CHAPTER III

THERMOELASTIC PROBLEM OF COMPOSITE BODIES
1. Introduction

In this paper we have analysed the thermal stresses and the temperature in a composite sphere on the basis of the Fourier's heat conduction equation and the equilibrium equation. Surface of the sphere is heated from zero initial temperature to a constant temperature and is supposed to be kept at that constant temperature subsequently. Laplace transform is used as a tool for the solution of the Fourier's equation. Numerical computations are done and represented in the graphs to show how the hoop stresses are affected by the physical properties and the position of the interface.

2. Nomenclatures

\[ T = \text{temperature} \]

\( \frac{K_1}{K_2} \) = the ratio of the thermal conductivity in the core to that in the shell (suffixes 1 and 2 were used to indicate the values of variables in the core and the shell regions)

\( k \) = diffusivity

\( \mu^2 = \frac{k_1}{k_2} \) = the ratio of the diffusivity in the core to that in the shell

\( r \) = radial co-ordinate

\( a \) = radius of a composite sphere

\( b \) = interface radius of a composite sphere

\( \rho = \frac{r}{a} \) = dimensionless radial co-ordinate

\( \rho_0 = \frac{b}{a} \) = dimensionless interface radius

\( t \) = time

\( \tau = \frac{kt}{a^2} \) = Fourier's number

\( \nu \) = Poisson's ratio

\( E \) = Young's modulus

\( \alpha \) = coefficient of linear thermal expansion.

3. Temperature

Let \( r, \theta, \phi \) be the spherical polar co-ordinates. When all quantities depend on time \( t \) and the radial co-ordinate \( r \).
and the linking term has been neglected, the heat conducting equation is

\[ \frac{\partial^2 T}{\partial t^2} + \frac{2}{\rho} \frac{\partial T}{\partial \rho} = \frac{1}{k} \frac{\partial T}{\partial t} \]  

(3.1)

Introducing dimensionless coordinate to (3.1), the equation becomes

\[ \frac{2}{\rho^2} \frac{\partial^2 T}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial T}{\partial \rho} = \frac{\partial T}{\partial t} \]  

(3.2)

As the above equation is linear, to obtain an elementary solution we have assumed that the initial temperature of the sphere is uniform. If \( T \) denotes the temperature measured from this equilibrium one, the initial condition is given as follows

\[ [T]_{t=0} = 0 \]  

(3.3)

Furthermore we assume the boundary condition

\[ [T]_{\rho=1} = \begin{cases} 0 & \tau < 0 \\ T_0 & \tau > 0 \end{cases} \]  

(3.4)
Let us now stipulate that the corresponding suffixes 1 and 2 used to indicate the values of functions within the core and the shell regions. To obtain the solution of equation (3.2), we introduce the Laplace transformation defined by the form

\[
\tilde{T} = \int_0^\infty T e^{-\mu} d\tau.
\]

The Laplace transformation of equation (3.2) can be represented under the consideration of equation (3.3), as follows for the core and the shell regions

\[
\frac{d^2 \tilde{T}_1}{d\rho^2} + \frac{2}{\rho} \frac{d \tilde{T}_1}{d\rho} = \mu \tilde{T}_1, \quad (0 \leq \rho < \rho_0)
\]

\[
\frac{d^2 \tilde{T}_2}{d\rho^2} + \frac{2}{\rho} \frac{d \tilde{T}_2}{d\rho} = \mu \tilde{T}_2, \quad (\rho_0 \leq \rho < 1)
\]

Boundary condition on a spherical surface is also given by the Laplace transformation of the equation (3.4) as

\[
[\tilde{T}_2]_{\rho=1} = \frac{I_0}{p}
\]
As the temperature distribution and the heat flow must be continuous across an interface between the shell and the core, the conditions on its surface are represented in the Laplace transform as

\[
\left[ \tilde{T}_1 \right]_{\rho = \rho_0} = \left[ \tilde{T}_2 \right]_{\rho = \rho_0} \quad (3.7)
\]

\[
K_1 \left[ \frac{dT_1}{d\rho} \right]_{\rho = \rho_0} = K_2 \left[ \frac{dT_2}{d\rho} \right]_{\rho = \rho_0} \quad (3.8)
\]

