MODEL ORDER REDUCTION OF LINEAR DYNAMIC SYSTEM AND CONTROLLER DESIGN

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By
Jasvir Singh Rana

Under the Supervision of

Prof. (Dr.) R.P. Agarwal
Prof. & Academic Adviser,
Shobhit University
Former Prof. & Dean, IIT Roorkee

Prof. (Dr.) Rajendra Prasad
Prof., Dept. of Electrical Engg.
IIT Roorkee,
Roorkee

Department of Electronics and Communication Engineering
Faculty of Engineering and Technology
Shobhit University, NH-58, Modipuram, Meerut – 250110
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CANDIDATE’S DECLARATION

I hereby declare that the work which is being presented in the thesis entitled “Model Order Reduction Of Linear Dynamic System And Controller Design” in partial fulfillment of the requirements for the award of the Degree of Doctor of Philosophy and submitted in the Department of Electronics and Communication Engineering of the Shobhit University, Meerut, is an authentic record of my own work carried out during period from July 2011 to September 2016 under the supervision of Prof.(Dr.) Rajendra Prasad and Prof.(Dr.) R.P. Agarwal

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other Institute.

(Jasvir Singh Rana)

This is to certify that the above statement made by the candidate is correct to the best of our knowledge.

Prof. (Dr.) R.P. Agarwal
Prof. & Academic Adviser, S.U.
Former Prof. & Dean, IIT Roorkee

Prof.(Dr.) Rajendra Prasad
Prof., Dept. of Electrical Engg.
IIT Roorkee, Roorkee
ABSTRACT

Model order reduction techniques to reduce the order of the system are one of the very challenging area of control system. The mathematical models of high order dynamic systems can be described either in state space form or in transfer function form which are also called time domain and frequency domain representations. In the state space representation a physical system is represented by set of first order differential equations. Similarly in the transfer function representation a physical system is represented as a rational function.

Many physical systems are translated into mathematical model through higher order differential equations. It is usually recommended to reduce the order of the model retaining the dominant behavior of the original system. This will help to make better understanding of the physical system, reduce computational complexity, reduce hardware complexity and simplifies the controller design. The proposed research works deals with the methods of approximating the transfer function of high order system by one of lower order system.

Mixed methods of model order reduction are proposed in frequency domain to reduce the higher order system into various lower order models. In mixed methods, the denominator of the reduced order model is obtained by one of the stability based reduction methods such as Mihailov Criterion, Routh stability Criterion, Modified pole clustering etc. and then the numerator of the reduced model is obtained by Pade approximation, Factor division method etc. The method preserves steady state value and stability of original system in the reduced order models for stable systems. The response of reduced order model obtained is compared on the basis of unit step response and frequency response. Also some of these methods are extended
for order reduction of multi input and multi output system and discrete time system.

The controller has been designed on the basis of approximate model matching, based on direct and indirect approaches, using the conventional method. The desired performance specifications of the plant have been translated into a reference model transfer function. In direct approach reduced order model is obtained for the original higher order system and a controller is designed for low order model. In indirect approach of controller design, a controller is designed for the high order system and the closed loop response of original high order system and high order controller with unity feedback is reduced and compared with reference model. The performance comparison of various models has been carried out using MATLAB 7.11.0.584 Pentium IV processor.
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In present day technology, there are a large number of problems which are highly complex and large in dimension. Physical systems such as aircraft, chemical plants, electrical power system networks, urban traffic networks, digital communication networks, economic systems and control system can be described mathematically by state space models or by transfer function models. In the most practical situations of control system, high order model for the system is obtained from theoretical considerations. The high order model posses so many problems in the analysis and design. For example it takes more computational time in the analysis and design, and large complicated hardware requirement etc. It is therefore, desirable to obtain a simplified/reduced model which retains the important properties of high order system.

1.1 Need for Model Order Reduction

The approximation of linear system plays an important role in many engineering problems, especially in control system design, where an engineer is faced with controlling a physical system for which an analytical model is represented by a high order linear system.

The main objective for obtaining the reduced order model is to have a better understanding of the system, to reduce computational and hardware complexity, to make feasible designs and to obtain simple control laws. Thus among several reasons for reducing the order of a system, a few are as follows:

(a) To simplify the understanding of the system.

(b) To reduce the computational complexity for analyzing the system.
(c) To economize in terms of hardware while realizing or designing the system.
(d) To reduce the computational time while applying the models.

**Motivations for Simplifications**

The motivation for deriving simplified models may be summarized as follows:

1. It simplifies the description and the analysis of the system.
2. It simplifies the computer simulation of the system.
3. It facilitates the controller design problems and yields controllers with simpler structures.
4. It reduces the computational effort in the analysis and design of control systems.

### 1.2 Problem Definition

The model of linear time invariant physical system is obtained in the form of high order differential equations by physical laws. These differential equations may be expressed in the form of a set of first order differential equations which is called a state space model and is given by \((A, B, C)\). If the Laplace transform of high order is taken with zero initial conditions then original system may be expressed in the form of high order transfer function \(G(s)\) which is the ratio of Laplace transform of output variable and input variable. The objective of proposed research work is to obtain reduced order model either in state space form \((A_r, B_r, C_r)\) or in the transfer function form \(R(s)\). The reduced order model will be utilized for the design of controller.

**Frequency Domain**

Let the transfer function of high order original system of the order ‘\(n\)’ be

\[
G_n(s) = \frac{N(s)}{D(s)} = \frac{a_0 + a_1 s + a_2 s^2 + \cdots + a_{n-1} s^{n-1}}{b_0 + b_1 s + b_2 s^2 + \cdots + b_n s^n}
\]

(1.1)

Where; \(a_i, b_i\) \(0 \leq i \leq n-1\) and \(b_i\) \(0 \leq i \leq n\) are known scalar constants.
\[ R_k(s) = \frac{N(s)}{D(s)} = \frac{c_0 + c_1 s + c_2 s^2 + \cdots + c_{k-1} s^{k-1}}{d_0 + d_1 s + d_2 s^2 + \cdots + d_k s^k} \]  

(1.2)

Where; \( c_i \) \( 0 \leq i \leq k-1 \) and \( d_i \) \( 0 \leq i \leq k \) are known scalar constants.

The objective is to realize the \( k^{\text{th}} \) order reduced model in the form of (1.2) from the original system (1.1) such that it retains the important features of the original high order system.

**Time Domain**

In time domain, the system can be described by the following state space equations

Original System

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

Reduced order system (\( k < n \))

\[
\begin{align*}
\dot{x}_k &= A_k x_k(t) + B_k u(t) \\
y_k &= C_k x_k(t) + D_k u(t)
\end{align*}
\]

Where,

- \( x(t) = n \times 1 \) state vector
- \( x_k(t) = k \times 1 \) state matrix
- \( u(t) = p \times 1 \) input vector
- \( A_k = k \times k \) system matrix
- \( A = n \times n \) system matrix
- \( B = n \times p \) input matrix
- \( B_k = k \times p \) input matrix
- \( C = m \times n \) output matrix
- \( C_k = m \times k \) output matrix
- \( C = m \times n \) output matrix
- \( D = m \times p \) transmission matrix
- \( D_k = m \times p \) transmission matrix
- \( y_k = m \times 1 \) vector matrix
- \( y(t) = m \times 1 \) Output vector

**1.3 Objective of The Thesis**

In the course of this period of study and research work, the main aim of the research work is to gradually understand and evaluate the importance as well as the advantages of both the conventional and modern model order reduction methods and thus evolve the new techniques/methods to improve upon the recent as well as established methods of design. The main focus of the work is to design and develop the new methods in frequency domain both
for continuous single input single output systems and multi input & multi output systems.

Finally, the reduction methods have been developed used to design a PID controller for control systems of high order plants.

1.4 ORGANIZATION OF THE THESIS

The proposed research work has been segmented into seven sections/chapters of the thesis as follows:

Chapter one deals with the mathematical modeling and simplification, motivations for simplification, statement of model reduction problem, application of reduced order models and its use in control engineering are discussed; the objective and organization of the thesis is also given.

In chapter two, various methods for model order reduction over the years by various scientists and researchers are reviewed briefly.

In most of the research work related to order reduction methods there is a common flaw that the reduced order models (ROMs) turn out to be unstable (stable) if the original high order system happens to be stable(unstable). In order to overcome these shortcomings, few mixed methods of order reduction are proposed which are described in chapter three for linear dynamics systems for single input and single output system. In these methods the denominator of the reduced order model has been derived by one of the stability based methods and the numerator of the reduced model is obtaining by other methods.

In chapter four methods developed as described in previous chapter has been applied for multi-input and multi output systems.
In fifth chapter the model reduction techniques using mixed methods are discussed and applied for order reduction of discrete time system.

In sixth chapter the model reduction method mixed methods as discussed in chapter 3 are used to design a PID controller and performance of the system has been discussed.

Seventh chapter concludes the work in the thesis and briefly discussed emerging trends.
CHAPTER 2
MODEL ORDER REDUCTION METHODS: AN OVERVIEW

2.1 INTRODUCTION

Reduced order modeling has wide applications in various fields of engineering and therefore order reduction methods have been discussed in details in literature [8, 9, 25, 29] during last few decades and in text books [49, 50, 58, 78]. The model order reduction techniques are broadly classified in frequency domain and time domain reduction methods.

2.2 FREQUENCY DOMAIN ORDER REDUCTION METHODS

The frequency domain order reduction techniques can be subdivided into three groups

(i) Classical Reduction Methods (CRM)
(ii) Stability Preservation Methods (SPM)
(iii) Stability Criterion Methods (SCM)

The methods of first group are based on classical theories of mathematical approximation such as continued fraction expansion and truncation, pade approximation, time moment matching etc. These methods are algebraic in nature and some cases may reduce unstable system to stable system and vice versa. The problems such as instability, non minimum phase behavior and accuracy in the mid and high frequency range of reduced order models limit the application of Classical Reduction Method (CRM).

Stability Preservation Methods (SPM) are stable reduction methods. These methods suffer from a serious drawback of flexibility when the reduced order model does not produce a good enough approximation. This group includes Routh approximations, Hurwitz polynomial approximation, Routh –
Hurwitz array and stability equation method [53, 76, 92, 99, 104]. The other SPM are dominant pole retention, reduction based on differentiation, the method using Mihailov Criterion also preserve stability and can be included in SPM [39, 47, 63, and 84].

Stability Criterion Methods (SCM) includes mixed methods. In these methods, the denominator of reduced order model is derived by one of stability preservation methods (SPM), while numerator coefficients of the reduced order model are determined by using one of the classical reduction methods (CRM), to improve the degree of accuracy at low frequency range [30, 41, 42, 91, 96, 105].

Some important frequency domain order reduction methods are briefly reviewed below:

**Continued Fraction Expansion (CFE) Method**

Chen and Shieh [17] proposed this method of obtaining reduced order model of linear time invariant SISO system. A detailed account of continued fraction expansion is available in Wall [35]. This approach does not require any knowledge of Eigen values or Eigen vectors and contains most of the essential characteristics of original system in first few terms. The basic principle gives rise to the derivation of simplified models by continued fraction expansion (CFE) is based on expanding the original higher order system using continued fraction expansions. As the quotients in continued fraction expansion descend lower and lower in position, they become less and less important as far as their influence on the performance of the system is concerned. This observation is general basis of the simplification techniques using continued fraction expansions. After truncating, the continued fraction expansion after some terms, and inversion the truncated CFE, results in a reduced order model. The various modifications and extensions have been
carried out by many authors. Davidson and Lucas [2] give CFE about a general point; Chen [16] extended CFE to MIMO systems.

**Moment Matching Method**

This model order reduction approach of moment matching was first introduced by Paynter [34] and Zarkian [102] applied this method. This method is based on determining a set of time functions of full order model and matching them with those for the reduced order model i.e. matching a few lower order moments of original high order system with that of the reduced order model. Matching of initial time - moments leads to better approximation at low frequency and matching of initial Markov parameters leads to better approximation at high frequencies. This method is the transient performance of the reduced order model may not always be satisfactory and also there is no guarantee of stability. This method has calculation difficulty for large no of large time constant.

**Pade Approximation Method**

Pade introduced this technique and Shamash [106] applied this method. This method is computationally simple, fits initial time moments and steady state value of original and reduced order model matches. For \( r \)th order model, ‘2\( r \)-1’ coefficients of power series expansion (about \( s=0 \)) of reduced order model matched with the corresponding coefficients of the original system. The disadvantage this method is that reduced order model may be unstable even though the original system is stable. Also it may sometimes approximate non –dominant poles of the system, thus giving bad approximation. To overcome this disadvantage, various alternatives methods have been suggested. Shamash [106] introduced a method of reduction based on retention of poles of high order system in reduced order model and concept of Pade approximation about more than one point.

**Routh Approximation Method**
This method of model order reduction was proposed by Hutton and Friendland [53]. The transfer function of high order system is initially reciprocated and then expanded in the $\beta$-$\beta$ canonical form for the denominator and numerator polynomials respectively. The $\beta$ table is prepared from denominator coefficients using well known Routh algorithm where $\beta$ table is prepared by similar algorithm using numerator coefficients in which $\beta$ coefficients are determined by using the $\beta$-table and successive elements of $\beta$-table. This method requires neither optimization nor Eigen values evaluation, but ensures system stability and the steady state values of reduced order models match with that of original system. It involves simple algebraic calculations of finite number of steps.

**Routh–Hurwitz Array Method**

This technique consists of obtaining the numerator and denominator polynomials of the reduced order model respectively from the numerator and denominator polynomials of the original system by forming the Routh Hurwitz stability arrays for numerator and denominator polynomials [99]. Using second and third rows of Routh stability array nth degree denominator polynomial, a polynomial of (n-1)th order can be constructed. Similarly (n-2)th order polynomial can be constructed from third and fourth row of the array and so on. The same procedure is repeated for reducing the numerator polynomial.

**Stability Equation Method**

This method was proposed by Chen et al [91]. In this method, numerator and denominator polynomials of high order system are separated into their even and odd parts. These are then factored and roots, which are closer to origin, are retained. In this method, polynomial is reduced by successively discarding the less significant factors. The transfer function of the reduced order model is constructed using these retained roots. This method
preserves stability in the reduced order model for stable original system and retains the first two time moments of the system, thus ensuring steady state response matching for impulse, step and ramp inputs.

**Polynomial Differentiation Method**

This method introduces by Gutman et al [63]. The reciprocal of numerator and denominator polynomials of high order transfer function are differentiated many times to yield the coefficients of reduced order transfer function. These reduced polynomials are reciprocated back and normalized. This method is computationally simple and is applicable to unstable and non-minimum phase systems. The only drawback of the method is that steady state does not match always.

**Truncation Method**

It was first introduced by Gustafson where successively lower order models are obtained by neglecting progressively higher order terms from numerator and denominator of high order system. This method is computationally simple. Shamash [108] extended this method for multivariable systems and compares the technique of Routh approximation and concluded this method is equally good as other methods.

**Dominant Pole Retention Method**

This approach was proposed by Davison [39]. In this method the reduced order model is always stable for a stable original system and dominant performance of the original system is also retained. The poles near to imaginary axis, known as dominant poles are retained in reduced order model and poles, far away from imaginary axis are neglected, as their effect on overall performance of the system is comparatively less. The disadvantage of this method is that if all the poles are very near to imaginary axis then it becomes difficult to distinguish which one is more dominant.

**Factor Division Method**
Lucas [98] introduced this technique for model order reduction. It allows retention of dominant poles and initial time moments in the reduced order model. It avoids calculation of system time moments and the solution of Pade equation [104, 107] by dividing out the unwanted poles factors. This method was extended by Lucas [97] to generate biased order models by retaining initial Markov parameters as well as time moments. The ideas of Lucas [97, 98] were extended and modified factor division approach [96] was developed.

**Mihailov Stability Criterion**

Mihailov stability criterion of model order reduction was described by D. Kranthi et al. [47]. D. Kranthi combined this criterion with the Factor division approach. This method is computationally simple and efficient. It avoids calculating the initial time moments Markov parameters of the original system. This method also ensures stability if original system is stable.

