Chapter 4

On Finite Sum of Polynomial Expressions

4.1 Introduction.

For a positive integer \( p \), to find the value of \( \sum_{r=1}^{n} r^p \), there are some methods, for example Principle of Undetermined coefficient [30]:

\[
\sum_{r=1}^{n} r^p = \frac{n^{p+1}}{p+1} + \frac{n^p}{2} + \frac{B_1 p n^{p-1}}{2!} - \frac{B_3 p (p-1)(p-2) n^{p-3}}{4!} \\
+ B_5 \frac{p(p-1)(p-2)(p-3)(p-4)}{6!} n^{p-5} - \cdots \tag{4.1.1}
\]

where \( B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_5 = \frac{1}{42}, B_9 = \frac{1}{30}, B_9 = \frac{5}{66}, \cdots \) are Bernoulli’s Numbers.

We are interested for \( p = 1, 2, 3, \cdots \) up to some finite values, say \( p = 7 \) only. Without using any formula, in section 2, we have obtained these results in a simple manner and in very less time.

R.H.S. of equation (4.1.1) is 0 if \( n = 0 \) and it is 1 when \( n = 1 \). Obviously \( n \) is a factor of R.H.S. equation (4.1.1), which is \( (p+1)^{th} \) degree polynomial in \( n \).

Notations: Let \( \sum_{r=1}^{n} g(r) = \sum_{n} g(n) \)

where \( g(r) \) be a polynomial in \( r \).
\[
\sum g(n) = \begin{cases} 
0 & \text{if } n = 0, \\
g(1) & \text{if } n = 1, 
\end{cases}
\]

\[
\sum_{r=1}^{n} r^p = \sum n^p
\]

Take \( \frac{d}{dn} \sum g(n) = \sum \frac{d}{dn} g(n) \), \( \int \sum g(n) = \sum \int g(n) \, dn \)

Let \( \Delta g(n) = g(n) - g(n - 1) \).

Then \( f^{p+1}(n) = 0 \) if \( f(n) \) is a polynomial in \( n \) upto degree \( p \), knowing \( p + 1 \) different values of \( f(n) \) for the values of \( n \), we can determine \( f(n) \) as a polynomial in \( n \) upto degree \( p \) by Newton forward difference interpolation formula.

By principle of mathematical induction one can easily verify the formulae obtained by equation (4.1.1)

### 4.2 Integral Technique.

If \( g(n) = \text{Polynomial in } n \text{ of degree } p \)
\[
= a_0 + a_1 n + a_2 n^2 + \cdots + a_p n^p,
\]

where \( a_i \)'s are known constants (real), then

\[
\sum g(n) = \text{Polynomial of degree } p + 1 \text{ and}
\int g(n) \, dn = a_0 n + \frac{1}{2} a_1 n^2 + \frac{1}{3} a_3 n^3 + \cdots + \frac{1}{p+1} a_p n^{p+1} + \text{constant } K
\]

\[
\Rightarrow \sum \int g(n) \, dn = \sum (a_0 n + \cdots + \frac{1}{p+1} a_p n^{p+1}) + \sum K
= (\text{Polynomial in } n \text{ of degree } p + 2) + K \sum 1
\]

\[
\Rightarrow \sum \int g(n) \, dn = \int (\text{some polynomial of degree } p + 1) \, dn + Kn
= \int f(n) \, dn + Kn
\]
where \( f(n) \) is some polynomial of degree \( p + 1 \).

"If \( \sum g(n) = f(n) \)" where \( g(n) \) is a \( p \)-degree polynomial in \( n \) and \( f(n) \) is a \( p + 1 \) degree polynomial in \( n \) (with \( f(0) = 0 \)), then adding \( K \) on R.H.S. and integrating with respect to \( n \) we get,

\[
\sum \int g(n)dn = \int f(n)dn + Kn
\]

where \( K \) is a real constant.

In L.H.S., under summation there may be a constant of integration and R.H.S. must be 0 for \( n = 0 \) (that is, in particular, constant term on R.H.S. is 0).