The general solutions of equations (3.5) are given as follows and the integral constants appearing in the solution are determined with the help of (3.7) and (3.8)

\[
\frac{\tilde{T}_1}{\tilde{T}_0} = \frac{K_2 \mu \rho_0 \sqrt{\rho \sinh \rho \sqrt{\rho}}}{P \rho \left\{ k_2 \psi'_2 (\rho_0) \sinh \rho_0 \rho - k_1 \psi'_1 (\rho_0) \sinh (\mu \rho_0 \sqrt{\rho} - \mu \sqrt{\rho}) \right\}} \quad (0 \leq \rho < \rho_0)
\]

\[
\frac{\tilde{T}_2}{\tilde{T}_0} = \frac{\cosh (\mu \rho_0 \sqrt{\rho} - \mu \sqrt{\rho}) + \sinh (\mu \sqrt{\rho} - \mu \sqrt{\rho})}{P \rho \left\{ k_2 \psi'_2 (\rho_0) \sinh \rho_0 \rho - k_1 \psi'_1 (\rho_0) \sinh (\mu \rho_0 \sqrt{\rho} - \mu \sqrt{\rho}) \right\}} \quad (\rho_0 \leq \rho < 1)
\]
4. Stress analysis

If the temperature in a body is known, the displacement and the thermal stresses in a hollow sphere initially undisturbed can be determined by the following equations.

For the core

\[
\frac{1 - \nu_1}{2 \alpha_1 E_1} \sigma_{\theta \theta} = \frac{1}{\rho^3} \int_0^\rho T_1 \rho^2 \, d\rho + \rho \left( \frac{1 - \nu_1}{1 + \nu_1} \right) \frac{C_1}{\alpha_1} \rho^3 + \frac{1}{\rho^3} \frac{(1 - \nu_1)}{1 + \nu_1} \frac{D_1}{\alpha_1} \quad (4.2a)
\]

where

\[
\psi_1(\rho) = \rho \sqrt{\rho} \cosh \rho_0 \sqrt{\rho} - \sinh \rho_0 \sqrt{\rho}
\]

\[
\psi_2(\rho) = \mu \rho \sqrt{\rho} \cosh (\mu \rho_0 \sqrt{\rho} - \mu \sqrt{\rho}) - \sinh (\mu \rho_0 \sqrt{\rho} - \mu \sqrt{\rho})
\]
For the shell

\[
\left(1 - \frac{\nu_2}{1 + \nu_2}\right) \frac{u_2}{a_1} \frac{1}{p^2} \int_{p_0}^{p} \left(1 - \frac{\nu_2}{1 + \nu_2}\right) \frac{C_2}{a_2} + \frac{1}{p^2} \frac{1 - \nu_2}{1 + \nu_2} \frac{D_2}{a_2} \right) \text{d}p
\]

\[
\frac{1 - \nu_2}{2 \alpha^2 E_2} \sigma_{n_2} = -\frac{1}{p^3} \int_{p_0}^{p} T_2 \frac{\sigma_{n_2}}{p^2} + \frac{1}{2} \frac{1 - \nu_2}{1 - 2\nu_2} \frac{C_2}{a_2} - \frac{1}{p^3} \frac{1 - \nu_2}{1 + \nu_2} \frac{D_2}{a_2} \right) \text{d}p
\]

\[
\frac{1 - \nu_2}{2 \alpha^2 E_2} \sigma_{\theta_2} = \frac{1}{p^3} \int_{p_0}^{p} T_2 \frac{\sigma_{\theta_2}}{p^2} + \frac{1 - \nu_2}{1 - 2\nu_2} \frac{C_2}{a_2} + \frac{1}{p^3} \frac{1 - \nu_2}{1 + \nu_2} \frac{D_2}{a_2} - T_2 \right) \text{d}p
\]

where \(C_1, D_1\) and \(C_2, D_2\) represent the integral constants for the core and shell region and are to be determined from the boundary conditions given below in the equations (4.3) to (4.5). The displacement and the stress components given by the equations (4.1a) to (4.2b) will be utilised after obtaining the inversion of the temperature distribution revealed by the equation (3.9).