**Least Square Method**

Shoji et al. [27] introduced the method of model order reduction based on least square matching of time moments of the original system. An attractive feature of this method is that it provides an extra degree of freedom in the design of reduced order model. Lucas and Beat [93] modified this method. Smith and Lucas [37] estimated the denominator coefficients by least square method and numerator coefficients by exact moment matching.

**2.3 Time Domain Order Reduction Methods**

The time domain order reduction methods require either the major knowledge of overall characteristics or the Eigen values and Eigen vectors of higher order systems.

Some important time domain order reduction methods are also briefly reviewed below:
Aggregation Method

In this method which retains the important Eigen values of the original system in the reduced model, the most general is the aggregation proposed by Aoki [52]. Aoki has shown usefulness of aggregation matrices for designing suboptimal controllers.

Singular Perturbation Method

The singular perturbation method introduced by Sannuti and Kokotovic [64]. It is particularly useful for simplifying a system having the time scale property. Then states are portioned into the slow and fast parts, and reduction is obtained by setting the derivatives of fast states to zero, so that the fast states can be eliminated. An important advantage of the method is that the physical nature of the problem is persevered. The main difficulty with this method is determining an appropriate partitioning of the state vector to obtain a suitable low order model.

Optimal Order Reduction Methods

This group of methods is based on obtaining a model of specified order such that its impulse or step response (or alternately its frequency response) matches that of the original system in optimum manners with no restriction on the location of the Eigen values. Such techniques aim minimizing a selected performance criterion which in general is a function of error between the response of the original high order system and its reduced order model. The parameter of reduced order model are then obtained either from the necessary conditions of optimality or by means of numerical algorithm. Anderson [45] proposed a geometric approach, based on orthogonal projection, to obtain a low order minimizing the integral square error in time domain. Sinha and Pille [60] proposed utilizing the matrix pseudo inverse for a least squares fit with samples of the response. Other criteria for optimization have been studied (Sinha and Bereznai [58], and suggested using the pattern search method of Hooke and Jeeves [65], where as Bandler,
Markettos and Sinha [44] have proposed using gradient methods, which require less computation time but the gradient of objective function has to be evaluated. The development of optimal order reduction is attributed to Wilson and Mishra [22]. Where the approximation have been studied for step and impulse responses.

2.4 CONCLUSION

The various model order reduction methods proposed and applied in the design and development by various researchers have been described. These methods are segmented into frequency domain and time domain reduction methods. Depending on the ways of methods is further classified as Classical reduction methods, stability preservation methods and stability criterion methods in frequency domain. The methods which are used and exploited further in the coming chapters have been described.
3.1 INTRODUCTION

Various model order reduction techniques/methods available in frequency and time domain have been described by researchers have advantages and disadvantages associated with them. A common disadvantage found with methods is that the reduced order models turn out to be unstable even if the original high order system is stable. The other drawbacks related with them are that they have low accuracy in mid and high frequency ranges and may exhibit non minimum phase characteristics. Experience in model order reduction shows that Pade approximation, continued fraction expansion method and moment matching method may produce unstable models for stable systems. A few methods which produce stable models are Routh approximation [53], Routh Hurwitz array [42], stability equation method [41, 91, 92], polynomial differentiation [15, 94], dominant pole retention [78], Modified Cauer Form [66], Mihailov stability criterion [72], Hurwitz Polynomial Approximation [76] and Factor Division Method [72, 97, 98].

There is another series of investigations reported in the literature[5, 13,15,107] referred to as combined or mixed methods, which use one of the above stable reduction methods to reduce the order of the denominator polynomial and use another method to obtain the terms of numerator polynomial. The mixed or combined methods proposed to be in the thesis overcome some of these disadvantages. It has been shown that the use of the combined methods is superior in comparison to the use of simple method [12].
3.2 Problem Statement

Let the transfer function of high order original system of the order ‘n’ be

\[ G_n(s) = \frac{N(s)}{D(s)} = \frac{a_0 + a_1 s + a_2 s^2 + \cdots + a_{n-1} s^{n-1}}{b_0 + b_1 s + b_2 s^2 + \cdots + b_n s^n} \]  

(3.1)

Where; \( a_i \), \( b_i \) \( 0 \leq i \leq n-1 \) and \( b_i \) \( 0 \leq i \leq n \) are known scalar constants.

\[ R_k(s) = \frac{N(s)}{D(s)} = \frac{c_0 + c_1 s + c_2 s^2 + \cdots + c_{k-1} s^{k-1}}{d_0 + d_1 s + d_2 s^2 + \cdots + d_k s^k} \]  

(3.2)

Where; \( c_i \) \( 0 \leq i \leq k-1 \) and \( d_i \) \( 0 \leq i \leq k \) are known scalar constants.

The objective is to realize the \( k^{th} \) order reduced model in the form of (3.2) from the original system (3.1) such that it retains the important features of the original high order system.

3.3 Proposed Reduction Methods

In the proposed research work of model reduction methods which combine the advantages of Mihailov Stability Criterion, Pade approximation, Modified Cauer form, factor division, modified pole clustering and basic characteristics are proposed. In the first method the Mihailov stability Criterion and Pade Approximations are combined, in second method order reduction using modified Cauer form and factor division method is carried out. In third method order reduction using modified pole clustering and factor division method is proposed, while in fourth method order reduction using basic characteristics and factor division method is proposed.

Method-1: Combination of Mihailov Stability Criterion and Pade Approximations Method

In the proposed method the denominator coefficient of the reduced order model is obtained by using advantages of the Mihailov Criterion [47] while the coefficients of the numerator are obtained by Pade approximations [12].
Mihailov Criterion Approximation

Determination of the denominator polynomial $D_k(s)$ for the $k^{th}$ order reduced model by using Mihailov criterion [47]. The brief procedure for getting $D_k(s)$ using Mihailov criterion is as follows:

Substituting $s = jw$ in $D(s)$ and separating into real and imaginary parts as

$$
D(jw) = b_0 + b_1(jw) + b_2(jw)^2 + \ldots \ldots + b_n(jw)^n
= (b_0 - b_2w^2 + b_4w^4 - \ldots) + j(b_1w - b_3w^3 + b_5w^5 - \ldots)
= \Phi(w) + j\psi(w) \tag{3.3}
$$

Where $w$ is the angular frequency in rad/sec.

Setting $\phi(w) = 0$ and $\psi(w) = 0$, the intersecting frequencies $w_1 = 0, \pm w_2, \pm w_3, \ldots, \pm w_n$ are obtained where $|w_1| < |w_2| < |w_3| < \ldots < |w_n|$

Similarly, substitute $s = jw$ in $D_k(s)$, which results

$$
D_k(jw) = \xi(w) + j\eta(w) \tag{3.4}
$$

Where $\xi(w) = d_0 - d_2w^2 + d_4w^4 + \ldots$

$$
\eta(w) = d_1w - d_3w^3 + d_5w^5 - \ldots
$$

If the reduced model is stable, its Mihailov frequency characteristic must intersect $k$ times with abscissa and ordinate alternatively in the same manner as that of the original system. In other words, $k$ roots of $\xi(w) = 0$ and $\eta(w) = 0$ must be real and positive and alternately distributed along the $w$ axis.

So, the first $k$ intersecting frequencies $0, w_1, w_2, w_3, \ldots, w_{k-1}$ are kept unchanged and are set to be then roots of $\xi(w) = 0$ and $\eta(w) = 0$. Therefore

$$
\begin{align*}
\xi(w) &= \lambda_1(w^2 - w_2^2) (w^2 - w_3^2) (w^2 - w_4^2) \\
\eta(w) &= \lambda_2(w^2 - w_2^2) (w^2 - w_3^2) (w^2 - w_4^2)
\end{align*} \tag{3.5}
$$

where, the values of the coefficients $\lambda_1, \lambda_2$ are computed from $\phi(0) = \xi(0)$ and
ψ(w₁) = η(w₁) respectively. By putting the values of λ₁ and λ₂ in (3.3) respectively, we get

\[ D_k(jw) = \xi(w) + j\eta(w) \]  

(3.6)

Now replacing \( jw \) by \( s \), the denominator \( D_k(s) \) is obtained as

\[ D_k(s) = d_0 + d_1 s + d_2 s^2 + d_3 s^3 + \ldots \ldots + d_k s^k \]  

(3.7)

**Pade Approximation Method**

Determination of the numerator coefficients of the reduced model by using the Pade approximation[12]

The original \( n \)th-order system can be expanded in power series about \( s = 0 \) as

\[ G_n(s) = \frac{\sum_{i=0}^{n-1} a_i s^i}{\sum_{i=0}^{n} b_i s^i} = e_0 + e_1 s + e_2 s^2 + \ldots \ldots \ ]  

(3.8)

The coefficients of the power series expansion can also be calculated as follows:

\[
\begin{align*}
    e_0 &= \frac{a_0}{b_0} \\
    e_i &= \frac{1}{b_0} [a_i - \sum_{j=1}^{i} a_j e_{i-j}] \quad \text{for} \ i > 0 \\
    a_i &= 0, \quad \text{for} \ i > n-1
\end{align*}
\]

(3.9)

The \( k \)th-order reduced model is taken as

\[ R_k(s) = \frac{N_k(s)}{D_k(s)} = \frac{\sum_{i=0}^{k-1} c_i s^i}{\sum_{i=0}^{k} d_i s^i} \]  

(3.10)

Here, \( D_k(s) \) is known through equations (3.3-3.7)

For \( N_k(s) \) of eqn (3.10) to be Pade approximants of \( G_n(s) \) of equation (3.9), we have

\[
\begin{align*}
    c_0 &= d_0 e_0 \\
    c_1 &= d_0 e_1 + d_1 c_0 \\
    \vdots \\
    c_{k-1} &= d_0 e_{k-1} + d_1 e_{k-2} + \ldots + d_{k-2} e_1 + b_{k-1} e_0
\end{align*}
\]

(3.11)
The coefficients $c_j; j=0,1,2 \ldots \ldots \ldots k-1$ can be found by solving the above $k$ linear equations.

Hence, the numerator $N_k(s)$ is obtained as

$$N_k(s) = c_0 + c_1 s + c_2 s^2 + \cdots + c_{k-1} s^{k-1}$$ (3.12)

**Analysis of Physical System**

A physical system is taken from the literature to illustrate the algorithm of the proposed method. The example is solved to get second order reduced model. An integral square error (ISE) in between the transient parts of the original and reduced model is calculated using MATLAB to measure the goodness of the reduced order model i.e. lower the ISE, closer the $R_k(s)$ to $G_n(s)$, which is given by

$$ISE = \int_0^\infty |y(t) - y_k(t)|^2 dt$$ (3.13)

Where, $y(t)$ and $y_k(t)$ are the unit step responses of original and reduced system respectively.

**Analysis of Physical System**

Consider a 4th-order system from the literature [85]

$$G_4(s) = \frac{N(s)}{D(s)} = \frac{24+24s+7s^2+s^3}{24+50s+35s^2+10s^3+s^4}$$

Putting $s = jw$ in the denominator $D(s)$

$$D(jw) = (24-35w^2+w^4)+jw(50-10w^2)$$

The intersecting frequencies are

$w_1 = 0, 0.8365, 2.2361, 5.8566$ (i=0,1,2,3)

Using the procedure given in step -1, the denominator of the second order model is taken as
\[ D_2(jw) = \lambda_1 (w^2 - 1.190362) + j\lambda_2 w, \]
\[ \lambda_1 = -34.2988 \]
\[ \lambda_2 = 43.0027 \]

Hence the denominator \( D_2(s) \) is obtained as
\[ D_2(s) = s^2 + 1.25377s + 0.6997009 \]

Using the Eqns. (3.8-3.11), the following coefficients are calculated
\[ e_0 = 1 \quad a_0 = 24 \]
\[ e_1 = -1.0833 \quad a_1 = 17.0035 \]

Therefore, finally 2\textsuperscript{nd} -order reduced model is obtained as
\[
R_2(s) = \frac{24+17.0035s}{24+43.0027s+34.2988s^2} = \frac{0.6994+0.4957s}{0.6994+1.2537s+s^2}
\]

The step responses of the reduced order model and the original system are compared in Fig.3.1 and the Bode plot comparison is shown in Fig. 3.2. The proposed method is compared with the well-known reduction methods and shown in the Table-3.1, from which it is concluded that this method is comparable in quality.

**Table- 3.1 Comparison of the Reduction Methods for ISE**

<table>
<thead>
<tr>
<th>Reduction Methods</th>
<th>Reduction Models</th>
<th>ISE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed Method</td>
<td>( R_2(s) = \frac{0.6994+0.4957s}{0.6994+1.2537s+s^2} )</td>
<td>0.03300</td>
</tr>
<tr>
<td>G.Parmar et al.[65]</td>
<td>( R_2(s) = \frac{8+24.1429s}{8+9s+s^2} )</td>
<td>0.04809</td>
</tr>
<tr>
<td>Mukherjee et al.[80]</td>
<td>( R_2(s) = \frac{4.457+11.3909s}{4.4357+4.2122s+s^2} )</td>
<td>0.05697</td>
</tr>
</tbody>
</table>
Fig. 3.1 Step Response Comparison of Original and Reduced Order System

**Equation:**

\[ R_2(s) = \frac{1.9906s + 7.0908s}{2 + 3s + s^2} \]

**Table:**

<table>
<thead>
<tr>
<th>Magnitude (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
<tr>
<td>-10</td>
</tr>
<tr>
<td>-20</td>
</tr>
<tr>
<td>-30</td>
</tr>
<tr>
<td>-40</td>
</tr>
</tbody>
</table>

**Phase (deg):**

<table>
<thead>
<tr>
<th>Phase (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
<tr>
<td>-45</td>
</tr>
<tr>
<td>-90</td>
</tr>
<tr>
<td>-135</td>
</tr>
</tbody>
</table>

**Figures:**

- **Step Response:**
  - ORIGINAL
  - REDUCED

- **Bode Diagram:**
  - ORIGINAL
  - REDUCED
**Fig.3.2** Frequency Response Comparison of Original and Reduced Order System

**TABLE-3.2 COMPARISON OF THE PERFORMANCE PARAMETERS OF ORIGINAL AND REDUCED ORDER SYSTEMS**

<table>
<thead>
<tr>
<th>System</th>
<th>Rise Time</th>
<th>Peak Time</th>
<th>Settling Time</th>
<th>Peak</th>
</tr>
</thead>
<tbody>
<tr>
<td>4&lt;sup&gt;th&lt;/sup&gt; Order(Original)</td>
<td>2.2602</td>
<td>6.9770</td>
<td>3.9307</td>
<td>0.9991</td>
</tr>
<tr>
<td>2&lt;sup&gt;nd&lt;/sup&gt; Order(Reduced)</td>
<td>2.2243</td>
<td>4.6025</td>
<td>6.2757</td>
<td>1.0389</td>
</tr>
</tbody>
</table>

**METHOD-2: ORDER REDUCTION USING MODIFIED CAUER FORM AND FACTOR DIVISION METHOD**

In this technique, the denominator polynomial of the reduced order model is determined by using the modified Cauer[46] form while the coefficients of the numerator are obtained by Factor Division Method[98]. This technique is simple and gives stable reduced models for the stable high-order system. The proposed method is described by solving a numerical example taken from the literature.

**Reduction Method**

Let the transfer function of higher order original system given by equation (3.1) of the order of n<sup>th</sup> and its k<sup>th</sup> order reduced order model (3.2) are expressed as

\[
G_n(s) = \frac{b_{11} + b_{12}s + b_{13}s^2 + \ldots + b_{1n}s^{n-1}}{a_{11} + a_{12}s + a_{13}s^2 + \ldots + a_{1n}s^{n-1} + s^n} \tag{3.14}
\]

Where; \( b_{ii} \) & \( a_{ii} \), \( 1 \leq i \leq n \) are known scalar constants.
\[ R_k(s) = \frac{N_k(s)}{D_k(s)} = \frac{b_{k,1} + b_{k,2}s^2 + \ldots + b_{k,k}s^{k-1}}{a_{k+1,1} + a_{k+1,2}s + \ldots + a_{k+1,k}s^{k-1} + s^k} \]  

(3.15)

Where; \( b_{m,j} \) and \( a_{m+1,j} \) \( 1 \leq j \leq m \) are known scalar constants.