Taking particular value of a constant of integration in L.H.S. under summation and putting \( n = 1 \), we can determine the value of \( K \) and hence we obtain a formula for \( \sum \int g(n)dn \). In particular taking \( g(n) = n^p \) where \( p \geq 0 \), we obtain the formulae for \( \sum n^{p+1}, \sum n^{p+2}, \cdots \).

4.2.1 Determination of \( \sum n^p \) for \( p = 1, 2, 3, \cdots \)

\( \sum 1 = 1 + 1 + \cdots \) upto \( n \) times. Therefore

\[
\sum 1 = n \tag{4.2.1}
\]

Adding constant \( K \) on R.H.S. of equation (4.2.1), integrating with respect to \( n \) and taking constant of integration 0 i.e. using integral technique,

\[
\sum n = \frac{n^2}{2} + Kn
\]

For \( n = 1 \), we have \( 1 = \frac{1}{2} + K \) and so \( K = \frac{1}{2} \).

Hence \( \sum n = \frac{n^2}{2} + \frac{n}{2} \) i.e. \( \sum n = \frac{n(n+1)}{2} \) \( \tag{4.2.2} \)
Again using integral technique,

\[ \sum \frac{n^2}{2} = \frac{n^3}{6} + \frac{n^2}{4} + K_1 n \]

i.e. \[ \sum n^2 = \frac{n^3}{3} + \frac{n^2}{2} + 2K_1 n \]

For \( n = 1 \), \( 1 = \frac{1}{3} + \frac{1}{2} + 2K_1 \) and so \( 2K_1 = \frac{1}{6} \)

Hence \[ \sum n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \] i.e. \[ \sum n^2 = \frac{n(n+1)(2n+1)}{6} \] (4.2.3)

Using integral technique

\[ \sum \frac{n^3}{3} = \frac{n^4}{12} + \frac{n^3}{6} + \frac{n^2}{12} + K_2 n \]

i.e. \[ \sum n^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} + 3K_2 n \]

For \( n = 1 \), \( 1 = \frac{1}{4} + \frac{1}{2} + \frac{1}{2} + 2K_2 \) and so \( 3K_2 = 0 \)

Hence \[ \sum n^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} \] i.e. \[ \sum n^3 = \left[ \frac{n(n+1)}{2} \right]^2 \] (4.2.4)

Again using integral technique

\[ \sum \frac{n^4}{4} = \frac{n^5}{20} + \frac{n^4}{8} + \frac{n^3}{12} + K_3 n \]

i.e. \[ \sum n^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} + 4K_3 n \]

For \( n = 1 \), \( 1 = \frac{1}{5} + \frac{1}{2} + \frac{1}{3} + 4K_3 \) and so \( 4K_3 = -\frac{1}{30} \)

Hence \[ \sum n^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30} \] i.e. \[ \sum n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \] (4.2.5)
CHAPTER 4. ON FINITE SUM OF POLYNOMIAL EXPRESSIONS

72

Using integral technique

\[ \sum \frac{n^5}{5} = \frac{n^6}{30} + \frac{n^5}{10} + \frac{n^4}{12} - \frac{n^2}{60} + K_4 n \]

i.e. \[ \sum n^5 = \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12} + 5K_4 n \]

For \( n = 1 \), \( 1 = \frac{1}{6} + \frac{1}{2} + \frac{5}{12} - \frac{1}{12} + 5K_4 \) and so \( 5K_4 = 0 \)

Hence \[ \sum n^5 = \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12} \] (4.2.6)

Using integral technique to equation (4.2.6)

\[ \sum \frac{n^6}{6} = \frac{n^7}{42} + \frac{n^6}{12} + \frac{n^5}{12} - \frac{n^3}{36} + K_5 n \]

i.e. \[ \sum n^6 = \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{3} + 6K_5 n. \]

For \( n = 1 \), \( 1 = \frac{1}{7} + \frac{1}{2} + \frac{1}{2} - \frac{1}{6} + 6K_5 \) and so \( 6K_5 = \frac{1}{42} \)

Hence \[ \sum n^6 = \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42} \] (4.2.7)

