CHAPTER 3

3 ANALYTICAL MODELING

3.1 INTRODUCTION

The main objectives of mathematical models are to understand the physics involved in the process and include all possible phenomena to estimate their effect on the process and their importance. It is very important to processes such as laser welding process, which often include all the four phase changes. This enables one to include complicated phenomena, such as conduction, convection, radiation and discontinuities at the liquid–vapour and solid–liquid interfaces.

3.2 PRINCIPLE OF HEAT CONDUCTION FOR A MOVING LASER BEAM BY CONTROL VOLUME METHOD

During laser beam welding process, heat can be transferred from one part of the materials to another part by three different ways viz conduction, convection and radiation. If heat transmitted by the actual movement of the heated particles, the processes is known as convection, which is prominent in the case of liquids and gases. Radiation, the heat transferred from material surface to the surroundings directly without the necessity of the intervening medium.

When laser beam irradiated on a metal slab that start to conduct heat energy, the atoms at the surface vibrate with higher amplitude (kinetic energy) and transmit the heat energy from one atom to another and so on without actual motion of the atoms, known as conduction.

According to the Fourier first law of heat conduction in a rectangular metal slab is
\[ Q = -KA \frac{dT}{dx} \] \hspace{1cm} (3.1)

Hence, heat transmitted per second by the metal slab between any two points at the distance \( \delta x \) given by

\[ Q = KA \frac{\partial^2 T}{\partial x^2} \delta x \] \hspace{1cm} (3.2)

The quantity of heat \( Q \) distributed in two different ways before reaching to the steady state temperature. Part of the heat used to raise the temperature of the metal slab through conduction followed by absorption and rest of the heat is lost due to radiation. The heat used per second to raise the temperature of the metal slab is

\[ = \text{Mass} \times \text{Specific heat} \times \text{Rate of raise of temperature} \]

\[ = (A\delta x)\rho Cp \frac{\partial T}{\partial t} \] \hspace{1cm} (3.3)

The heat lost per second due to radiation in from the surface of the metal slab

\[ = EP\delta x T \] \hspace{1cm} (3.4)

Where \( E \) is emissivity power of the surface, \( P \) is perimeter and \( T \) is average excess of temperature of the metal slab between any two points considered.

To obtain the temperature distribution as function of time, we consider control volume as shown in Fig. 3.1.

The law of conservation of energy, the rate of change of internal energy \((Q_g = H/K)\), which is also known as the internal heat generation must be equal to the sum of the net rate of heat flux per unit volume across its faces. In addition, any heat sources or sinks within it per volume such as chemical reactions or current passing through it (Joule effect).
Fig. 3.1 Heat flow in a control volume

Hence, the heat balance on the differential element can be stated that rate of heat conduction and the internal heat generation are equal to the heat convection and radiation.

\[
\left( KA \frac{\partial^2 T}{\partial x^2} \delta x \right) + Q_g = (A \delta x) \rho S \frac{\partial T}{\partial t} + EP \delta x T \quad \text{(or)}
\]

\[
\left( \frac{\partial^2 T}{\partial x^2} \right) + Q_g = \frac{\rho S}{K} \frac{\partial T}{\partial t} + \frac{EP}{KA} T \quad \text{(3.5)}
\]

The rectilinear flow of heat along a rectangular metal slab in three dimensions in terms of Cartesian coordinates is

\[
\left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + Q_g = \frac{\rho S}{K} \frac{\partial T}{\partial t} + \frac{EP}{KA} T \quad \text{(3.6)}
\]

In a more compact form,

\[
\nabla^2 T + Q_g = \frac{1}{\alpha} \frac{\partial T}{\partial t} + \frac{EP}{KA} T \quad \text{(3.7)}
\]

Here, \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \) is Laplacian Operator.

Thermal diffusivity or Thermometric conductivity (\( \alpha \)) is defined as ratio of thermal conductivity to thermal capacity per unit volume.

\[
\alpha = \frac{K}{\rho S} \quad \text{(3.8)}
\]

Under steady state flow of heat \( \frac{\partial T}{\partial t} = 0 \), therefore, the equation (3.7) can be reduced as follows,

\[
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \quad \text{(3.9)}
\]
Equation (3.9) is known as Laplacian transformation.

In the case of transient heat conduction, the surface of the material is assumed that it is impervious to heat flow has arbitrary of temperature in the direction of x and y. Since, there is no heat flow in z direction actually along the thickness direction to laser beam incident on materials surface i.e., temperature gradient is independent of z. From the assumptions,

\[
\frac{\partial^2 T}{\partial z^2} = 0 \quad \text{and} \quad \frac{EP}{KA} T = 0
\]

Hence, the equation (3.7) can reduced as follows,

\[
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + Q_g = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (3.10)
\]

Now, the above expression is reduced to the two dimensional equation for rectilinear flow of heat along a rectangular metal slab in terms of Cartesian coordinates. In this simplified lumped parameter energy balance equations are presented that enable quick estimates to be made of energy requirements for a given laser welding process. This differential equation with its associated boundary conditions can be solved using analytical or numerical methods such as the finite difference, finite element or control volume methods (S.Murugan 1998).

