infinite future. It is interesting to note that the cosmological model exists only when one of d's is non-zero.

The Hubble parameter \(H\) corresponding to the metric (6.33) is

\[
H = \frac{Q_T}{Q} = -\frac{1}{2} \left( \frac{d_1e^{-T} + 2d_2e^{-2T}}{c^2 + d_1e^{-T} + d_2e^{-2T}} \right)
\]

which determines the present rate of expansion of the universe. However, \(H(0) = \text{constant and} \ H \rightarrow 0\) as \(T \rightarrow \infty\), which indicates that the rate of expansion is accelerated or decelerated depending on the signature of the parameters.

Also the deceleration parameter \(q = -\frac{Q_{TT}Q}{Q_T^2}\) for the model (6.33) becomes
\[ q = \left[ \frac{1 + 2c^2(d_2e^{-2T} + 4d_2e^{-2T}) + 2d_1d_2e^{-3T}}{(d_1e^{-T} + 2d_2e^{-2T})^2} \right] \]

\( q(0) \) is constant and at infinite future \( q \) is not defined. Thus, the model does not represent a steady state model.

6.5. Conclusion

A non-singular Bianchi type VI, cosmological model constructed here starts evolving at \( T = 0 \) and \( x = 0 \) with constant volume and expands throughout the evolution till infinite future. As far as matter is concerned, the model does not admit either Big bang or Big crunch during evolution from initial epoch to infinite future. It is also interesting to note that the model is Minkowskian with unity volume at both initial and infinite future for \( d_1, d_2 = 0 \)

(i) the model is a flat model with unit volume at both initial and infinite future.

(ii) the total pressure \( p_T \) and total density \( \rho_T \) vanish.

(iii) the matter density \( \rho_m \) vanishes for \( \Lambda_0 = 11 \), but \( \rho_m \neq 0 \) which leads to unphysical situation.

(iv) \( \rho_m = -\frac{2}{kc^4t^2} \) and \( \rho_m = 0 \) for \( \Lambda_0 = 9 \), which indicates that dust model does not survive.