Here each boundary conditions are written as follows

\[
(\sigma_{n_2})_{p=1} = 0 \quad \text{on the spherical surface} \quad (4.3)
\]
On the interface of a composite sphere

\[ (U_1)_{\rho=R} = (U_2)_{\rho=R} \]  
\[ (\sigma_{nn})_{\rho=R} = (\sigma_{nn})_{\rho=R} \]  

5. Inversion integral

The inversion integral of results obtained in sections 3 and 4 will be the solutions of temperature, displacement and thermal stresses. The series expansion of \( T_1 \) and \( T_2 \) are rather complicated, so we consider only the solutions obtained from the inversion theorem. Here \( T_1 \) and \( T_2 \) contain the doubled function of the complex variable \( \rho \) and the many valued function like \( \sinh \rho \sqrt{\rho} \), \( \sinh(\mu \rho_0 \sqrt{\rho} - \mu \sqrt{\rho}) \), \( \sinh \rho \sqrt{\rho} \).

The contribution to the inversion integral from large circular portion of a contour goes to zero as the radius tends to infinity (Carslaw, H.S. (1959)). Consequently, the inversion process can be reduced to integrals round the poles, the evaluation of residues. The integrands in this case have single poles only at the origin and \( \rho = -\omega_m^2 \) (\( m = 1, 2, \ldots \)) in the \( \rho \)-plane respectively, where \( \pm \omega_m \) are the roots of the transcendental equation.
The roots of this equation (5.1) are the roots of

\[ k_2 \{ 1 - \mu P_0 \cot (\mu P_0 - \mu) \alpha_m \} + k_1 \{ P_0 \alpha_m \cot P_0 \alpha_m - 1 \} = 0 \]  

(5.2)

together with the roots of

\[ \sin P_0 \alpha_m = 0 \quad , \quad \sin (\mu P_0 - \mu) \alpha_m = 0 \]  

(5.3)

Thus we get from (3.9).

\[ \frac{T_2}{T_0} = \frac{2 \mu P_0}{F} \sum_{m=1}^{\infty} \frac{1}{\phi(\alpha_m)} \sin P_0 \alpha_m \sin (\mu P_0 - \mu) \alpha_m \sin P_0 \alpha_m e^{-\alpha_m T} \]  

(5.4)

\[ \frac{T_2}{T_0} = \frac{2 \mu P_0}{F} \sum_{m=1}^{\infty} \frac{1}{\phi(\alpha_m)} \sin (2\mu P_0 - \mu - \mu P) \alpha_m \sin P_0 \alpha_m e^{-\alpha_m T} \]  

(5.5)

where
\[ \Phi(\alpha_m) = \mu P_0 \alpha_m (\mu P_0 - \mu) \sin^2 P_0 \alpha_m - K P_0^2 \alpha_m \sin^2 (\mu P_0 - \mu) \alpha_m \]
\[ + [(k-1)/(\alpha_m^2)] \sin^2 (\mu P_0 - \mu) \alpha_m \sin^2 P_0 \alpha_m \quad (5.6) \]

and
\[ K = k_1/k_2 \]

If \( \frac{\mu P_0 - \mu}{P_0} \) is rational, let us suppose
\[ \frac{\mu P_0 - \mu}{P_0} = \frac{n}{5} \] (in its lowest term)

Then the equation \((5.3)\) have the common positive roots
\[ \alpha_m = \frac{5n}{P_0} \quad n = 1, 2, 3, \ldots \quad (5.7) \]