### 3.3 TEMPERATURE DISTRIBUTION DURING A MOVING LASER BEAM DURING WELDING

Generally, the solution of heat flow equations for any welding conditions is a complicated problem. In order to find analytical solutions to the equations, it is therefore necessary to make many simplifying assumptions. To make the problem more tractable analytically, the assumptions are made as follows,

1. The work piece material is assumed to be homogeneous and isotropic.
2. Heat conduction through the work piece is usually much greater than any heat
exchange with the surroundings by natural convection or radiation. Here, it is assumed that the work piece surfaces are adiabatic; that is, there is no heat loss or gain by either convection or radiation.

3. The heat source considered is a moving line that goes through the entire plate thickness uniformly.

4. A Gaussian distribution at TEM$^{00}$ spatial mode is considered and it is the most suitable specifically for welding.

5. The moving heat source is analysed using a coordinate system that is attached to the heat source.

6. In a realistic model, the thermal conductivity and specific heat are considered as functions of temperature. But, this model is simplified by considering an average value of thermo-physical coefficients of the sample materials such as thermal conductivity and specific heat is independent of temperature.

7. During welding, melting occurs and convective heat transfer occurs with the weldpool in addition to conductive heat transfer, however, it is assume that no phase changes occur; that is, the effect of latent heat of fusion is not considered in the present study.

Let us, therefore, consider a coordinate system moving with the heat source along the x-axis, as shown in Fig. 3.2. The corresponding governing equation is obtained by a coordinate transformation from the plate to the heat source, with x being replaced by $\xi$, y by $y'$, z by $z'$, and t by t', i.e,

$$\xi = x - u_x t, \ y' = y, \ z' = z, \ t' = t$$

Where, $u_x$ is the traverse velocity of the heat source in the x-direction (mm/s).
Since the heat source is uniform through the thickness, there can be no change in temperature in the thickness direction. The heat source considered a line that goes through the entire plate thickness uniformly. Thus, heat input to the weld joint as power per unit thickness. Now let $r = \sqrt{\xi^2 + y^2}$, the radius of a cylinder drawn around the heat source.

Thus, we have $\frac{\partial T}{\partial x} = 0$ for all $z$. The temperature distribution in a plate for a moving line heat source given as

$$T - T_0 = \frac{Q}{2\pi k\alpha} \exp\left(-\frac{u_x \xi}{2k}\right) K_0\left(\frac{u_x r}{2k}\right) \quad (3.11)$$

Where $K_0(\chi)$ is the modified Bessel function of the second kind of order zero and $Q$ is the heat developed within the weld plates is given in the expression (Steven T. Yang 2010) as follows

$$Q(x, y, z) = Q_0 (1 - R_c) \cdot \frac{A_c}{\pi \sigma_x \sigma_y} \cdot e^{-\frac{(x-x_0)^2 + (y-y_0)^2}{2\sigma_x^2 + 2\sigma_y^2}} \cdot e^{-\lambda z}$$

The reflection ($R_c$) and absorption ($A_c$) coefficients are assumed to be constants. The
planar surface of the slabs incident to the laser beam is assumed to be aligned with the X-Y plane of the global coordinate system. The top planar surface is aligned with \( z = 0 \). Hence the effect of absorption can be simulated by the term \( \exp(-Ac \cdot \text{abs}(z)) \). The centre of the beam can be easily shifted by changing \( x_0 \) and \( y_0 \). The beam width and astigmatism can be easily controlled by the standard deviation parameters; \( \sigma_x \) and \( \sigma_y \).

Equation (3.11) is the derivative from the famous Rosenthal heat equation for the moving heat sources, before Carslaw and Jaeger (1959) who first derived them. Temperature distribution may be represented by a family of isotherms drawn around the instantaneous heat source position in (x-y plane).

### 3.4 TEMPERATURE DISTRIBUTION DURING LASER WELDING PROCESS

#### 3.4.1 TEMPERATURE DISTRIBUTION IN RADIAL DIRECTION FROM THE CENTRELINE WELDS

A family of isotherms drawn around an instantaneous laser source position in radial directions (x-y plane) for temperature distribution in similar 316L stainless steel and mild steel welding process at a given mean power of 2100 W as shown in Fig. 3.3 and 3.4 respectively. Fig. 3.3 and 3.4 compares the effect of thermal conductivity in terms of isotherms for a 316L stainless steel and mild steel when other processing conditions are same.
Fig. 3.3 Temperature Distribution contours in 316L stainless steel during laser beam welding

Fig. 3.4 Temperature Distribution in mild steel during laser beam welding
3.4.2 TEMPERATURE DISTRIBUTION IN DEPTH DIRECTION FROM THE SURFACE AT THE POINT OF LASER IRRADIATION

The laser- matter interaction modelling is of great importance that will allow using laser as an interesting tool for industrial manufacturing especially for the welding process. If, we observe this area coaxial to the laser beam it is possible to have idea about keyhole behaviour. The temperature distribution of the AISI 316L stainless steel, mild steel and AISI 316L stainless steel-mild steel attained from the top surface to the entire depth were given as shown in Fig. 3.5, 3.6 and 3.7 i.e., in Z - axis.