The objective of order reduction is to realize the \( k \)th order reduced model in the form of (3.15) from the original system (3.14) such that it retains the important features of the original high order system.

The proposed reduction procedure consists of the following two steps:

**Step-1:** Determination of the denominator polynomial \( D_k(s) \) for the \( k \)th order reduced model by using modified Cauer form [46]: The brief procedure for getting \( D_k(s) \) using modified Cauer form is as follows:

The given higher order system \( G_n(s) \)

Without loss of generality, the coefficient of highest power of \( s^n \) equations (3.14) & (3.15) can always made unity and numerator is one degree lower than the denominator.

\( G_n(s) \) can be expanded into a Cauer type continued fraction about \( s=0 \) and \( s=\infty \) alternately as follows

\[ G_n(s) = \frac{1}{s} \frac{1}{\frac{1}{b_{21}} - \frac{1}{s} \frac{1}{\frac{1}{b_{22}} - \ldots - \frac{1}{s} \frac{1}{\frac{1}{b_{2 \cdot n-1}} - \frac{1}{s} \frac{1}{\frac{1}{b_{2 \cdot n}}}}}} \]

Where

\[ a_{21} = \frac{a_{12}b_{12}}{b_{11}}, \quad b_{21} = b_{11} - \frac{b_{1n}a_{21}}{a_{2n}} \]

\[ a_{2,n-1} = a_{1,n} - \frac{a_{1n}b_{1n}}{b_{11}}, \quad b_{2,n-1} = b_{1,n-1} - \frac{b_{1n}a_{2,n-1}}{a_{2n}} \]

\[ a_{2,n} = 1 \]

By continuing the above sequence of expansion we get the following form
Where the quotients $h_1$, $h_2$, .............. $H_2$, $H_1$ are evaluated from the following modified array

\[
\begin{array}{cccc}
  a_{11} & a_{12} & a_{1,n-1} & a_{1,n} \\
  b_{11} & b_{12} & a_{1,n-1} & b_{1,n} \\
  a_{21} & a_{22} & a_{2,n-1} & 1 \\
  b_{21} & b_{22} & b_{2,n-1} & \\
  a_{31} & a_{32} & 1 \\
  \vdots & \vdots & \vdots & \vdots \\
  b_{n-1,1} & b_{n-1,2} & & \\
  a_{n,1} & & & \\
  b_{n,1} & & & \\
  1 & & & \\
\end{array}
\]  

(3.18)

For $j=1,2,\ldots,n-1$,

\[
H_j = b_{j,n-1}
\]

(3.19)

Here the coefficients in equation (3.18) form the first two rows and remaining elements starting with third row are obtained from the recursive relations.

\[
\begin{align*}
  a_{j+1,k} &= a_{j,k+1} - h_j b_{j,k+1} \\
  b_{j+1,k} &= b_{j,k} - H_j a_{j+1,k} \\
\end{align*}
\]

(3.20)

where \( h_j = \frac{a_{j,1}}{b_{j,1}} \).
It is to be noted that the end elements of all the odd rows can be written directly as

\[ a_{1,n+1}=a_{2,n}=a_{3,n-1}=\ldots=a_{n+1,1}=1 \]  \hspace{1cm} (3.21)

A reduced model of order \( k \), is obtained by truncating equation (3.17) after the first \( 2k \)th terms and inverting it to yield the transfer function formed out of the values of the quotients \( h_j \) and \( H_j \) derived in equation (3.19) are achieved by constructing the inversion table as follows

\[
\begin{pmatrix}
    a_{11} & a_{12} & \ldots & a_{1,k} \\
    b_{11} & b_{12} & \ldots & b_{1,k} \\
    a_{21} & 1 & \ldots & b_{2,k} \\
    b_{21} & b_{22} & \ldots & b_{2,k} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{k,1} & a_{k,2} & \ldots & a_{k,k-1} & 1 \\
    b_{k,1} & b_{k,2} & \ldots & b_{k,k-1} & b_{k,k} \\
    a_{k+1,1} & a_{k+1,2} & \ldots & a_{k+1,k-1} & 1 \\
    \end{pmatrix}
\]  \hspace{1cm} (3.22)

\[
H_k = \frac{a_{11}}{b_{11}}, \quad h_k = \frac{b_{11}}{a_{21}}
\]

\[
H_{k-1} = \frac{1}{b_{22}}, \quad h_{k-1} = \frac{b_{21}}{a_{31}}
\]

\[
H_1 = \frac{1}{b_{k,k}}, \quad h_1 = \frac{b_{k,1}}{a_{k+1,1}}
\]

Starting with \( a_{11}=1 \), the elements of the 2nd, third and subsequent rows are evaluated recursively from.

\[ a_{j+1,1}= h_{k+1,j} \cdot b_{j,1} \quad \text{for} \quad j=1,2,\ldots,m \]
\( a_{j,m} = a_{j-1,k-1} + h_{k+2-j} b_{j-1,m} \)

\( m = 2, 3, \ldots \ldots \ldots j-1 \)

& \( b_{j,m} = b_{j-1,m} + H_{k+1-j} a_{j,m} \)

\( j = 2, 3 \ldots \ldots \) and \( m = 1, 2 \ldots \ldots j-1 \)

It is also evident that the end elements can be written by inspection according as

\[
\begin{align*}
    a_{jj} &= 1 & j = 1, 2, \ldots \ldots k+1 \\
    b_{jj} &= H_{k+1-j} & j = 1, 2, \ldots \ldots k
\end{align*}
\]  

(3.23)

There must be \((2k+1)\) rows in complete array. The required denominator of the reduced order model equation (3.15) can be written from \((2k+1)\)th row of the array as

\[ D_k(s) = a_{k+1} + a_{k+1,k} s^{k-1} + \ldots \ldots + a_{k+1,2} s + a_{k+1,1} \]  

(3.24)

**Step-2** Determination of the numerator of \(k\)th order reduced model using Factor Division algorithm [98]. After obtaining the reduced denominator, the numerator of the reduced model is determined as follows

\[
\tilde{N}(s) = \frac{N(s)}{D(s)} \times D_k(s) = \frac{N(s)}{D(s)/D_k(s)}
\]

(3.25)

Where \(D_k(s)\) is reduced order denominator

There are two approaches for determining of numerator of reduced order model.

(i) By performing the product of \(N(s)\) and \(D_k(s)\) as the first row of factor division algorithm and \(D_k(s)\) as the second row up to \(s^{k-1}\) terms are needed in both rows.

(ii) By expressing \(N(s)D_k(s)/D(s)\) as \(N(s)/[D(s)/D_k(s)]\) and using factor division algorithm twice; the first time to find the term up to \(s^{k-1}\) in
the expansion of \( \frac{D(s)}{D_k(s)} \) (i.e., put \( D(s) \) in the first row and \( D_k(s) \) in the second row, using only terms up to \( s^{k-1} \)), and second time with \( N(s) \) in the first row and the expansion \( \frac{[D(s)/D_k(s)]}{s^{k-1}} \) in the second row.

Therefore the numerator \( N_k(s) \) of the reduced order model \( (R_k(s)) \) in eq.(3.15) will be the series expansion of

\[
\frac{N(s)}{D_k(s)} \frac{\sum_{i=0}^{k-1} c_i s^i}{\sum_{i=0}^{k-1} d_i s^i}
\]

About \( s=0 \) up to term of order \( s^{k-1} \).

This is easily obtained by modifying the moment generating \([98]\) which uses the familiar routh recurrence formulae to generate the third, fifth, and seventh etc rows as,

\[
\begin{array}{c}
\alpha_0 = \frac{d_0}{f_0} < \frac{d_0}{f_0} \quad \frac{d_1}{f_1} \quad \frac{d_2}{f_2} \ldots \quad \frac{d_{k-1}}{f_{k-1}} \\
\alpha_1 = \frac{g_0}{f_0} < \frac{g_0}{f_0} \quad \frac{g_1}{f_1} \quad \frac{g_2}{f_2} \ldots \quad \frac{g_{k-2}}{f_{k-2}} \\
\alpha_2 = \frac{l_0}{f_0} < \frac{l_0}{f_0} \quad \frac{l_1}{f_1} \quad \frac{l_2}{f_2} \ldots \quad \frac{l_{k-3}}{f_{k-3}} \\
\vdots \\
\alpha_{k-2} = \frac{p_0}{f_0} < \frac{p_0}{f_0} \quad \frac{p_1}{f_1} \\
\alpha_{k-1} = \frac{q_0}{f_0} < \frac{q_0}{f_0}
\end{array}
\]

Where

\[
g_i = d_{i+1} - a_0^* f_{i+1} \quad \text{and} \quad l_i = g_{i+1} - a_1^* f_{i+1} \quad i = 0, 1, 2, \ldots
\]

\[
q_0 = p_1 - a_{k-2}^* f_1
\]

Therefore, the numerator \( N_m(s) \) of eq.(3.15) is given by

\[
N_k(s) = \sum_{i=0}^{k-1} \alpha_i s^i
\]

(3.28)
Analysis of physical System

The proposed method explains by considering the physical system based on the published literature. The goodness of the proposed method is measured by calculating integral square error (ISE) between the transient parts of the original and reduced model using MATLAB. The ISE should be minimum for better approximation i.e close the $R_k(s)$ to $G_n(s)$, which is given by

$$ISE = \int_0^\infty |y(t) - y_k(t)|^2 dt$$  \hspace{1cm} (3.29)

Where, $y(t)$ and $y_k(t)$ are the unit step responses of original and reduced system respectively.

Example:- Consider a 4th-order system from the literature [89]

$$G_4(s) = \frac{24 + 24s + 7s^2 + s^3}{24 + 50s + 35s^2 + 10s^3 + s^4}$$

The denominator of reduced order model $R_k(s)$ can be evaluate using equations 3.19, 3.20, & 3.21 for k=2. Make modified Routh array & evaluate the quotients $h_1, h_2, h_3, h_4$ and $H_1, H_2, H_3, H_4$.

Modified Routh Array

\[
\begin{array}{cccccc}
24 & 50 & 35 & 10 & 1 \\
24 & 24 & 7 & 1 \\
26 & 28 & 9 & 1 \\
-2 & -4 & -2 \\
-24 & -17 & 1 \\
-50 & -38 & \\
1.24 & 1 \\
-2.88 & \\
1 & \\
\end{array}
\]

$$h_1 = \frac{24}{24} = 1, \quad H_1 = \frac{1}{1} = 1$$
With the knowledge of the first four quotients \((k=2), h_1=1, H_1=1, h_2=-13, H_2=-2\) and with the help of equation (3.22) construct the inversion table as follows

**Inversion Table**

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>-2</td>
</tr>
<tr>
<td>26</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>27</td>
<td>1</td>
</tr>
</tbody>
</table>

Hence the denominator \(D_2(s)\) of reduced order model \(R_2(s)\) is obtained as

\[D_2(s) = s^2 + 27s + 24\]

Using the factor division algorithm the following coefficients of numerator \(N_2(s)\) of reduced order model are calculated

Consider \(D_4(s)/D_2(s)\) gives

\[
\begin{array}{c|c}
24 & 50 \\
24 & 27 \\
23 & \\
24 & \\
\end{array}
\]

\[
\begin{array}{c}
\alpha_0 = \frac{24}{24} = 1 , \\
\alpha_1 = \frac{23}{24} ,
\end{array}
\]

Now considering \(N_4(s)/D_4(s)/D_2(s)\)

\[
\begin{array}{c|c}
24 & 24 \\
1 & 23 \\
1 & 24 \\
1 & 1 \\
\end{array}
\]

\[
\begin{array}{c}
\alpha_0 = \frac{24}{1} = 24 , \\
\alpha_1 = \frac{1}{1} = 1
\end{array}
\]

Thus Reduced Numerator is given as
\[ N_2(s) = 24 + s \]

Thus the Reduced model is given by

\[ R_2(s) = \frac{24 + s}{24 + 27s + s^2} \]

Fig. 3.3 Step Response Comparison of Original and Reduced Order System
**Fig.3.4** Frequency Response Comparison of Original and Reduced Order System

**TABLE- 3.3 COMPARISON OF THE REDUCTION METHODS FOR ISE**

<table>
<thead>
<tr>
<th>Reduction Methods</th>
<th>Reduction Models</th>
<th>ISE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed Method</td>
<td>$R_2(s) = \frac{24 + s}{24 + 27s + s^2}$</td>
<td>0.00110</td>
</tr>
<tr>
<td>G.Parmar et al.[30]</td>
<td>$R_2(s) = \frac{8 + 24.11429s}{9 + 9s + s^2}$</td>
<td>0.04809</td>
</tr>
<tr>
<td>Mukherjee et al.[80]</td>
<td>$R_2(s) = \frac{4.457 + 11.3909s}{24 + 27s + s^2}$</td>
<td>0.05697</td>
</tr>
<tr>
<td>Mittal et al[1]</td>
<td>$R_2(s) = \frac{1.9906 + 7.0908s}{2 + 3s + s^2}$</td>
<td>0.2689</td>
</tr>
</tbody>
</table>

**METHOD-3: ORDER REDUCTION USING MODIFIED POLE CLUSTERING AND FACTOR DIVISION METHOD**
In this technique, the denominator polynomial of the reduced order model is determined by using the modified pole clustering [14] while the coefficients of the numerator are obtained by Factor Division Method [98]. This technique is simple and gives stable reduced models for the stable high-order system.

**Statement of the Problem**

Let the transfer function of high order original system $G_n(s)$ of the order ‘$n$’ of equation (3.1) and reduced order model $R_k(s)$ of equation(3.2) may express as

$$G_n(s) = \frac{N(s)}{D(s)} = \frac{c_0+c_1s+c_2s^2+\ldots+c_{n-1}s^{n-1}}{d_0+d_1s+d_2s^2+\ldots+d_n s^n} \quad (3.30)$$

Where $c_i$ (i varies 0 to n-1) and $d_j$ (j varies 0 to n) are constants

$$R_k(s) = \frac{N_k(s)}{D_k(s)} = \frac{e_0+e_1s+e_2s^2+\ldots+e_{k-1}s^{k-1}}{f_0+f_1s+f_2s^2+\ldots+f_k s^k} \quad (3.31)$$

Where $e_i$ (i varies 0 to k-1) and $f_j$ (j varies 0 to k) are unknown scalar constants.

The objective is to realize the $k^{th}$ order reduced model in the form of (3.31) from the original system (3.30) such that it retains the important features of the original high-order system.

**Reduction Method**

The reduction procedure for getting the reduced models consists of the following two steps:

**Step-1:** Determination of the denominator polynomial for say the $k^{th}$ order reduced model using modified pole clustering [14]:

The following rules are used for clustering the poles of the original system given in frequency domain:
• Separate clusters should be made for real and complex poles.
• Clusters of the poles in the left half $s$-plane should not contain any pole of the right half $s$-plane and vice-versa.
• Poles on the $jw$-axis have to be retained in the reduced order model.
• Poles at the origin have to be retained in the reduced order model.

The denominator polynomial by using the modified pole clustering is as follows:

Let there be $r$ number of real poles in $i^{th}$ cluster are $p_1, p_2, p_3, \ldots, p_r$

Where $|p_1| < |p_2| < |p_3| \ldots \ldots \ldots < |p_r|$, and then modified cluster centre $p_{ci}$ can be obtained by using the algorithm of modified pole clustering suggested in this method. Let $m$ pair of complex conjugate poles in the $j^{th}$ cluster be

$[(\alpha_1 \pm j\beta_1), (\alpha_2 \pm j\beta_2) \ldots \ldots (\alpha_m \pm j\beta_m)]$ where $|\alpha_1| < |\alpha_2| < |\alpha_3| \ldots \ldots \ldots < |\alpha_m|.$

Now using the same algorithm separately for real and imaginary parts of the complex conjugate poles, the modified cluster centre is obtained, which is written as

$\Phi_{ej} = A_{ej} \pm jB_{ej}$

Where $\Phi_{ej} = A_{ej} + jB_{ej}$ and $\Phi_{ej} = A_{ej} - jB_{ej}$

An interactive computer oriented algorithm [15] has been developed, which automatically finds the modified cluster centre and is given as follows:

Step-1 Let $r$ real poles in a cluster be $|p_1| < |p_2| < |p_3| \ldots \ldots \ldots < |p_r|,$
Step-2 Set $j = 1.$
Step-3 Find pole cluster centre $C_j = \left[\sum_{i=1}^{r} (1) / |p_i| \right]^{-1}$
Step-4 Set $j = j+1.$
Step-5 Now find a modified cluster centre from $C_j = \left[\left(\frac{-1}{|p_j|} + \frac{-1}{|p_{j-1}|} \right) / 2\right]^{-1}$
**Step-6** Is \( r = j \)? If No, and then go to step-4 otherwise go to step-7.