These roots of \((5.1)\) give rise to additional terms
\[ \frac{2 T_0}{\rho \pi (n \sigma - s)} \sum_{n=1}^{\infty} (-1)^{n(n+1)} \sin \frac{s \pi n}{P_0} \sin^2 \frac{s \pi n}{P_0} \quad (5.8) \]

where
\[ \sigma = \frac{k_1}{k_2 \mu} \]

and
\[ \frac{2 T_0 \sigma}{\rho \pi (n \sigma - s)} \sum_{n=1}^{\infty} \sin \frac{2 \pi n (\mu P_0 - \mu P_\theta)}{n} \sin \frac{s \pi n}{P_0} \quad (5.9) \]

in the expressions for \( \frac{T_i}{T_0} \) and \( \frac{T_2}{T_0} \) respectively.
Now we substitute the conditions (4.3), (4.4), (4.5) upon (4.1a and b) and (4.2a and b) and thus we obtain

\[
\frac{C_1}{\alpha_2} = \tau_0 \left[ f'(q(x) + 2 \bar{F}(x) \bar{F}) \left( \frac{1-\gamma_1}{1+\gamma_1} \right)^2 + 2 \bar{F} \left( q(x) + q(x) \right) \right] \\
\left[ 2 \rho_0 \left( \frac{1-\gamma_2}{1+\gamma_2} \right) + \left( \frac{1-\gamma_2}{1+\gamma_2} \right) \right] / \left[ \rho_0 \left( \frac{1-\gamma_2}{1+\gamma_2} \right) + \left( \frac{1-\gamma_2}{1+\gamma_2} \right) \right]
\]

\[
\left( \frac{1-\gamma_1}{1+\gamma_1} \right) - 2 \rho_0 \bar{F} \left( \frac{1-\gamma_1}{1+\gamma_1} \right) \left( \frac{1-\gamma_2}{1+\gamma_2} \right)
\]

and for the core \( U_1 = 0 \) at \( \rho = 0 \) and therefore, \( D_1 = 0 \)

\[
\frac{C_2}{\alpha_2} = \tau_0 \left[ 2 \bar{F}(x) \rho_0 \left( \frac{1-\gamma_1}{1+\gamma_1} \right) + \left( \frac{1-\gamma_1}{1+\gamma_1} \right) + \bar{q}(x) \rho_0 \left( \frac{1-\gamma_1}{1+\gamma_1} \right) \right] / \left[ \rho_0 \left( \frac{1-\gamma_2}{1+\gamma_2} \right) + \left( \frac{1-\gamma_2}{1+\gamma_2} \right) \right]
\]

\[
\left( \frac{1-\gamma_1}{1+\gamma_1} \right) - 2 \tau_0 \bar{F}(x) \left( \frac{1-\gamma_2}{1+\gamma_2} \right) \left( \frac{1-\gamma_1}{1+\gamma_1} \right) - 2 \rho_0 \bar{F} \left( \frac{1-\gamma_1}{1+\gamma_1} \right) \left( \frac{1-\gamma_2}{1+\gamma_2} \right)
\]

\[
\frac{D_2}{\alpha_2} = \left( \tau_0 \left( \frac{1-\gamma_1}{1+\gamma_1} \right) \right) \left[ \bar{F}_0 \left( \frac{1-\gamma_1}{1+\gamma_1} \right) + \left( \frac{1-\gamma_1}{1+\gamma_1} \right) + \bar{q}(x) \rho_0 \left( \frac{1-\gamma_1}{1+\gamma_1} \right) \right] / \left[ \rho_0 \left( \frac{1-\gamma_1}{1+\gamma_1} \right) + \left( \frac{1-\gamma_1}{1+\gamma_1} \right) \right]
\]

\[
\left( \frac{1-\gamma_1}{1+\gamma_1} \right) - 2 \tau_0 \bar{F}(x) \left( \frac{1-\gamma_1}{1+\gamma_1} \right) \left( \frac{1-\gamma_1}{1+\gamma_1} \right) - 2 \rho_0 \bar{F} \left( \frac{1-\gamma_1}{1+\gamma_1} \right) \left( \frac{1-\gamma_1}{1+\gamma_1} \right)
\]