![Fig. 3:5 Thermal profile of 316L stainless steel joint during laser welding with symmetrical spot position.](image-url)
Fig. 3:6 Thermal profile of mild steel joint during laser welding with symmetrical spot position.

Fig. 3:7 Temperature Distribution in 316L stainless steel-mild steel joint during laser beam welding with an offsetted position.
3.5 PEAK TEMPERATURE

The point experiences the peak temperature at a given point shortly after it passed by the heat source (S. Murugan 1998) can obtain from equation (3.11). The point is furthest from the radial distances (or line of motion of the heat source) is at its peak temperature at that instant. Using equation (3.11) and considering temperatures in terms of distance from the fusion zone boundary, the peak temperature for a plate (point source) is

\[
\frac{1}{T_{p}-T_0} = \frac{2\pi Ke}{Q\mu_x} \left\{2 + \left(\frac{\mu_x Y_{HAZ}}{2a}\right)^2\right\} + \frac{1}{T_{m}-T_0} \quad (3.12)
\]

Where \(e\) = natural exponent = 2.71828, \(T_p\) is the peak or maximum temperature at a distance \(Y\) from the fusion boundary and \(Y\) is the distance from the fusion boundary at the work piece surface. Equations (3.12) are applicable to single-pass processes and have to be applied to each pass by itself if necessary.

Fig. 3:8 Temperature Distribution across the 316L stainless steel as a function of thickness at \(Z=2\) mm
Fig. 3:9 Temperature Distribution across the mild steel-316L stainless steel as a function of thickness at Z=2 mm

Fig. 3:10 Temperature Distribution across the mild steel-316L stainless steel with an offsetted beam position.
Fig. 3.8 shows the predicted peak temperature for the similar 316 L stainless steel joints at the instant laser beam position at 2 mm in welding direction and at the point 2 mm in depth direction. Similarly, Fig. 3.9 shows the predicted peak temperature for the 316 L stainless steel-mild steel joints at the instant laser beam position at 2 mm in welding direction and at the depth direction. Fig. 3.10 shows the temperature distribution across the 316L stainless steel- mild steel joints without and with an offset-ted beam position respectively.

They are useful for estimating the heat-affected zone size and for showing the effect of preheat on the HAZ size. It is evident from the equations that all parameters being constant, preheating increases the size of the HAZ. In addition, the size of the HAZ is proportional to the net energy input. Thus, high-intensity processes such as laser welding generally have a smaller HAZ. A high intensity energy source results in a lower total heat input because the energy used in melting the metal is concentrated in a small region.

The equation (3.12) that gives the higher computed distance from the fusion zone or higher peak temperature at a given location is the more accurate of the two.

### 3.6 THERMAL CYCLES

Fig. 3.11 and 3.12 shows the thermal cycles predicted during the laser beam welding of similar 316 L stainless steel joint and mild steel joint respectively at the various points along the centreline weld.
Fig. 3:11 Thermal cycles during 316L stainless steel laser welding process

Fig. 3:12 Thermal cycles during mild steel laser welding process
3.7 COOLING RATES

The heat and the fluid flow that occur during laser processing influence the microstructures (through the grain structure and phases that are formed), residual stresses (through the thermal stresses that result from differential strains), and distortions that evolve during the process. These in turn affect the mechanical properties and thus the quality of the process (Shahram Sarkani 2000). Hence, knowledge of cooling rate is very important for materials that are polymorphic in nature, for example, steels. In the general case, the cooling rate at any position at any time can be obtained by differentiating equation (3.11) with respect to time. In the centreline-cooling rate of the 2 mm thick materials, the cooling rate is proportional to the square of the temperature rise above the initial temperature.

\[
\frac{\partial T}{\partial t} = -\frac{2\pi\alpha \mu_x}{Q}(T - T_0)^2 \quad (3.13)
\]

Fig. 3:13 Cooling rate of 316L stainless steel joint as a function of distance from Y= 0 to 2.0 mm
Fig. 3.14 Cooling rate of mild steel joint as a function of distance from Y= 0 to 2.0 mm

Fig. 3.13 and 3.14 shows the cooling curves calculated across both 316 L stainless steel and mild steel joints as a function of distance from y = 0 mm to 2 mm.