**Step-7** Take a modified cluster centre of the \( k^{th} \) -cluster as \( p_{ek} = c_j \)

For synthesizing the \( k^{th} \) -order denominator polynomial, one of the following cases may occur

**Case-1**

If all modified cluster centers are real, then denominator polynomial of the \( k^{th} \) order reduced model can be obtained as

\[
D_k(s) = (s - p_{e1})(s - p_{e2}) \ldots \ldots (s - p_{ek})
\]  

(3.32)

Where \( p_{e1}, p_{e2} \ldots \ldots p_{ek} \) are \( 1^{st}, 2^{nd}, \ldots \ldots k^{th} \) modified cluster centre respectively.

**Case-2:**

If all modified cluster centers are complex conjugate then denominator polynomial of the \( k^{th} \) -order reduced model can be obtained as

\[
D_k(s) = (s - \Phi_{e1})(s - \Phi_{e1}) \ldots \ldots (s - \Phi_{ek/2})(s - \Phi_{ek/2})
\]  

(3.33)

**Case-3**

If some cluster center is real and some are complex conjugate. For example \( k-2 \) cluster centers are real and one pair of cluster center is complex conjugate, then \( k^{th} \) -order denominator can be obtained as

\[
D_k(s) = (s - p_{e1})(s - p_{e2}) \ldots \ldots (s - p_{e(k-2)})(s - \Phi_{e1})(s - \Phi_{e1})
\]  

(3.30)

Hence, the denominator polynomial \( D_k(s) \) is obtained as

\[
D_k(s)=d_0+d_1s+\ldots+d_ks^k
\]  

(3.31)

**Step-2** Determination of the numerator of \( k^{th} \) order reduced model using Factor Division algorithm [98]. After obtaining the reduced denominator, the numerator of the reduced model is determined as
follows
\[ \tilde{N}(s) = \frac{N(s)}{D(s)} \times D_k(s) = \frac{N(s)}{D(s)} / D_k(s) \] (3.32)

Where \( D_k(s) \) is reduced order denominator

There are two approaches for determining of numerator of reduced order model.

(i) By performing the product of \( N(s) \) and \( D_k(s) \) as the first row of factor division algorithm and \( D_k(s) \) as the second row up to \( s^{k-1} \) terms are needed in both rows.

(ii) By expressing \( N(s)D_k(s)/D(s) \) as \( N(s)/[D(s)/D_k(s)] \) and using factor division algorithm twice; the first time to find the term up to \( s^{k-1} \) in the expansion of \( D(s)/D_k(s)(\text{i.e. put } D(s) \text{ in the first row and } D_k(s) \text{ in the second row, using only terms up to } s^{k-1}) \), and second time with \( N(s) \) in the first row and the expansion \([D(s)/D_k(s)]\) in the second row.

Therefore the numerator \( N_k(s) \) of the reduced order model \( R_k(s) \) in eqn. (3.24) will be the series expansion of
\[ \frac{N(s)}{D(s)} / D_k(s) = \frac{\sum_{i=0}^{k-1} e_i s^i}{\sum_{i=0}^{k-1} f_i s^i} \] (3.33)

About \( s=0 \) up to term of order \( s^{k-1} \).

This is easily obtained by modifying the moment generating which uses the familiar routh recurrence formulae to generate the third, fifth, and seventh etc rows as,
\[
\alpha_0 = \frac{d_0}{f_0} < \frac{e_0}{f_0} \quad \frac{e_1}{f_1} \quad \frac{e_2}{f_2} \quad \ldots \ldots \ldots \quad \frac{e_{k-1}}{f_{k-1}}
\]
\[
\alpha_1 = \frac{g_0}{f_0} < g_0 \quad g_1 \quad g_2 \quad \ldots \ldots \ldots \quad g_{k-2}
\]
\[ \frac{\alpha_2}{f_0} = \frac{l_0}{f_0} \quad \frac{l_0}{l_1} \quad \frac{l_1}{l_2} \quad \ldots \quad \frac{l_{k-3}}{l_{k-2}} \]

\[ \frac{\alpha_{k-2}}{f_0} = \frac{p_0}{f_0} \quad \frac{p_0}{p_1} \quad \frac{p_1}{p_2} \quad \ldots \quad \frac{p_{k-2}}{p_{k-3}} \]

\[ \frac{\alpha_{k-1}}{f_0} = \frac{q_0}{f_0} \quad \frac{q_0}{q_1} \quad \frac{q_1}{q_2} \quad \ldots \quad \frac{q_{k-2}}{q_{k-3}} \]

\[ g_i = d_{i+1} \cdot \alpha_{0^*} f_{i+1} \quad \text{and} \quad l_i = g_{i+1} \cdot \alpha_{1^*} f_{i+1} \quad i = 0, 1, 2 \ldots \]

\[ q_0 = p_1 - \alpha_{k-2}^* f_1 \]

Therefore, the numerator \( N_k(s) \) of the reduced order model is given by

\[ N_k(s) = \sum_{i=0}^{k-1} \alpha_i s^i \quad \text{(3.34)} \]

**Method for Comparison**

In order to check the accuracy of the proposed method, the comparison between step response data of the original system and the reduced system is done.

**Analysis of Physical System**

The proposed method explains by considering a physical system, taken from the literature. The goodness of the proposed method is measured by calculating step response data between the original and the reduced model using MATLAB.

**Example:** Consider an eight-order system from the literature [14]

\[ G_8(s) = \frac{N_8(s)}{D_8(s)} \]

Where,

\[ N_8(s) = 40320 + 185760s + 222088s^2 + 122664s^3 + 36380s^4 + 5982s^5 + 514s^6 + 18s^7 \]

And

\[ D_8(s) = 40320 + 109584s + 118124s^2 + 67284s^3 + 22449s^4 + 4536s^5 + 546s^6 + 36s^7 + s^8 \]
The poles are: -1, -2, -3, -4, -5, -6, -7, -8

Let the 2nd-order reduced model is required to be realized, for this purpose only two real clusters are required.

Let the first and second cluster consists the poles (-1, -2, -3, -4) and (-5, -6, -7, -8) respectively.

The modified cluster centers are computed as $P_{e1}=-1.06371$, $P_{e2}=-5.13271$

Using modified clustering the denominator polynomial $D_{2}(s)$ is obtained as $D_{2}(s)=5.45971+6.19642s+s^{2}$

Using the factor division algorithm, the following coefficients of numerator $N_{2}(s)$ of reduced order model are calculated.

Consider $D_{8}(s)/D_{2}(s)$ gives

\[
\begin{align*}
\alpha_{0} &= 7385.0076 < 40320 \quad 109584 \\
&\quad 5.45971 \quad 6.19642
\end{align*}
\]

\[
\begin{align*}
\alpha_{1} &= 11689.8867 < 63823.39 \\
&\quad 5.45971
\end{align*}
\]

Now considering $N_{8}(s)/D_{8}(s)/D_{2}(s)$

\[
\begin{align*}
\alpha_{0} &= 5.45971 < 40320 \quad 185760 \\
&\quad 7385.0076 \quad 11689.8867
\end{align*}
\]

\[
\begin{align*}
\alpha_{1} &= 16.51137 < 121936.6086 \\
&\quad 7385.0076
\end{align*}
\]

Thus Reduced Numerator is given as $N_{2}(s) = 5.45971+16.51137s$

Thus the Reduced model is given as $R_{2}(s) = \frac{5.45971+16.51137s}{5.45971+6.19642s+s^{2}}$
Fig. 3.5 Step Response Comparison of Original and Reduced Order System

Fig. 3.6 Frequency Response Comparison of Original and Reduced Order System

TABLE 3.4 COMPARISONS OF THE PERFORMANCE PARAMETERS ORIGINAL AND REDUCED ORDER SYSTEMS

<table>
<thead>
<tr>
<th>System</th>
<th>Rise Time($t_r$)</th>
<th>Peak</th>
<th>Settling</th>
<th>Peak</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
METHOD-4: ORDER REDUCTION USING BASIC CHARACTERISTICS AND FACTOR DIVISION METHOD

In this method, the denominator polynomial of the reduced order model is obtained by using the basic characteristics [89] of the higher order system which are maintained in the reduced model while the coefficients of the numerator are obtained by using factor division method [98]. This method is fundamentally simple and generates stable reduced models if the original high-order system is stable. The proposed method is illustrated with the help of the numerical example taken from the literature.

Statement of the Problem

Let the transfer function of high order original system $G_n(s)$ of the order ‘$n$’ of equation (3.1) and reduced order model $R_k(s)$ of equation(3.2) expressed as

$$G_n(s) = \frac{N(s)}{D(s)} = \frac{g_0 + g_1 s + g_2 s^2 + \cdots + g_{n-1} s^{n-1}}{h_0 + h_1 s + h_2 s^2 + \cdots + h_n s^n} \quad (3.37)$$

where $g_i; 0 \leq i \leq n - 1$ and $h_i; 0 \leq i \leq n$ known scalar constants.

Let the transfer function of the reduced model of the order ‘$k$’ be

$$R_k(s) = \frac{N_k(s)}{D_k(s)} = \frac{c_0 + c_1 s + c_2 s^2 + \cdots + c_{k-1} s^{k-1}}{d_0 + d_1 s + d_2 s^2 + \cdots + d_k s^k} \quad (3.38)$$

where ; $c_j; 0 \leq j \leq k - 1$ and $d_j; 0 \leq j \leq k$ are unknown scalar constants.

The aim of this work is to realize the $k^{th}$ order reduced model in the form of (3.38) from the original system (3.37) such that it retains the important features of the original high-order system.

<table>
<thead>
<tr>
<th></th>
<th>Time($t_p$)</th>
<th>Time($t_s$)</th>
<th>Overshoot($M_p$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original System</td>
<td>0.0597</td>
<td>0.4599</td>
<td>4.3580</td>
</tr>
<tr>
<td>Reduced System</td>
<td>0.0528</td>
<td>0.5858</td>
<td>4.8905</td>
</tr>
</tbody>
</table>
**Proposed Reduction Method**

The reduction procedure for getting the $k^{th}$ order reduced models consists of the following two steps:

**Step-1:** Determination of the denominator polynomial for the $k^{th}$ order reduced model using basic characteristics [89] of original system by the following procedure

- Firstly determine the basic characteristics of original system
- Then assume damping ratio $\zeta=0.99$ for an aperiodic or almost periodic system, and number oscillations before the system settles=1
- Determine the natural frequency ($\omega_n$) using
  \[ \omega_n = \frac{4}{\zeta T_s} \]
  
- Obtain the reduced order denominator as
  \[ D_2(s) = s^2 + 2\ast \omega_n + \omega_n^2 \quad (3.39) \]

**Step-2:** Determination of the numerator of $k^{th}$ order reduced model using Factor Division algorithm [98]

After obtaining the reduced denominator, the numerator of the reduced model is determined as follows

\[ N_k(s) = \frac{N(s)}{D(s)} \times D_k(s) = \frac{N(s)}{D(s)/D_k(s)} \]

Where $D_k(s)$ is reduced order denominator

There are two approaches for determining of numerator of reduced order model.

(i) By performing the product of $N(s)$ and $D_k(s)$ as the first row of factor division algorithm and $D(s)$ as the second row up to $s^{k-1}$ terms are needed in both rows.
(ii) By expressing \(N(s)D_k(s)/D(s)\) as \(N(s)/[D(s)/D_k(s)]\) and using factor
division algorithm twice; the first time to find the term up to \(s^{k-1}\) in
the expansion of \(D(s)/D_k(s)\) (i.e. put \(D(s)\) in the first row and \(D_k(s)\) in
the second row, using only terms up to \(s^{k-1}\)), and second time with
\(N(s)\) in the first row and the expansion \([D(s)/D_k(s)]\) in the second
row.

Therefore the numerator \(N_k(s)\) of the reduced order model \(R_k(s)\) in eq.(3.38)
will be the series expansion of

\[
\frac{N(s)}{D(s)} = \frac{\sum_{i=0}^{k-1} c_i s^i}{\sum_{i=0}^{k} d_i s^i}
\]

About \(s=0\) up to term of order \(s^{k-1}\).

This is easily obtained by modifying the moment generating which uses the
familiar routh recurrence formulae to generate the third, fifth, and seventh etc
rows as,

\[
\begin{array}{c}
\alpha_0 = \frac{g_0}{h_0} < \frac{g_0}{h_0} \quad \frac{g_1}{h_1} \quad \frac{g_2}{h_2} \quad \ldots \quad \frac{g_{k-1}}{h_{k-1}} \\
\alpha_1 = \frac{l_0}{h_0} < \frac{l_0}{h_0} \quad \frac{l_1}{h_1} \quad \frac{l_2}{h_2} \quad \ldots \quad \frac{l_{k-2}}{h_{k-2}} \\
\alpha_2 = \frac{m_0}{h_0} < \frac{m_0}{h_0} \quad \frac{m_1}{h_1} \quad \frac{m_2}{h_2} \quad \ldots \quad \frac{m_{k-3}}{h_{k-3}} \\
\vdots
\end{array}
\]

Where

\[
l_i = g_{i+1} - \alpha_0 \cdot h_{i+1}, \quad i=0,1,2, \ldots \\
m_i = l_{i+1} - \alpha_1 \cdot h_{i+1}, \quad i=0,1,2, \ldots 
\]
Therefore, the numerator $N_k(s)$ of eq.(3.38) is given by

$$N_k(s) = \sum_{i=0}^{k-1} \alpha_i s^i$$  \hspace{1cm} (3.40)

**Method for Comparison**

In order to check the quality of the proposed method the quantitative comparison in term of rise time ($t_r$), settling time ($t_s$) and maximum overshoot ($M_p$) with the original system has been done.

**Analysis of Physical System**

The proposed method explains by considering physical system taken from the literature. The goodness of the proposed method is measured by calculating the rise time ($t_r$), settling time ($t_s$) and maximum overshoot ($M_p$) and compare with the original system.