where
\[ F(\alpha) = 2\mu P_0 \sum_{m=1}^{\infty} \frac{1}{\Phi(\alpha_m)} \sin^2 p_0 \alpha_m e^{-\alpha_m^2 t} \cdot (1/\mu \alpha_m)[\cos^2 (\mu P_0 - \mu) \alpha_m - (1/\mu \alpha_m) \sin (\mu P_0 - \mu) \alpha_m] \]

\[ + \frac{2Q}{n(\pi \sigma - s)} \sum_{n=1}^{\infty} \frac{n}{n} \cdot (p_0/\mu \sin n) \cdot \cos^2 (\mu P_0 - \mu)(\sin n/p_0) \]

\[ \frac{2h}{n(\pi \sigma - s)} \sum_{n=1}^{\infty} (\frac{\sin n}{p_0})^2 e^{-\alpha_m^2 t} \cdot \sin (\mu P_0 - \mu)(\sin n/p_0) \]

\[ q(\alpha) = 2\mu P_0 \sum_{m=1}^{\infty} \frac{1}{\Phi(\alpha_m)} \sin (\mu P_0 - \mu) \alpha_m \sin p_0 \alpha_m e^{-\alpha_m^2 t} \cdot \frac{(\sin^2 p_0 \alpha_m - p_0 \cos p_0 \alpha_m)}{\alpha_m^2} \]

\[ + \frac{2}{n(\pi \sigma - s)} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cdot (\frac{\sin n}{p_0})^2 e^{-\alpha_m^2 t} \cdot \left[ (\frac{p_0}{\sin n})^2 \sin n \cdot \frac{\frac{p_0^2}{\sin n} \cos n}{\alpha_m^2} \right] \]

\[ \xi = \frac{(1-\nu_1)(1+\nu_2)\alpha_1}{(1+\nu_1)(1-\nu_2)\alpha_1} \]

\[ \gamma = \frac{(1-\nu_1)\alpha_2 E_2}{(1-\nu_2)\alpha_1 E_1} \]

6. Expressions for temperature, displacement and stresses

For the core

\[ \frac{T_1}{T_0} = 2\mu P_0 \sum_{m=1}^{\infty} \frac{1}{\Phi(\alpha_m)} \sin p_0 \alpha_m \sin (\mu P_0 - \mu) \alpha_m \sin p_0 \alpha_m e^{-\alpha_m^2 t} \]
\[
\left(1 - \frac{\nu_1}{1 + \nu_1}\right) \frac{U_{11}}{a \alpha_1 T_0} = \frac{1}{\rho_3} q(\alpha) + \frac{1}{2} \frac{(1 - \nu_1)}{a \alpha_1} C_1 \tag{6.2}
\]

\[
\frac{1 - \nu_1}{2 \alpha_1 E_1 T_0} \sigma_{\theta 1} = - \frac{1}{\rho_3} q(\alpha) + \frac{1}{2} \frac{(1 - \nu_1)}{a \alpha_1} C_1 \tag{6.3}
\]

\[
\frac{1 - \nu_1}{2 \alpha_1 E_1 T_0} \sigma_{\theta 1} = \frac{1}{\rho_3} q(\alpha) + \frac{(1 - \nu_1)}{2(1 - 2 \nu_1)} C_1 \tag{6.4}
\]

For the shell

\[
\frac{T_2}{T_0} = \frac{2 \mu P_0}{P} \sum_{m=1}^{\infty} \frac{1}{\phi(\alpha_m)} \sin(2 \mu P_0 - \mu - \mu \rho) \alpha_m \sin P_0 \alpha_m e^{-\alpha_m^2 T} \tag{6.5}
\]
$$(1 - \nu_2) \frac{u_2}{(1 + \nu_2) \alpha_2 T_o} = \frac{1}{\rho^3} G(\alpha_m) + \rho \frac{1 - \nu_2}{(1 + \nu_2) \alpha_2} \frac{C_2}{\alpha_2} + \frac{1}{\rho^2} \frac{1 - \nu_2}{(1 + \nu_2) \alpha_2} \frac{D_2}{\alpha_2}$$  \qquad (6.6)$$