Equation (3.13) strictly give the centreline cooling rates behind a point or line source of heat moving in a straight line at constant velocity on a flat surface and are very accurate for cooling rates at temperatures that are significantly below the melting temperature. Fortunately, the temperatures at which cooling rates are of metallurgical interest, especially for steels and its alloys, are well below the melting point, and the estimates from these equations are then reasonably accurate. Furthermore, since the centreline cooling rate is higher than in the heat-affected zone, these equations also fairly well represent cooling rates in the regions of metallurgical interest.
3.8 THERMAL COOLING CYCLES

Fig. 3.15 and 3.16 shows the variation of temperature with time on both 316L stainless steel and mild steel joints at different distances from the fusion boundaries. This is referred to as the thermal cooling cycle diagram and can be obtained by substituting the relation $\tau = \xi / u_x$ into equation (3.11). This will result in equations of temperature as a function of time (S.Murugan 1998),

![Temperature vs Time Graph](image)

Fig. 3:15 Thermal cooling cycle of 316 L stainless steel from centreline Y= 0 to 5.0 mm
The obtained thermal cooling cycle diagrams states that the peak temperature decreases rapidly with increasing distance from the centreline in both 316L stainless steel and mild steel joints. However, the heat flow is considered only in the solid part of the work piece, because, the predicted results are more accurate in the solid part of the work piece.

3.9 VOLUMETRIC HEAT SOURCE MODEL

The volumetric heat source model developed by (Jao-Hwa Kuang 2012) adopted for prediction of plasma/vapour zone, melt zone and keyhole shape as shown in the Fig. 4.17. The focus spot diameter is calculated as 451 µm using the specifications of considered for Nd: YAG laser source with the wavelength of $\lambda = 1.064$ µm for TEM$_{00}$ mode for $M^2 = 1$, focal distance 200 mm and diameter of an optical fibre $D = 600$ µm.
In developing the proposed keyhole heat transfer model, the following assumptions are applied: (1) all of the laser energy in the keyhole zone is absorbed. (2) The weldpool penetration depth $h$ is approximated as the cylindrical and conical keyhole heights. (3) The pulsed laser input energy intensity exceeds the critical value required to form a keyhole and (4) The effects of complex physical phenomena such as capillary and thermocapillary forces, the vapour pressure, vapour condensation at the free surface, the shielding gas flow, and presence of a mushy zone, and so on are included implicitly in the keyhole penetration depth determination.

\[ r_v^2 = \left( \frac{4.746 \Delta E}{\pi h \rho (\Delta T_s C_s + L_m + \Delta T_f C_t + L_V)} \right) r_0^2 \tag{3.14} \]

Fig. 3:17 Schematic of the proposed volumetric heat source model

Furthermore, this cylindrical zone assumed to comprise an annular liquid zone (at a temperature higher than the melting point of the material) and a cone-like vaporization zone (at a temperature higher than the vaporization temperature of the material).

‘$r_v$’ the radius of the conical vapour region of the cylindrical heat source is given by

‘$r_l$’ the radius of the annular liquid region of the cylindrical heat source is given by
\[ r_l^2 = -\left( \frac{\frac{1}{3} \pi r_0^2 \rho h v r_0^2}{1.582(\Delta E) + (\pi^2 h\rho(\Delta T_s C_s + L_m + \Delta T_l C_l + L_v) r_0^2)} \right) \quad (3.15) \]

Figs. 3.18 and 3.19 shows the boundaries of the vapour-melt pool and melt pool and heat affected zone predicted for 316 L stainless steel and 316L stainless steel-mild steel joints. The shape and size of the weldpool obtained for above combinations reveals the temperature distributions at various thicknesses of the joint as discussed above. Therefore, swallow structure of the keyhole i.e., wider top and narrower towards bottom is justified from these observations.

![Fig. 3:18 The predicted melt and vapour region for 316 L stainless steel joint](image-url)
These of result of the predicted penetration depth may lesser than an actual penetration depth due to the various phenomenons those takes place during laser-material interaction may produce additional heat than the actual heat supplied in terms of laser power. However, this validation helps to compare keyhole profile observed between the simulated and the experimental profile.

Thus, these mathematical models predicted the temperature distribution, peak temperature, cooling rate, and thermal cycles for AISI 316L stainless steel and AISI 1018 mild steel and their dissimilar combinations. As mentioned above, these predictions more accurate only in the solid part of the workpiece, because, thermophysical properties such as density, heat capacity, melting point etc… were considered for solid state. However, vapour, and melt regions of keyhole (using volumetric heat source model) are predicted separately with appropriate assumptions in thermophysical properties for liquid and vapour states.

Fig. 3:19 The predicted melt and vapour region in 316L stainless steel-mild steel joint