**Example**: Consider a 4th-order system from the literature [104]

$$G(s) = \frac{24 + 24s + 7s^2 + s^3}{24 + 50s + 35s^2 + 10s^3 + s^4}$$

**Step 1**: Determination of Denominator of reduced order

Denominator of reduced order model is determine using following basic characteristics of original system

- Rise Time: 2.260
- Settling Time: 3.9307
- Settling Min: 0.9002
- Settling Max: 0.9991
- Overshoot: 0
- Undershoot: 0
- Peak: 0.9991
Peak Time: 6.9770

Assume =0.99 for an aperiodic or almost periodic system, and number oscillations before the system settles=1

since $\omega_n = \frac{4}{T_s}$

Therefore $\omega_n = \frac{4}{0.99*3.93} = 1.0281$

The Reduced denominator is given by

$$D_2(s) = s^2 + 2 * \omega_n * s + \omega_n^2$$

$$= s^2 + 2.0356s + 1.0569$$

**Step 2:** Now using the factor division method the numerator of reduced order model is given as

Consider $D_4(s)/D_2(s)$

$$\alpha_0 = 22.707 \begin{bmatrix} 24 & 50 \\ 1.0569 & 2.0356 \end{bmatrix}$$

$$\alpha_1 = 3.574 \begin{bmatrix} 3.7776 \\ 1.0569 \end{bmatrix}$$

Now Considering $N_4/D_4(s)/D_2(s)$

$$\alpha_0 = 1.05694 \begin{bmatrix} 24 & 24 \\ 22.707 & 3.574 \end{bmatrix}$$

$$\alpha_1 = 0.8906 \begin{bmatrix} 20.2224 \\ 22.707 \end{bmatrix}$$

Thus Reduced Numerator is given as

$$N_2(s) = 1.05694 + 0.8906s$$

Thus the Reduced model is given as

$$R_2(s) = \frac{1.05694 + 0.8906s}{1.0569 + 2.0356s + s^2}$$
Fig. 3.7 Step Response Comparison of Original and Reduced Order System

Fig. 3.8 Frequency Response Comparison of Original and Reduced Order System

Table 3.5: Comparison of the Performance Parameters of Original and Reduced Order Systems

<table>
<thead>
<tr>
<th>System</th>
<th>Rise Time</th>
<th>Peak</th>
<th>Settling</th>
</tr>
</thead>
<tbody>
<tr>
<td>ORIGINAL</td>
<td>REDUCED</td>
<td>ORIGINAL</td>
<td>REDUCED</td>
</tr>
</tbody>
</table>
3.4 Conclusion

It is shown that analysis and plots show the use of that the combined method is superior in comparison to the use of any of conventional single methods; the mixed methods of model reduction which combine the advantages of Mihailov stability criterion and Pade approximation, Modified Cauer form, & Factor Division, Modified pole clustering & Factor Division and Basic characteristics & factor division have been discussed. In the first section the method of Mihailov stability criterion and Pade approximations are combined while a combination of Modified Cauer form and factor division is carried out in the second. In section three and four combination of modified pole clustering and factor division method and basic characteristics and factor division method are discussed respectively. Few numerical examples of linear time invariant (LTI) continuous system are taken up to illustrate these methods for Single input single output (SISO) systems. It can see from these examples that step responses of the reduced and original models match at not only at the steady state but also at the transient state as well.

<table>
<thead>
<tr>
<th>Order</th>
<th>t_r (sec.)</th>
<th>Overshoot(M_p)</th>
<th>time (T_s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4th</td>
<td>2.2602</td>
<td>0</td>
<td>3.9307</td>
</tr>
<tr>
<td>2nd</td>
<td>2.3592</td>
<td>0</td>
<td>4.1170</td>
</tr>
</tbody>
</table>

Chapter 4

New Composite Methods for Reduced Order Modeling: Multi Input and Multi Output System
4.1 Introduction

Systems with more than one input and/or more than one output are known as Multi-Input Multi-Output systems, or they are frequently known by the abbreviation MIMO. Almost all physical systems are multi input and multi output system by nature. So this chapter emphasis the extension of previous developed methods for multi input/multi output system.

4.2 Problem Definition

Consider the nth order linear time invariant general MIMO system [90] described in frequency domain by the rectangular function matrix

\[
[G(s)] = \begin{bmatrix}
g_{11}(s) & g_{12}(s) & \ldots & \ldots & g_{1r}(s) \\
g_{21}(s) & g_{22}(s) & \ldots & \ldots & g_{2r}(s) \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
g_{m1}(s) & g_{m2}(s) & \ldots & \ldots & g_{mr}(s)
\end{bmatrix}
\]

\[
= \frac{1}{D(s)} \begin{bmatrix}
n_{11}(s) & n_{12}(s) & \ldots & \ldots & n_{1r}(s) \\
n_{21}(s) & n_{22}(s) & \ldots & \ldots & n_{2r}(s) \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
n_{m1}(s) & n_{m2}(s) & \ldots & \ldots & n_{mr}(s)
\end{bmatrix}
\]

\[
= \frac{A_0 + A_1 s + \ldots + A_{n-1} s^{n-1}}{b_0 + b_1 s + \ldots + b_n s^n}
\]

Where \(D(s) = b_0 + b_1 s + \ldots + b_n s^n\) \hspace{1cm} (4.1)

Where \(A_i\) matrices of are appropriate dimension and \(b_i\) are scalar constants. The elements of the transfer function matrix \([G(s)]\) are rational functions of \(s\) of the form

\[
g_{it}(s) = \frac{n_{it}(s)}{D(s)} = \frac{a_0 + a_1 s + \ldots + a_{n-1} s^{n-1}}{b_0 + b_1 s + \ldots + b_n s^n}
\]

\hspace{1cm} (4.2)
The corresponding $r^{th}$ order model is of the form

$$[R(s)] = \frac{C_0 + C_1 s + \ldots + C_{k-1} s^{k-1}}{d_0 + d_1 s + \ldots + d_k s^k} \frac{n_{11}(s)}{D(s)}$$

(4.3)

Where $C_i$ are square matrices and $d_i$ are scalar constants.

First a common denominator is obtained of the Transfer matrix and then numerator terms of each element of Transfer function matrix are obtained.

**4.3 ORDER REDUCTION USING MODIFIED CAUER FORM AND FACTOR DIVISION METHOD FOR MULTI INPUT AND MULTI OUTPUT SYSTEM**

In this method denominator reduced using Modified Cauer Form and Numerator polynomial obtain by factor division method.

**Analysis of Physical System**

Consider the sixth order two input two output MIMO system taken from [90] described by the transfer function matrix $[G(s)]$

$$[G(s)] = \begin{bmatrix} \frac{2(s+5)}{(s+1)(s+10)} & \frac{(s+4)}{(s+2)(s+5)} \\ \frac{(s+10)}{(s+1)(s+20)} & \frac{(s+6)}{(s+2)(s+3)} \end{bmatrix} = \begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{bmatrix}$$

$$= \frac{1}{D(s)} \begin{bmatrix} n_{11}(s) & n_{12}(s) \\ n_{21}(s) & n_{22}(s) \end{bmatrix}$$

Where the common denominator $D(s)$ is

$$D_6(s) = (s+1)(s+2)(s+3)(s+5)(s+10)(s+20)$$

$$= s^6 + 41s^5 + 571s^4 + 3491s^3 + 10060s^2 + 13100s + 6000$$

$$= s^6 + 41s^5 + 571s^4 + 3491s^3 + 10060s^2 + 13100s + 6000$$
And numerators are

\[
\begin{align*}
n_{11}(s) &= 2s^5 + 70s^4 + 762s^3 + 3610s^2 + 7700s + 6000 \\
n_{12}(s) &= s^5 + 38s^4 + 459s^3 + 2182s^2 + 4160s + 2400 \\
n_{21}(s) &= s^5 + 30s^4 + 331s^3 + 1650s^2 + 3700s + 3000 \\
n_{22}(s) &= s^5 + 42s^4 + 601s^3 + 3660s^2 + 9100s + 6000
\end{align*}
\]

Determination denominators of reduced order system using Modified Cauer form

Now consider

\[
g_{11}(s) = \frac{n_{11}(s)}{D_6(s)} = \frac{2s^5 + 70s^4 + 762s^3 + 3610s^2 + 7700s + 6000}{s^6 + 415s^5 + 571s^4 + 3491s^3 + 10060s^2 + 13100s + 6000}
\]

The denominator of reduced order model \( R_k(s) \) can be evaluate using equations 3.19, 3.20, & 3.21 for \( k=2 \). Make modified Routh array & evaluate the quotients \( h_1, h_2, h_3, h_4, h_5, h_6, \) and \( H_1, H_2, H_3, H_4, H_5, H_6 \)

Modified Routh Array

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
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<td>13100</td>
<td>10060</td>
<td>3491</td>
<td>571</td>
<td>41</td>
<td>1</td>
</tr>
<tr>
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<td>7700</td>
<td>3610</td>
<td>1762</td>
<td>70</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>5400</td>
<td>6450</td>
<td>2729</td>
<td>501</td>
<td>39</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>-8</td>
<td>-240</td>
<td>-1848</td>
<td>-5200</td>
<td>-4800</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-15550</td>
<td>-12444671</td>
<td>-3509499</td>
<td>-3239961</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1.56 \times 10^{10}</td>
<td>-1.68 \times 10^{10}</td>
<td>-5.97 \times 10^9</td>
<td>-7.46 \times 10^7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1195129.8</td>
<td>3515449.86</td>
<td>3239953.28</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-2.42 \times 10^{14}</td>
<td>-2.62 \times 10^{14}</td>
<td>-8.92 \times 10^{13}</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-2221549.002</td>
<td>-2799434.66</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-2.49 \times 10^{20}</td>
<td>-1.98 \times 10^{20}</td>
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</tr>
<tr>
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<tr>
<td>-2.045 \times 10^{25}</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
h_1 = \frac{6000}{6000} = 1, \quad H_1 = \frac{2}{1} = 2
\]

\[
h_2 = \frac{5400}{-8} = -675, \quad H_2 = \frac{-4800}{1} = -4800
\]

\[
h_3 = 9.96 \times 10^{-7}, \quad H_3 = -7.46 \times 10^7
\]

\[
h_4 = 4.94 \times 10^{-9}, \quad H_4 = -8.92 \times 10^{13}
\]
h$_5$=8.92$\times$ 10$^{-15}$, \quad H$_5$= -1.98$\times$ 10$^{20}$,  

h$_6$=5.05$\times$ 10$^{-20}$, \quad H$_6$= -2.045$\times$ 10$^{25}$,  

With the knowledge of the first four quotients (k=2), h$_1$=1, H$_1$=2, h$_2$=-675, H$_2$=-4800 and with the help of equation (3.22) construct the inversion table as follows:

Inversion Table

\[
\begin{array}{ccc}
1 & 1 \\
-4800 & 1 \\
3240000 & 1 \\
3235200 & 1 \\
3235200 & 3240001 & 1 \\
\end{array}
\]

Hence the denominator d$_{11}(s)$ of reduced order model r$_{11}(s)$ is obtained as

\[D_{11}(s) = s^2 + 3240001s + 3235200\]

Now by using the factor division algorithm the following coefficients of numerator N$_2(s)$ of reduced order model are calculated

Consider D$_6(s)$/D$_2(s)$ gives

\[
\begin{array}{ccc}
6000 & 6000 & 13100 \\
3235200 & 3240001 \\
7091.09 & \\
3235200 & \\
\end{array}
\]

\[\alpha_0 = \frac{6000}{3235200} = 1.85 \times 10^{-3}, \quad \alpha_1 = \frac{7091.09}{3235200} = 2.19 \times 10^{-3}\]

Now considering N$_6(s)/D_6(s)/D_2(s)$

\[
\begin{array}{cc}
6000 & 7700 \\
1.85 \times 10^{-3} & 2.19 \times 10^{-3} \\
597.29 & \\
1.85 \times 10^{-3} & \\
\end{array}
\]

\[\alpha_0 = \frac{6000}{1.85 \times 10^{-3}} = 3243243.24, \quad \alpha_1 = \frac{597.29}{1.85 \times 10^{-3}} = 322859.45\]

Thus Reduced Numerator is given as
Rn_{11}(s) = 3243243.24 + 322859.45s

Thus the Reduced model is given as
\[ r_{11}(s) = \frac{322859.45s + 3243243.24}{s^2 + 3240001s + 3235200} \]

Now consider \( g_{12}(s) \)
\[ g_{12}(s) = \frac{n_{12}(s)}{D_{6}(s)} = \frac{s^5 + 38s^4 + 459s^3 + 2182s^2 + 4160s + 2400}{s^6 + 415s^5 + 571s^4 + 3491s^3 + 10060s^2 + 13100s + 6000} \]

The modified Routh array

The denominator of reduced order model \( R_k(s) \) can be evaluated using equations 3.19, 3.20, & 3.21 for \( k=2 \). Make modified Routh array & evaluate the quotients \( h_1, h_2, h_3, h_4, h_5, h_6 \) and \( H_1, H_2, H_3, H_4, H_5, H_6 \)

Modified Routh Array

\[
\begin{array}{cccccc}
6000 & 13100 & 10060 & 3491 & 571 & 41 \\
2400 & 4160 & 2182 & 459 & 38 & 1 \\
2700 & 4605 & 23435 & 476 & 38.5 & 1 \\
-0.5 & -17 & -163.5 & -445 & 300 & 1 \\
-87195 & -880556.5 & -2403238 & -1619961.5 & 1 \\
-485988895 & -7209715635 & -264166967 & -26158500.5 & 1 \\
-751201.4 & -23563263.61 & -1615268.213 & 1 \\
-4.23 \times 10^{13} & -6.16 \times 10^{14} & -1.96 \times 10^{13} & 1 \\
-126233782.23 & -1267193.805 & 1 \\
-2.48 \times 10^{19} & -2.47 \times 10^{20} & 1 \\
-124461605 & 1 \\
-3.07 \times 10^{28} & 1 \\
\end{array}
\]

\[ h_1 = \frac{6000}{2400} = 2.4, \quad H_1 = \frac{1}{1} = 1 \]
\[ h_2 = \frac{2700}{-0.5} = -5400, \quad H_2 = -\frac{300}{1} = -300 \]
\[ h_3 = 1.79 \times 10^{-4}, \quad H_3 = -26158500.5 \]
\[ h_4 = 1.77 \times 10^{-8}, \quad H_4 = -1.96 \times 10^{13} \]
\[ h_5 = 5.09 \times 10^{-13}, \quad H_5 = -2.47 \times 10^{20} \]
With the knowledge of the first four quotients \((k=2)\), \(h_1=2.4, H_1=-1\) \(h_2=-5400, H_2=-300\) and with the help of equation (3.22) construct the inversion table as follows

**Inversion Table**

\[
\begin{array}{ccc}
1 & -300 \\
1620001 & 1 \\
1619700 & 1 \\
3887280 & 1620002.4 & 1 \\
\end{array}
\]

Hence the denominator \(d_{11}(s)\) of reduced order model \(r_{11}(s)\) is obtained as

\[D_{12}(s) = s^2 + 1620002.4s + 3887280\]

Now by using the factor division algorithm the following coefficients of numerator \(N_2(s)\) of reduced order model are calculated

Consider \(D_6(s)/D_2(s)\) gives

\[
\begin{array}{ccc}
6000 & 13100 & \checkmark \\
3887280 & 1620002.4 \\
10599.53 \\
3887280 \\
\end{array}
\]

\[
\alpha_0 = \frac{6000}{3887280} = 1.54 \times 10^{-3}, \quad \alpha_1 = \frac{10599.53}{3887280} = 2.72 \times 10^{-3}
\]

Now considering \(N_{12}(s)/D_6(s)/D_2(s)\)

\[
\begin{array}{ccc}
2400 & 4160 & \checkmark \\
1.54 \times 10^{-3} & 2.72 \times 10^{-3} \\
-78.86 \\
1.54 \times 10^{-3} \\
\end{array}
\]

\[
\alpha_0 = \frac{2400}{1.54 \times 10^{-3}} = 1558441.5, \quad \alpha_1 = \frac{-78.86}{1.54 \times 10^{-3}} = -51272.72
\]

Thus Reduced Numerator is given as
\( r_{12}(s) = 1558441.5 - 51272.72s \)

Thus the Reduced model is given as
\[
\frac{r_{12}(s)}{s^2 + 1620002.41s + 3887280}
\]

Now consider \( g_{21}(s) \)
\[
g_{21}(s) = \frac{n_{21}(s)}{D_6(s)} = \frac{s^5 + 30s^4 + 331s^3 + 1650s^2 + 3700s + 3000}{s^6 + 415s^5 + 571s^4 + 3491s^3 + 10060s^2 + 13100s + 6000}
\]

The modified Routh array

The denominator of reduced order model \( R_k(s) \) can be evaluate using equations 3.19, 3.20, & 3.21 for \( k=2 \). Make modified Routh array and evaluate the quotients \( h_1, h_2, h_3, h_4, h_5, h_6 \) and \( H_1, H_2, H_3, H_4, H_5, H_6 \)