$$\frac{1 - \nu_2}{2 \alpha_2 E_2 T_o} \sigma_{n2} = - \frac{1}{\rho^3} G(\alpha_m) + \frac{\rho}{2} \frac{(1 - \nu_2)}{(1 - 2 \nu_2) \alpha_2} C_2 + \frac{1}{\rho^3} \frac{(1 - \nu_2)}{(1 + \nu_2) \alpha_2} D_2$$  \qquad (6.7)$$

$$\frac{1 - \nu_2}{2 \alpha_2 E_2 T_o} \sigma_{n2} = - \frac{1}{\rho^3} G(\alpha_m) \frac{(1 - \nu_2)}{(1 - 2 \nu_2) \alpha_2} C_2 + \frac{1}{\rho^3} \frac{(1 - \nu_2)}{(1 + \nu_2) \alpha_2} D_2$$

$$- \frac{2 \mu \rho_0}{\rho} \sum_{m=1}^{\infty} \frac{1}{\Phi(\alpha_m)} \sin(2 \mu \rho_0 - \mu - \mu \rho) \alpha_m \sin^2 \alpha_m \zeta$$

$$- \frac{2 \sigma_{n2}}{\rho \pi (\alpha_2)} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(2 \mu \rho_0 - \mu - \mu \rho) (\sin \rho_0) \zeta$$  \qquad (6.8)$$

where \( \frac{C_1}{\alpha_1}, \frac{C_2}{\alpha_2}, \frac{D_2}{\alpha_2} \) are given by (5.10), (5.11) and (5.12) respectively and

$$G(\alpha_m) = \int_{T_2} T_2 \rho \, d \rho = 2 \mu \rho_0 \sum_{m=1}^{\infty} \frac{1}{\Phi(\alpha_m)} \sin^2 \rho_0 \alpha_m \zeta \left\{ (1/\mu \alpha_m) \rho \cos(2 \mu \rho_0 - \mu - \mu \rho) \alpha_m 
- (\rho_0/\mu \alpha_m) \cos(\mu \rho_0 - \mu) \alpha_m + (1/\mu \alpha_m^2) \sin(2 \mu \rho_0 - \mu - \mu \rho) \alpha_m 
- (1/\mu \alpha_m^2) \sin(\mu \rho_0 - \mu) \alpha_m \right\}.$$
7. Numerical results

On the basis of the analytical results we have calculated the hoop stress $\sigma_{h2}$ at the interface of the composite sphere. The study of which is important from the viewpoint of fracture theory. To calculate the thermal stress using the equation (6.8), first we must get the roots of the transcendental equation (5.2). To facilitate this calculation we have taken the parameter $\mu = p_0/(1-p_0)$ and the materials for the shell regions having almost same conductivities. Thus in the three case of $\mu = 0.66, 1.0, 1.27$ corresponding to $p_0 = 0.4, 0.5, 0.56$ we determine the roots of the transcendental equation (5.2). We have also considered $\gamma_1 = \gamma_2 = \gamma, \alpha_1 = \alpha_2 = \alpha$ and $E_1/E_2 = 0.5$. 
Thereafter we have examined the effects of the parameters $\mu$ and $f_0$ on the hoop stress $\sigma_{\theta z}$ when $f=1$ for the values of $\gamma = 0.25, 0.3, 0.35$ respectively, and calculated the variations of hoop stress with time, as shown in the figures 1 to 3. From the graphs it is found that the dimensionless hoop stress $\frac{1}{2\gamma_2\epsilon_2} \sigma_{\theta z}$ takes the maximum value at the Fourier number $\tau=0$ and their values decrease and tending towards zero as the Fourier number increases. Also the dimensionless hoop stress at a particular time takes the bigger value with larger values of $\mu$. Due to this reason, the larger the parameter $\mu$ becomes, more predominant its effect may be.
FIG. 1 HOOP STRESS FOR $\rho = 0.4, \mu = 0.66$
FIG. 2 HOOP STRESS FOR $P_e=0.5, \mu=1.0$
FIG. 3 HOOP STRESS FOR $\rho_o = 0.56, \mu = 1.27$