Modified Routh Array

\[
\begin{array}{cccccccc}
6000 & 13100 & 10060 & 3491 & 571 & 41 & 1 \\
3000 & 3700 & 1650 & 331 & 30 & 1 \\
5700 & 6760 & 2829 & 511 & 39 & 1 \\
-9 & -180 & -1179 & -3060 & -2700 \\
-107240 & -743871 & -1937489 & -1709961 & -1 \\
-4616897760 & -5231321479 & -2008451880 & -289548009 \\
-622361.6618 & -1890837.246 & -1703235.46 \\
-4.9317 \times 10^{14} & -5.47 \times 10^{14} & -1.80 \times 10^{14} \\
-1200958.166 & -1476004.022 & 1 \\
-2.65 \times 10^{20} & -2.16 \times 10^{20} \\
-497109.8 & 1 \\
-1.07 \times 10^{26} & & & & & & & \\
1 & & & & & & & \\
\end{array}
\]

\[
h_1 = \frac{6000}{2400} = 2.4, \quad H_1 = \frac{1}{1} = 1
\]

\[
h_2 = \frac{2700}{-0.5} = -5400, \quad H_2 = \frac{-300}{1} = -300
\]

\[
h_3 = 2.32 \times 10^{-5}, \quad H_3 = -289548009
\]

\[
h_4 = 1.26 \times 10^{-5}, \quad H_4 = -1.80 \times 10^{14}
\]

\[
h_5 = 4.53 \times 10^{-15}, \quad H_5 = -2.16 \times 10^{20}
\]
With the knowledge of the first four quotients \((k=2)\), \(h_1=2\), \(H_1=-1\) \(h_2=-633.33\), \(H_2=-2700\) and with the help of equation (3.18) construct the inversion table as follows

**Inversion Table**

\[
\begin{array}{ccc}
1 & & \\
-2700 & 1 & \\
1709991 & 1 & \\
1707291 & 1 & \\
3414582 & 1709993 & 1 \\
\end{array}
\]

Hence the denominator \(d_{11}(s)\) of reduced order model \(r_{21}(s)\) is obtained as
\[D_{21}(s) = s^2 + 1709993s + 3414582\]

Now by using the factor division algorithm the following coefficients of numerator \(N_2(s)\) of reduced order model are calculated

Consider \(D_6(s)/D_2(s)\) gives
\[
\begin{array}{ccc}
6000 & 13100 & \\
3414582 & 1709993 & \\
10095.25 & \\
3414582 & \\
\end{array}
\]

\[\alpha_0 = \frac{6000}{3414582} = 1.76 \times 10^{-3}, \quad \alpha_1 = \frac{10095.25}{3414582} = 2.95 \times 10^{-3}\]

Now considering \(N_{12}(s)/D_6(s)/D_2(s)\)
\[
\begin{array}{ccc}
3000 & 3700 & \\
1.76 \times 10^{-3} & 2.95 \times 10^{-3} & \\
& -1328040 & \\
& 1.76 \times 10^{-3} & \\
\end{array}
\]

\[\alpha_0 = \frac{3000}{1.76 \times 10^{-3}} = 1704545.455, \quad \alpha_1 = \frac{-1328040}{1.76 \times 10^{-3}} = -754772.73\]

Thus Reduced Numerator is given as
\[ R_{21}(s) = 1704545.455 - 754772.73s \]

Thus the Reduced model is given as
\[ r_{21}(s) = \frac{-754772.73s + 1704545.455}{s^2 + 1709993s + 3414582} \]

Now consider \( g_{22}(s) \)
\[ g_{21}(s) = \frac{n_{22}(s)}{D_6(s)} = \frac{s^5 + 42s^4 + 601s^3 + 3660s^2 + 9100s + 6000}{s^6 + 415s^5 + 571s^4 + 3491s^3 + 10060s^2 + 13100s + 6000} \]

**The modified Routh array**

The denominator of reduced order model \( R_k(s) \) can be evaluate using equations 4, 5, & 6 for \( k=2 \). Make modified Routh array & evaluate the quotients \( h_1, h_2, h_3, h_4, h_5, h_6 \), and \( H_1, H_2, H_3, H_4, H_5, H_6 \)

**Modified Routh Array**

\[
\begin{array}{cccccc}
6000 & 13100 & 10060 & 3491 & 571 & 41 \\
6000 & 9100 & 3660 & 601 & 42 & 1 \\
4000 & 6400 & 2890 & 529 & 40 & 1 \\
2 & 72 & 770 & 2700 & 2000 & 1 \\
-137600 & -1537110 & -5399471 & -3999960 & 1 \\
799922700 & 2.15 \times 10^{13} & 3074220072 & 275200002 & 1 \\
-369649875.6 & -5346593.904 & -3995226.514 & 1 \\
1.09 \times 10^{19} & 1.49 \times 10^{15} & -1.017 \times 10^{17} & 1 \\
-50531.12 & -546291.4 & 1 \\
-2.55 \times 10^{22} & -5.13 \times 10^{21} & 1 \\
-550962.11 & 1 \\
-2.83 \times 10^{27} & 1 \\
\end{array}
\]

\( h_1 = \frac{6000}{6000} = 1 \), \( H_1 = \frac{1}{1} = 1 \)

\( h_2 = \frac{2700}{-0.5} = -5400 \), \( H_2 = \frac{-300}{1} = -300 \)

\( h_3 = -1.72 \times 10^{-5} \), \( H_3 = 275200002 \)

\( h_4 = -3.3912 \times 10^{-11} \), \( H_4 = 1.017 \times 10^{17} \)

\( h_5 = -9.10 \times 10^{-19} \), \( H_5 = -5.13 \times 10^{21} \)

\( h_6 = 1.94 \times 10^{-22} \), \( H_6 = -2.83 \times 10^{27} \)
With the knowledge of the first four quotients \((k=2)\), \(h_1=1\), \(H_1=-1\) \(h_2=-2000\), \(H_2=2000\) and with the help of equation (3.18) construct the inversion table as follows:

**Inversion Table**

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2000</td>
<td></td>
</tr>
<tr>
<td>4000000</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>4002000</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>4002000</td>
<td>4002001</td>
<td>1</td>
</tr>
</tbody>
</table>

Hence the denominator \(d_{11}(s)\) of reduced order model \(r_{21}(s)\) is obtained as:

\[ D_{21}(s) = s^2 + 4002001s + 4002000 \]

Now by using the factor division algorithm the following coefficients of numerator \(N_2(s)\) of reduced order model are calculated.

Consider \(D_6(s)/D_2(s)\) gives:

\[
\begin{array}{c|c|c|c}
6000 & 13100 & \setminus \\
4002000 & 4002001 & 7099.99 \\
7099.99 & & 4002000 \\
\end{array}
\]

\[
\alpha_0 = \frac{6000}{4002000} = 1.49 \times 10^{-3}, \quad \alpha_1 = \frac{7099.99}{4002000} = 1.77 \times 10^{-3}
\]

Now considering \(N_{22}(s)/D_6(s)/D_2(s)\):

\[
\begin{array}{c|c|c|c}
6000 & 9100 & \setminus \\
1.49 \times 10^{-3} & 1.77 \times 10^{-3} & 1972.48 \\
1972.48 & & 1.49 \times 10^{-3} \\
\end{array}
\]

\[
\alpha_0 = \frac{6000}{1.49\times10^{-3}} = 4026845.638, \quad \alpha_1 = \frac{1972.48}{1.49\times10^{-3}} = 1323812.081
\]

Thus Reduced Numerator is given as:

\[ Rn_{22}(s) = 1323812.081s + 4026845.638 \]
Thus the Reduced model is given as

\[ r_{22}(s) = \frac{1323812.081s + 4026845.638}{s^2 + 4002001s + 4002000} \]

**Fig. 4.1: Comparison of Step Response of G_{11} and R_{11}**

**Fig. 4.2: Comparison of Step Response of G_{12} and R_{12}**
Fig. 4.3: Comparison of Step Response of $G_{21}$ and $R_{21}$

Fig. 4.4: Comparison of Step Response of $G_{22}$ and $R_{22}$
4.4 ORDER REDUCTION USING BASIC CHARACTERISTICS AND FACTOR DIVISION METHOD FOR MULTI INPUT AND MULTI OUTPUT SYSTEM

In this method denominator is reduced system is obtain by using basic characteristics [89] and numerator is obtain by using factor division method [98].

Analysis of Physical System
Consider the sixth order two input two output MIMO system taken from[90] described by the transfer function matrix $G(s)$

$$G(s) = \frac{2(s+5)}{(s+1)(s+10)} \frac{(s+4)}{(s+2)(s+5)} \frac{(s+2)(s+3)}{(s+6)} = \begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{bmatrix}$$

$$= \frac{1}{D(s)} \begin{bmatrix} n_{11}(s) & n_{12}(s) \\ n_{21}(s) & n_{22}(s) \end{bmatrix}$$

Where the common denominator $D(s)$ is

$$D(s) = (s+1)(s+2)(s+3)(s+5)(s+10)(s+20)$$

$$= s^6 + 41s^5 + 571s^4 + 3491s^3 + 10060s^2 + 13100s + 6000$$

And numerators are

$$n_{11}(s) = 2s^5 + 70s^4 + 762s^3 + 3610s^2 + 7700s + 6000$$
$$n_{12}(s) = s^5 + 38s^4 + 459s^3 + 2182s^2 + 4160s + 2400$$
$$n_{21}(s) = s^5 + 30s^4 + 331s^3 + 1650s^2 + 3700s + 3000$$
$$n_{22}(s) = s^5 + 42s^4 + 601s^3 + 3660s^2 + 9100s + 6000$$

Now considering

$$g_{11}(s) = \frac{n_{11}(s)}{D6(s)} = \frac{2s^5 + 70s^4 + 762s^3 + 3610s^2 + 7700s + 6000}{s^6 + 41s^5 + 571s^4 + 3491s^3 + 10060s^2 + 13100s + 6000}$$
Determination of denominator using system basic characteristics

Rise time = 2.2254
Settling time = 3.7944
Peak Overshoot = 0.9998
Peak Time = 8.5950
Assume $\xi = 0.99$

Natural frequency $= \frac{4}{\xi \times \text{settling time}} = \frac{4}{0.99 \times 3.7944} = 1.065$

Thus denominator

$$D_{11}(s) = s^2 + 2\xi \omega_n s + \omega_n^2 = s^2 + 0.99 \times 1.065 + (1.065)^2 = s^2 + 2.1087s + 1.342$$

Determination of numerator by factor division

**Step-I**

Considering $\frac{D_{11}(s)}{D_2(s)}$

<table>
<thead>
<tr>
<th></th>
<th>6000</th>
<th>13100</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.342</td>
<td>2.1087</td>
</tr>
<tr>
<td></td>
<td>3672.13</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.342</td>
<td></td>
</tr>
</tbody>
</table>

$$\alpha_0 = \frac{6000}{1.342} = 4470.2, \quad \alpha_1 = \frac{3672.13}{1.342} = 2736.3$$

**Step-II**

$$\frac{n_{11}(s)}{D_{11}(s)/D_2(s)}$$

<table>
<thead>
<tr>
<th></th>
<th>6000</th>
<th>7700</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4470.9</td>
<td>2736.37</td>
</tr>
<tr>
<td></td>
<td>4027.85</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4470.9</td>
<td></td>
</tr>
</tbody>
</table>

$$\alpha_0 = \frac{6000}{4470.9} = 1.342, \quad \alpha_1 = \frac{4027.85}{4470.9} = 0.9009$$
Thus reduced order model

\[ r_{11} = 1.342 + 0.9009s \]

Thus reduced order model

\[ R_{11}(s) = \frac{r_{11}(s)}{D_{11}(s)} = \frac{0.9009s + 1.342}{s^2 + 2.1087s + 1.342} \]

Now considering

\[ g_{12}(s) = \frac{n_{12}(s)}{d_{12}(s)} = \frac{s^5 + 38s^4 + 459s^3 + 2182s^2 + 4160s + 2400}{s^6 + 41s^5 + 571s^4 + 3491s^3 + 10060s^2 + 13100s + 6000} \]

Determination of numerator by factor division

Step-II

\[
\begin{array}{ccc}
2400 & 4160 \\
4470.9 & 2736.3 \\
2691.13 & \\
4470.9 & \\
\end{array}
\]

\[
\alpha_0 = \frac{2400}{4470.9} = 0.537, \quad \alpha_1 = \frac{2691.13}{4470.9} = 0.6019
\]

\[ r_{12} = 0.537 + 0.6019s \]

Thus reduced order model

\[ R_{12}(s) = \frac{r_{12}(s)}{D_{11}(s)} = \frac{0.6019s + 0.537}{s^2 + 2.1087s + 1.342} \]

Now considering

\[ g_{21}(s) = \frac{n_{21}(s)}{D_6(s)} = \frac{s^5 + 30s^4 + 331s^3 + 1650s^2 + 3700s + 3000}{s^6 + 41s^5 + 571s^4 + 3491s^3 + 10060s^2 + 13100s + 6000} \]
Determination of numerator by factor division

Step-II

\[ \frac{n_{21}(s)}{D_6(s)/D_2(s)} \]

\[
\begin{array}{c|c}
3000 & 3700 \\
4470.9 & 2736.3 \\
1863.9 & \\
4470.9 & \\
\end{array}
\]

\[ \alpha_0 = \frac{3000}{4470.9} = 0.6710, \quad \alpha_1 = \frac{1863.9}{4470.9} = 0.4168 \]

\[ r_{21} = 0.6710 + 0.4168s \]

Thus reduced order model

\[ R_{21}(s) = \frac{r_{21}(s)}{D_{11}(s)} = \frac{0.4168s + 0.6710}{s^2 + 2.1087s + 1.342} \]

Now considering

\[ g_{22}(s) = \frac{n_{22}(s)}{D_6(s)} = \frac{s^5 + 42s^4 + 601s^3 + 3660s^2 + 9100s + 6000}{s^6 + 41s^5 + 571s^4 + 3491s^3 + 10060s^2 + 13100s + 6000} \]

Determination of numerator by factor division

Step-II

\[ \frac{n_{22}(s)}{D_6(s)/D_2(s)} \]

\[
\begin{array}{c|c}
6000 & 9100 \\
4470.9 & 2736.3 \\
5427.85 & \\
4470.9 & \\
\end{array}
\]

\[ \alpha_0 = \frac{6000}{4470.9} = 1.342, \quad \alpha_1 = \frac{5427.85}{4470.9} = 1.2140 \]
Thus reduced order model

\[ R_{22}(s) = \frac{r_{22}(s)}{D_{11}(s)} = \frac{1.2140s + 1.342}{s^2 + 2.1087s + 1.342} \]

Fig. 4.5 Comparison of Step Responses of \( G_{11} \) and \( R_{11} \)

Fig. 4.6 Comparison of Frequency Responses of \( G_{11} \) and \( R_{11} \)
Fig. 4.7 Comparison of Step Responses of $G_{12}$ and $R_{12}$

Fig. 4.8 Comparison of Frequency Responses of $G_{12}$ and $R_{12}$
Fig. 4.9 Comparison of Step Responses of $G_{21}$ and $R_{21}$

Fig. 4.10 Comparison of Frequency Responses of $G_{21}$ and $R_{21}$
Fig. 4.11 Comparison of Step Responses of $G_{22}$ and $R_{22}$

Fig. 4.12 Comparison of Frequency responses of $G_{22}$ and $R_{22}$
4.5 Conclusion

This chapter deals with the extension of two methods, Modified cauer form, & Factor Division, Basic characteristics & factor division as described in chapter 3 for single input and single output system for multi input and multi output system. Few numerical examples of linear time invariant (LTI) continuous system are taken up to illustrate these methods for multi input multi output (MIMO) systems. It is obvious from these examples that step responses of the reduced and original models match at the steady state but also at the transient state very well.
CHAPTER 5
REDUCTION OF LINEAR TIME INVARIANT DISCRETE TIME SYSTEMS BY MIXED METHODS

5.1 INTRODUCTION

In previous chapters it has been discussed about brief review and proposed new methods for continuous systems for order reduction of SISO and MIMO. As per concern of reduction of higher order discrete time systems have same arguments as for continuous systems. However large usage of fast digital computers in design and implementation of control systems have increased interest in the model reduction of discrete systems as well. Therefore, this chapter the reduction of discrete time systems methods is consider separately.

The reduced orders of discrete time system can be achieved by transform it into continuous time system and then by applying continuous time reduction methods. Firstly discrete time system converted into continuous time system by using suitable transformation such as the bilinear transformation \( z = \frac{1+w}{1-w} \), homographic transformation \( z = p(A+Bp) \) and \( p \) & \( w \) are constant.

The transformation of a \( z \) plane into \( w \) plane by using bilinear transformation and application of continuous time reduction methods used for obtaining reduced order model which do not zero initial value of step response even through initial condition of the original step response is zero. Therefore synthetic division separately on the numerator and denominator polynomials.

In the proposed methods the reduced denominator polynomial in continuous domain (\( w \) domain) is derived using a stability reduction method and numerator is then obtained by some other appropriate method. Again
this reduced order system converted into discrete system by using inverse bilinear transformation, separately on numerator and denominator polynomials to give the desired result. A steady state correction is applied to match the final the final values of responses of original and reduced systems.

In this chapter are two different methods are proposed for reduction of discrete time systems. The denominator polynomial is obtained using basic characteristics, modified Cauer method while numerator is obtained using factor division method.

5.2 Problem Statement

Given a high order discrete -time stable system $G_0(z)$ of order ‘n’, described by the z-transfer function

$$G_0(z) = \frac{N(z)}{D(z)} = \frac{a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}}{b_0 + b_1 z + \cdots + b_n z^n} \quad (5.1)$$

And reduced model $R(z)$ of order ‘k’ (k<n) as given eqn. (5.2) is obtained from eqn. (5.1) and has transfer function

$$R(z) = \frac{N_k(z)}{D_r(z)} = \frac{c_0 + c_1 z + \cdots + c_{k-1} z^{k-1}}{d_0 + d_1 z + \cdots + d_k z^k} \quad (5.2)$$

In such a way that $G_0(z)$ and $R(z)$ exhibit almost same responses and characteristics on the application of same input. $a_i$, $b_i$, $c_i$ and $d_i$ are scalar constants.

5.3 Discrete System Reduction Using Basic Characteristics And Factor Division Method

The system basic characteristics [89] are used to reduce the transformed denominator polynomial and reduced numerator polynomial is obtained by using Factor Division Method [98]. The reduce system is
converted back into discrete system using inverse bilinear transformation. The method demonstrated by an example.

### Reduction Method
The method carried out in this section is same in all the following methods of this chapter except step 3 and is obtain in following steps:

**Step-1**

Apply bilinear transformation \( z = \frac{1 + w}{1 - w} \) separately in the numerator and denominator polynomial of 5.1 using synthetic division. This convert \( G_0(z) \) into \( G(w) \) as

\[
N(w) = N(z) \bigg|_{z = \frac{1 + w}{1 - w}} \quad \text{and} \quad D(w) = D(z) \bigg|_{z = \frac{1 + w}{1 - w}}
\]

\[
N(w) = \frac{N(z)}{(1 - w)^n} = 0 \quad \text{(5.3)}
\]

\[
D(w) = \frac{D(z)}{(1 - w)^n} = 0 \quad \text{(5.4)}
\]

**Step-2:**

Determination of the denominator polynomial for the \( k^{th} \) order reduced model using basic characteristics of original system by the following procedure

- Firstly determine the basic characteristics of original system
- Then assume damping ratio \( \zeta = 0.99 \) for an aperiodic or almost periodic system, and number oscillations before the system settles = 1
- Determine the natural frequency \( \omega_n \) using
  \[
  \omega_n = \frac{4}{w \cdot T_w}
  \]
- Obtain the reduced order denominator as
  \[
  D_2(w) = w^2 + 2^* \omega_n + \omega_n^2
  \]

**Step-2:** Determination of the numerator of \( k^{th} \) order reduced model using Factor Division algorithm [98]

After obtaining the reduced denominator, the numerator of the reduced
model is determined as follows

\[ N_k(w) = \frac{N(w)}{D(w)} \times D_k(w) = \frac{N(w)}{D(w)} \frac{1}{D_k(w)} \]  \hspace{1cm} (5.5)

Where \( D_k(w) \) is reduced order denominator

There are two approaches for determining of numerator of reduced order model

(i) By performing the product of \( N(w) \) and \( D_k(w) \) as the first row of factor division algorithm and \( D(w) \) as the second row up to \( w^{k-1} \) terms are needed in both rows.

(ii) By expressing \( \frac{N(w)D_k(w)}{D(w)} \) as \( \frac{N(w)}{[D(w)/D_k(w)]} \) and using factor division algorithm twice; the first time to find the term up to \( s^{k-1} \) in the expansion of \( \frac{D(w)}{D_k(w)} \) (i.e. put \( D(w) \) in the first row and \( D_k(w) \) in the second row, using only terms up to \( w^{k-1} \), and second time with \( N(w) \) in the first row and the expansion \( \frac{D(w)}{D_k(w)} \) in the second row.

Therefore the numerator \( N_k(w) \) of the reduced order model \( R_k(w) \) in eq.(5.2) will be the series expansion of

\[ \frac{N(w)}{D(w)} \frac{1}{D_k(w)} = \frac{\sum_{i=0}^{k-1} c_i w^i}{\sum_{i=0}^{k} d_i w^i} \]

About \( w=0 \) up to term of order \( w^{k-1} \).

This is easily obtained by modifying the moment generating which uses the familiar routh recurrence formulae to generate the third, fifth, and seventh etc rows as,
\[ \alpha_0 = \frac{g_0}{h_0} \quad \frac{g_0}{h_0} \quad \frac{g_1}{h_1} \quad \frac{g_2}{h_2} \quad \cdots \quad \frac{g_{k-1}}{h_{k-1}} \]
\[ \alpha_1 = \frac{l_0}{h_0} \quad \frac{l_0}{h_0} \quad \frac{l_1}{h_1} \quad \frac{l_2}{h_2} \quad \cdots \quad \frac{l_{k-2}}{h_{k-2}} \]
\[ \alpha_2 = \frac{m_0}{h_0} \quad \frac{m_0}{h_0} \quad \frac{m_1}{h_1} \quad \frac{m_2}{h_2} \quad \cdots \quad \frac{m_{k-3}}{h_{k-3}} \]

\[ \alpha_{k-2} = \frac{p_0}{h_0} \quad \frac{p_0}{h_0} \quad \frac{p_1}{h_1} \]
\[ \alpha_{k-1} = \frac{q_0}{h_0} \quad \frac{q_0}{h_0} \]

Where

\[ l_i = g_{i+1} - \alpha_0 \cdot h_{i+1}, \quad i=0,1,2, \ldots \]

\[ m_i = l_{i+1} - \alpha_1 \cdot h_{i+1}, \quad i=0,1,2, \ldots \]

\[ p_0 = p_1 - \alpha_{k-2} h_1 \]

Therefore, the numerator \( N_k(w) \) of eq.(5.3) is given by

\[ N_k(w) = \sum_{i=0}^{k-1} \alpha_i w^i \quad (5.6) \]

**Step 3**
Construct reduced order model

\[ R(w) = \frac{N_k(w)}{D_k(w)} \]

**Step 4**
Applying the inverse bilinear transformation

\[ w = \frac{z-1}{z+1} \quad \text{on} \quad R(w) \quad \text{to obtain} \quad R(z) \]

**Step 5**
Remove steady state error by evaluating the gain correction factor

\[ K = \frac{g_0(z)}{R(z)} \bigg|_{z=1} \]
Analysis of Physical System

Consider the original 4th order discrete system [89] is

\[
G_0(z) = \frac{N(z)}{D(z)} = \frac{0.54377z^3 - 0.40473 + 0.31921z - 0.216608}{z^4 - 1.361178z^3 + 0.875599z^2 - 0.55120z + 0.282145}
\]

This system is to be reduced to 2nd order ROM. Applying bilinear transformation separately on Numerator and Denominator, using synthetic division, the equivalent continuous system becomes

\[
G(w) = \frac{1.4879w^3 + 1.0778w^2 + 1.568w + 0.24526}{4.0707w^4 + 4.429w^3 + 5.9417w^2 + 1.2503w + 0.24476}
\]

Settling time

\[
t_w = 33.76667
\]
\[
\xi = 0.99
\]
\[
t_w = \frac{4}{\xi * w_n}
\]
\[
w_n = \frac{4}{\xi * t_w} = \frac{4}{0.99 * 33.7667} = 0.1196
\]

\[
D_2(w) = w^2 + 2 * \xi * w_n w + w_n^2
\]
\[
= w^2 + 2 * 0.99 * 0.1196w + (0.1196)^2
\]
\[
= w^2 + 0.236w^2 + 0.0143
\]

\[
D_4(w)/D_2(w)
\]
\[
\begin{array}{cc}
0.24476 & 1.2503 \\
0.0143 & 0.236 \\
-2.78 & \\
0.0143 & \\
\end{array}
\]

\[
\alpha_0 = 17.11, \alpha_1 = -194.40
\]
N₄(w)/D₄(w)/D₂(w)

\[
\begin{align*}
0.24476 & \quad 1.568 \\
17.11 & \quad -194.40 \\
4.35 & \\
17.11 \\
\end{align*}
\]

\[\alpha_0 = 0.0143 \quad \alpha_1 = 0.2542\]

\[
R_2(w) = \frac{0.2542w + 0.0143}{w^2 + 0.236w + 0.0143}
\]

\[
R_2(z) = R_2(w) \bigg| _{w = \frac{z-1}{z+1}} = \frac{0.268z - 0.2399}{1.2503z^2 - 1.9714z + 0.7783}
\]

After steady state correction

\[
R_2(z) = \frac{0.382z - 0.3420}{1.2503z^2 - 1.9714z + 0.7783}
\]

Fig. 5.1 Comparison of Step Responses of Original and Reduced Discrete System

5.4 Reduction of Discrete Systems Using Combination of Modified Cauer Form and Factor Division Method

In this method of model reduction which combines the advantages of Modified Cauer Form and factor division method. The denominator is obtained using MCF and numerator is obtained using Factor Division Method.
The Steps taken to reduce the original system are same as in section 5.2 except step 2 where the MCF is used to determine denominator coefficient and it is described in Method-2 of chapter 3

**Analysis of Physical System**

Consider the original 4th order discrete system from the literature [101]

\[
G_o(z) = \frac{N(z)}{D(z)} = \frac{0.54377z^3 - 0.40473 + 0.31921z - 0.216608}{z^4 - 1.361178z^3 + 0.875599z^2 - 0.55120z + 0.282145}
\]

This system is to be reduced to 2nd order ROM. Applying bilinear transformation separately on Numerator and Denominator, using synthetic division, the equivalent continuous system becomes

\[
G(w) = \frac{1.4879w^3 + 1.0778w^2 + 1.568w + 0.24526}{4.0707w^4 + 4.429w^3 + 5.9417w^2 + 1.2503w + 0.24476}
\]

Determination using denominator using MCF

<table>
<thead>
<tr>
<th></th>
<th>0.24476</th>
<th>1.2503</th>
<th>5.9417</th>
<th>4.492</th>
<th>4.0707</th>
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<tr>
<td></td>
<td>0.24526</td>
<td>1.568</td>
<td>1.0778</td>
<td>1.4879</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-0.31450</td>
<td>4.9359</td>
<td>3.0071</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-3.396</td>
<td>-5.7761</td>
<td>0.7113</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>5.4708</td>
<td>2.9412</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
h_1 = \frac{0.24476}{0.24526} = 0.99796, \quad H_1 = \frac{1.4879}{1} = 1.4879
\]

\[
h_2 = \frac{-0.31450}{-3.396} = 0.0926, \quad H_2 = \frac{0.7113}{1} = 0.7113
\]
Inversion Table

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0.7113</td>
<td>1</td>
</tr>
<tr>
<td>0.658</td>
<td>1</td>
</tr>
<tr>
<td>0.8092</td>
<td>1.4879</td>
</tr>
<tr>
<td>0.8075</td>
<td>1.5504</td>
</tr>
</tbody>
</table>

Thus denominator

\[ D_2(w) = w^2 + 1.5504w + 0.8075 \]

Determination of Numerator using Factor division

\[ D_4(w)/D_2(w) \]

\[
\begin{array}{ll}
0.24476 & 1.2503 \\
0.8075 & 1.5504 \\
0.7803 & \\
0.8075 & \\
\end{array}
\]

\[ \alpha_0 = 0.3031, \alpha_1 = 0.9963 \]

\[ N_4(w)/D_4(w)/D_2(w) \]

\[
\begin{array}{ll}
0.24526 & 1.568 \\
0.3031 & 0.9963 \\
0.7618 & \\
0.3031 & \\
\end{array}
\]

\[ \alpha_0 = 0.8092, \alpha_1 = 2.513 \]

\[ R_2(w) = \frac{2.513w + 0.08092}{w^2 + 1.5504w + 0.8075} \]

\[ R_2(z) = R_2(w) \big| w = \frac{z-1}{z+1} \]

\[ = \frac{3.3142z-1.7118}{3.3575z^2-0.259z-0.7429} \]

After steady state correction

\[ R_2(z) = \frac{3.397z-1.7154}{3.357z^2-0.259z-0.7729} \]
5.5 CONCLUSION

In this chapter two different methods have been proposed for reducing the discrete control systems namely reduction using basic characteristics and factor division method and reduction of discrete systems using combination of modified Cauer form & factor division method. All these methods may be called as indirect methods as the given discrete system is first transformed into $w$ domain and reduction is carried out by continuous time methods. The inverse bilinear transformation finally leads to reduced order model. The denominator is determined using basic characteristics and Modified Cauer form while numerator is determined using Modified Cauer form.

Fig. 5.2 Comparison of Step Responses of Original and Reduced Discrete System
6.1 INTRODUCTION
In previous chapters several model order reduction methods have been developed both in frequency and time domain which approximate certain characteristics of original system. The quality of reduced order model is judged by the way it is utilized and the degree of its success in representing the desired characteristics of the system. One of the main objective of order reduction is to design a controller of low order which can effectively control the original high order system so that overall system is of low order and is easy to understand. It is thus important that the model order reduction methods should reduce the high order controller to a low order controller without incurring too much error.

There are two common approaches for controller design. One is direct approach is to obtain the controller on the basis of reduced order model called process reduction [10] and second is indirect approach is to obtain a controller for full order system and then to reduce closed loop response of higher order controller with unity feedback called controller reduction[10]

---

**Fig.6.1 Schematic of Approaches for Controller Design**
In the proposed method a reduced order model has been obtained by pole clustering and factor division algorithm. The controller is designed for the reduced order model and is connected in cascade with the original system to obtain the desired specifications. The proposed method assures the stability of the system. The proposed method is illustrated by a numerical example

---

**Fig.6.2 (a) Block Schematic of Closed Loop Configuration of High Order System with Controller**

**Fig.6.2 (b) Block Schematic of Closed Loop Configuration of Reduced Order System with Reduced Order Controller**

**Fig.6.2 (c) Block Schematic of Open Loop Configuration of Reference Model**

### 6.2 Choice of Reference Model

The reference model may be chosen to meet the following design specifications [68]
• The time domain specifications such as rise time overshoot, settling time and study state error.
• The frequency domain specifications such as bandwidth, cut off rate, gain margin and phase margin.
• The complex domain specifications such as damping ratio, damping factor, un-damped natural frequency and location of closed poles.

The controller should be designed such that the closed loop response of controlled System approximately same as reference model.

6.3 STATEMENT OF THE PROBLEM

PID Controller Transfer Function

PID controller can be mathematically represented as [55]

$$u(t) = k[e(t) + \frac{1}{T_i} \int_0^t e(\tau) d\tau + T_d \frac{de(t)}{dt}] \tag{6.1}$$

Where $u(t)$ and $e(t)$ denote the control & error signals of the system, $k$ is the proportion gain, $T_i$ and $T_d$ represents the integral & derivative time constants respectively. The corresponding PID controller transfer function is given as

$$G_c(s) = k[1 + \frac{1}{T_i s} + T_d s] \tag{6.2}$$

Equation (6.2) can be rewritten as

$$G_c(s) = k_1 + \frac{k_2}{s} + k_3 s \tag{6.3}$$

$K_1, K_2$ and $k_3$ are represents the proportional, integral and derivative gain values of the controller.

Higher Order Transfer Function

Let higher order system or process whose performance is
unsatisfactory may be described by the transfer function

\[ G_n(s) = \frac{N(s)}{D(s)} = \frac{a_0 + a_1 s + a_2 s^2 + \cdots + a_{n-1} s^{n-1}}{b_0 + b_1 s + b_2 s^2 + \cdots + b_n s^n} \]  

(6.4)

Where \( a_i, b_i \ 0 \leq i \leq n-1 \) and \( b_i \ 0 \leq i \leq n \) are known scalar constants.

And a reference model having the desired performance is given.

### Lower Order Transfer Function

To find a \( k^{th} \) lower order model for the above continuous system, where \( k < n \) in the following form, such that the lower order model retains the characteristics of the original system and approximates its response as closely as possible for the same type of inputs.

\[ R_k(s) = \frac{N(s)}{D(s)} = \frac{c_0 + c_1 s + c_2 s^2 + \cdots + c_{k-1} s^{k-1}}{d_0 + d_1 s + d_2 s^2 + \cdots + d_k s^k} \]  

(6.5)

Where; \( c_i \ 0 \leq i \leq k-1 \) and \( d_i \ 0 \leq i \leq k \) are known scalar constants.

Objective is to derive a controller such that the performance of the augmented process matches with that of the reference model. To reduce the computational complexities and difficulties of implementation, the higher order of the system is reduced into lower second order system. And PID controller is also derived for reduced order system.

### 6.4 REDUCTION METHOD

The reduction procedure for getting the \( k^{th} \)-order reduced models consists of the following two steps:

#### Determination of Denominator

Determination of the denominator polynomial for the \( k^{th} \) -order reduced model using pole clustering techniques [101]. The criterion for grouping the poles in one particular cluster is based on relative distance between the poles
and desired order in the process of reduced order modeling. Since each cluster may be finally replaced by a single real pole, the following rules are used for clustering the poles to get the denominator polynomial for reduced order models.

i. Separate cluster should be made for real poles and complex pole.

ii. Poles on the \( jw \) -axis have to be retained in the reduced order model.

The cluster centre can be formed using a simple method known as ‘inverse distance measure’, which is explained as follows:

Let, \( r \) real poles in one cluster then the Inverse Distance Measure (IDM) criterion identifies the cluster centre as

\[
p_c = \left( \frac{1}{\sum_{i=1}^{r} \frac{1}{p_i}} \right)^{-1}
\]  

(6.6)

where \( p_c \) is cluster centre from \( r \) real poles of the original system.

Then denominator polynomial for order reduced model can be obtained as

\[
D_k(s) = (s - p_{c1})(s - p_{c2}) \ldots \ldots \ldots \ldots \ldots \ldots (s - p_{ck})
\]  

(6.7)

Where \( p_{c1}, p_{c2}, \ldots, p_{ck} \) are 1\(^{st}\), 2\(^{nd}\), \ldots, k\(^{th}\) cluster centre

**Determination of Numerator**

Determination of the numerator of \( k^{th} \) order reduced model using Factor Division algorithm [98]. After obtaining the reduced denominator, the numerator of the reduced model is determined as follows

\[
N_k(s) = \frac{N(s)}{D(s)} \times D_k(s) = \frac{N(s)}{\frac{D(s)}{D_k(s)}}
\]  

(6.8)

Where \( D_k(s) \) is reduced order denominator

There are two approaches for determining of numerator of reduced order model.
(i) By performing the product of \( N(s) \) and \( D_k(s) \) as the first row of factor division algorithm and \( D(s) \) as the second row up to \( s^{k-1} \) terms are needed in both rows.

(ii) By expressing \( N(s)D_k(s)/D(s) \) as \( N(s)/[D(s)/D_k(s)] \) and using factor division algorithm twice; the first time to find the term up to \( s^{k-1} \) in the expansion of \( D(s)/D_k(s) \) (i.e. put \( D(s) \) in the first row and \( D_k(s) \) in the second row, using only terms up to \( s^{k-1} \)), and second time with \( N(s) \) in the first row and the expansion \([D(s)/D_k(s)]\) in the second row.

Therefore the numerator \( N_k(s) \) of the reduced order model \( (R_k(s)) \) in eq.(6.5) will be the series expansion of

\[
\frac{N(s)}{D(s)} = \frac{\sum_{i=0}^{k-2} c_i s^i}{\sum_{i=0}^{k} d_i s^i}
\]

About \( s=0 \) up to term of order \( s^{k-1} \).

This is easily obtained by modifying the moment generating[98].which uses the familiar routh recurrence formulae to generate the third, fifth, and seventh etc rows as,

\[
\begin{align*}
\alpha_0 & = \frac{g_0}{h_0} < \frac{g_0}{h_0} \frac{g_1}{h_1} \frac{g_2}{h_2} \cdots \frac{g_{k-1}}{h_{k-1}} \\
\alpha_1 & = \frac{l_0}{h_0} < \frac{l_0}{h_0} \frac{l_1}{h_1} \frac{l_2}{h_2} \cdots \frac{l_{k-2}}{h_{k-2}} \\
\alpha_2 & = \frac{m_0}{h_0} < \frac{m_0}{h_0} \frac{m_1}{h_1} \frac{m_2}{h_2} \cdots \frac{m_{k-3}}{h_{k-3}} \\
\alpha_{k-2} & = \frac{p_0}{h_0} < \frac{p_0}{h_0} \frac{p_1}{h_1} \\
\alpha_{k-1} & = \frac{q_0}{h_0} < \frac{q_0}{h_0}
\end{align*}
\]
Where
\[ l_i = g_{i+1} - \alpha_0 \cdot h_{i+1}, \quad i=0,1,2, \ldots \]
\[ m_i = l_{i+1} - \alpha_1 \cdot h_{i+1}, \quad i=0,1,2, \ldots \]

Therefore, the numerator \( N_k(s) \) of eq. (6.5) is given by
\[
N_k(s) = \sum_{i=0}^{k-1} \alpha_i \cdot s^i \quad (6.10)
\]

**General Algorithm for designing the PID controller**

**Step 1:** Construction of a reference model \( M(s) \) on the basis of specifications whose closed loop system must approximate to that of the original closed loop response. Let it be specified as:
\[
M(s) = \frac{a_0 + a_1 s + \ldots + a_m s^m}{b_0 + b_1 s + \ldots + b_n s^n} \quad (6.11)
\]

**Step 2:** Determine an equivalent open loop specification model \( \overline{M(s)} \)
\[
\overline{M(s)} = \frac{M(s)}{1 + M(s)} \quad (6.12)
\]

**Step 3:** Specified the structure of the controller. Assume the controller structure is given by
\[
R_c(s) = \frac{p_0 + p_1 s + \ldots + p_k s^k}{q_0 + q_1 s + \ldots + q_l s^l} \quad (6.13)
\]

**Step 4:** For determining the unknown controller parameters, the response of the closed loop system is matched with reference model as
\[
R_c(s) R_P(s) = \overline{M(s)}
\]
\[
R_c(s) = \overline{M(s)} \frac{R_c(s)}{R_c(s)} = \sum_{i=0}^{\infty} e_i s^i \quad (6.14)
\]

Where \( e_i \) are the power series expansion coefficients about \( s=0 \).
Step 5: The unknown control parameters $p_i$ and $q_i$ are obtained by equating the equation (6.13) and (6.14) in Padé sense

\[ p_0 = q_0 e_0 \]
\[ p_1 = q_0 e_1 + q_1 e_0 \]
\[ p_2 = q_0 e_2 + q_1 e_1 + q_2 e_0 \]
\[ \vdots \]
\[ p_i = q_0 e_i + q_1 e_{i-1} + \cdots q_i e_0 \]
\[ 0 = q_0 e_{i+1} + q_1 e_i + \cdots q_{i+1} e_0 \]
\[ \vdots \]
\[ 0 = q_0 e_{i+j} + q_1 e_{i+j-1} + \cdots q_j e_0 \]

The controller having the desired structure is obtained by solving above linear equations

Step 6: After obtaining the controller parameters, the close loop transfer function can be obtained as

\[ R_{cl} = \frac{R_c(s)R_k(s)}{1 + R_c(s)R_k(s)} \]

Analysis of Physical Student

The proposed method has been applied by taking numerical values from the literature [71].

Example-: Consider fourth order system [71]

\[ G(s) = \frac{72 + 54s + 12s^2 + s^3}{100 + 180s + 97s^2 + 18s^3 + s^4} \]

Step-1

Determination of Denominator of reduced order

Two cluster centers from the real poles -1,-2,-5,-10 can be formed equation since
$P_c1=-1.01$

$P_c2=-2.003$

$D_2(s)=(s+1.01)(s+2.003)$

$D_2(s)=2.023+3.013s+s^2$

**Step-2**

Using the factor division method the numerator of reduced order model has been obtained

Consider $D_4(s)/D_2(s)$

\[
\alpha_0 = 49.4 < \frac{100}{2.023} \quad \frac{180}{3.013}
\]

\[
\alpha_1 = 15.75 < \frac{31.5}{2}
\]

Considering $N_4/D_4(s)/D_2(s)$

\[
\alpha_0 = 1.46 < \frac{72}{49.4} \quad \frac{54}{15.75}
\]

\[
\alpha_1 = 0.627 < \frac{31.005}{49.4}
\]

Thus Reduced Numerator is given as

$N_2(s) = 1.46+0.627s$

The Reduced model will be as follows

$R_2(s) = \frac{1.46+0.627s}{2.023+3.013s+s^2}$

**PID Controller Design Using Reduced Order Model**

Consider a reference model

$M(s) = \frac{4.242s+25}{s^2+7.07s+25}$

The equivalent open loop transfer function is

$\bar{M}(s) = \frac{4.242s+25}{s^2+2.828s}$
The reduced controller transfer function is

\[ \frac{M(s)}{R_2(s)} = \frac{12.285 + 10.641s - 1.236s^2}{s} \]

It is compared by PID controller transfer function

\[ K_1 + \frac{K_2}{s} + K_3s \]

And the value of controller parameters are obtained as

\[ K_1=10.641, K_2=12.285, K_3=-1.236 \]

Corresponding closed loop transfer function is

\[ \frac{R_{CL}(s)}{s} = \frac{s + 5.114s^2 + 23.28s + 17.7}{s^3 + 8.114s^2 + 25.28s + 17.7} \]

![Step Response](image)

**Fig. 6.3** Comparison of original and reduced order system for step response
Fig. 6.4 Comparison of desired reference model and close loop system for step response

**Table 6.1 A Comparison of Different Parameters of the Systems**

<table>
<thead>
<tr>
<th>Sr.No</th>
<th>Parameter</th>
<th>G(s)</th>
<th>R₂(s)</th>
<th>R₈CL(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Rise Time(Sec)</td>
<td>2.3149</td>
<td>2.2800</td>
<td>0.2915</td>
</tr>
<tr>
<td>2</td>
<td>Settling Time(Sec)</td>
<td>4.0227</td>
<td>4.0028</td>
<td>1.1268</td>
</tr>
<tr>
<td>3</td>
<td>Peak Time(Sec)</td>
<td>7.7084</td>
<td>7.7392</td>
<td>0.6601</td>
</tr>
<tr>
<td>4</td>
<td>Overshoot</td>
<td>0</td>
<td>0</td>
<td>5.9253</td>
</tr>
<tr>
<td>5</td>
<td>Peak</td>
<td>0.7196</td>
<td>0.7214</td>
<td>1.05912</td>
</tr>
</tbody>
</table>

6.5 Conclusion

In this chapter rational controller has been designed for different systems and suitability of the proposed reduction method has been examined. The controllers are designed using direct and indirect approaches. The controller
parameters have been found using approximate model matching techniques in Pade sense. Results of closed loop step and frequency response of original system and reduced system are compared with the response of reference model. It has been observed that closed loop step response is in close agreement with the step response of reference model.
CHAPTER 7
CONCLUSIONS AND EMERGING TRENDS

7.1 INTRODUCTION
The present research work deals with the development of new methods for model order reduction of linear dynamic system for single input and single output as well as multi input and multi output system. The proposed methods deal with the frequency domain model order reduction methods. The suitability of one of the proposed method has been examined for the controller design.

7.2 CONCLUSIONS
The first chapter gives the basic introduction about the model order reduction. The need of model order reduction and application of order reduction is also mentioned. The statement of model order reduction for SISO and MIMO systems is given in time and frequency domain.

In chapter 2, a wide variety of model order reduction methods in both time and frequency domain proposed by several authors during last few decades, are reviewed. The reduced orders models obtain by individual methods differ from each other in quality. Individual method has own advantages and disadvantages. So their advantages and disadvantages are removed by using mixed methods.

In chapter 3, the mixed methods of model order reduction which combine the advantages of Mihailov stability criterion, Padé approximation, Modified Cauer form, factor division, Modified Pole Clustering, Order Reduction using basic characteristics have been discussed. In the first section the method of Mihailov stability criterion and Padé approximation, Modified Cauer form and factor division, Modified Pole Clustering and Factor division, Order Reduction using basic characteristics and Factor division have been
discussed. Few numerical examples of linear time invariant (LTI) continuous system are taken up to illustrate these methods for single input single output. It can be seen from the illustrative examples that step responses of the original models and reduced models match in the transient and steady state as well.

In Chapter 4 proposed methods for SISO as discussed in chapter 3 are extended for MIMO systems. In this chapter two methods, first one is order reduction using modified Cauer form and factor division, second one is order reduction using basic characteristics and factor division methods for Multi Input And Multi Output System are discussed.

In chapter 5, there are two methods are proposed for reducing the discrete time systems namely Discrete time system order reduction using basic characteristics and factor division and Order reduction of discrete systems using combination of modified cauer form and factor division method. These methods may be called as indirect approach as given discrete system is first transformed into w-domain and then order reduction is carried out by continuous time methods. The inverse bilinear transformation finally leads to the reduced order model. The denominator of reduced order model is determined using basic characteristics (Rise time, Peak time, Settling time and Peak overshoot) and Modified Cauer Form while the numerator is determined using Factor Division Methods.

In chapter 6, controllers have been designed for different systems and suitability of the proposed reduced method is examined. The two approaches are proposed for designing of controllers using approximate model matching techniques base on direct and indirect approaches. The reduced order models are obtained by conventional methods as discussed in chapter 3. The controller parameters have been found using approximate model matching techniques in Pade sense and thus PID controllers have been
design. These methods in general assure exact matching in the steady state region whereas matching in the transient zone is also very close. It is illustrated that the closed loop response of original and reduced systems are in close agreement with that of reference model.

Finally this chapter concludes the work in the thesis and some work which can be carried out in days to come has been discussed. It is hoped that the contribution made in the thesis will be helpful in control engineering environment. All the work in the thesis is carried out in MATLAB 7.11.0.584 Pentium IV processor.

7.3 EMERGING TRENDS

The work in this area can be extended in various directions. The methods proposed in third chapter can be extended for reduction of discrete time system. Criterions could be devised to select the order of reduced order model that accurately represent the original system by the proposed method. The reduction methods have been proposed by extension of SISO methods to reduce MIMO systems by considering each elements of the transfer function matrix separately. This approach works well if there is a common denominator of the system, but if this is not the case then successive application SISO method to MIMO system may lead to a reduced system whose order may be equal or greater than the high order system in some cases. Therefore SISO methods can be extended for MIMO systems in such a way that all elements of the transfer matrix are handled simultaneously.

Chapter 3 deals with the various model order reduction techniques which are proposed. These methods can be utilized for order reduction of some power system, chemical processes and VLSI interconnects etc.

The chapter 6 considers the applications of reduction method for controller design. Extensive investigation can be undertaking from
application present overview. In this chapter the controller parameters have been found using approximate model matching technique. So some soft computing techniques such as GA, PSO etc can be used for determining the controller parameters.

The recent trends in VLSI industry towards miniature designs, low power consumption, high speed digital circuits with increased integration of analog circuits with digital blocks have made the signal integrity analysis a challenging task. A lot of research work has to be under taken to device the model order reduction a viable tool in the field of linear time invariant high speed VLSI circuit and theory aspect.
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**PUBLICATIONS BASED ON PROPOSED WORK**