Again using integral technique to equation (4.2.7)

\[ \sum \frac{n^7}{7} = \frac{n^8}{56} + \frac{n^7}{14} + \frac{n^6}{12} - \frac{n^4}{24} + \frac{n^2}{84} + K_6 n \]

i.e. \[ \sum n^7 = \frac{n^8}{8} + \frac{n^7}{2} + \frac{7n^6}{12} - \frac{7n^4}{14} + \frac{n^2}{12} + 7K_6 n. \]

For \( n = 1 \), \( 1 = \frac{1}{8} + \frac{1}{2} + \frac{7}{12} - \frac{7}{24} + \frac{1}{12} + 7K_6 \) and so \( 7K_6 = 0 \)

Hence \[ \sum n^7 = \frac{n^8}{8} + \frac{n^7}{2} + \frac{7n^6}{24} - \frac{7n^4}{24} + \frac{n^2}{12}. \] (4.2.8)
4.2.2 Determination of $\sum g(n)$ when $g(n)$ is a polynomial.

Applying the integral technique successively on $\sum 1 = n$, we get

$$\sum (n + a) = \frac{n^2}{2} + Kn \quad (4.2.9)$$

$$\sum \left(\frac{n^2}{2} + an + b\right) = \frac{n^3}{6} + \frac{Kn^2}{2} + K_1n \quad (4.2.10)$$

$$\sum \left(\frac{n^3}{6} + \frac{an^2}{2} + bn + c\right) = \frac{n^4}{24} + \frac{Kn^3}{6} + \frac{K_1n^2}{2} + K_2n \quad (4.2.11)$$

and so on up to the degree of a polynomial in the L.H.S under summation as the degree of $g(n)$. Here $a, b, c, \cdots$ and $K, K_1, K_2, \cdots$ are constants.

Comparing L.H.S in the last step with $\sum g(n)$ we fix the values of $a, b, c, \cdots$ and using $n = 1$ in equations (4.2.9), (4.2.10), (4.2.11), \cdots we find the values of $K, K_1, K_2, \cdots$ and hence we get the formula for $\sum g(n)$.

**Example 4.2.1.** Put $a = -\frac{1}{2}$ in equation (4.2.9).

Then $\sum (2n - 1) = n^2 + 2Kn$

Taking $n = 1$, we get $1 = 1 + 2K$ and so $2K = 0$. Therefore

$$\sum (2n - 1) = n^2 \quad (4.2.12)$$

**Example 4.2.2.** To determine $\sum n(n + 1)$, take $a = \frac{1}{2}, b = 0$ in equations (4.2.9) and (4.2.10).

Then for $n = 1$, $\frac{3}{2} = \frac{1}{2} + K$

i.e. $K = 1$ and $1 = \frac{1}{6} + \frac{1}{2} + K_1$ i.e. $K_1 = \frac{1}{3}$

Hence $\sum \left(\frac{n^2}{2} + \frac{n}{2}\right) = \frac{n^3}{6} + \frac{n^2}{2} + \frac{n}{3}$

i.e. $\sum n(n + 1) = \frac{n(n + 1)(n + 2)}{3}$

**Example 4.2.3.** To determine $1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2$, use integral technique for equation (4.2.12),

we get, $\sum \frac{(2n - 1)^2}{4} = \frac{n^3}{3} + Kn$
Taking \( n = 1 \), \( \frac{1}{4} = \frac{1}{3} + K \) and so \( K = -\frac{1}{12} \).

Hence \( \sum (2n - 1)^2 = 4\left[ \frac{n^3}{3} - \frac{n}{12} \right] = \frac{n}{3}(4n^2 - 1) \)

**Example 4.2.4.** To determine \( \sum n(n + 1)(n + 2) \) i.e. \( \sum (n^3 + 3n^2 + 2n) \)

Take \( a = 1, b = \frac{1}{3} \) and \( c = 0 \) in equations (4.2.9), (4.2.10), (4.2.11).

Taking \( n = 1 \), we get \( K = \frac{3}{2}, K_1 = \frac{11}{12} \) and \( K_3 = \frac{1}{4} \).

Hence equation (4.2.11) gives \( \sum \left( \frac{n^3}{6} + \frac{n^2}{2} + \frac{n}{3} \right) = \frac{n^4}{24} + \frac{n^3}{4} + \frac{11n^2}{24} + \frac{n}{4} \)

i.e. \( \sum (n^3 + 3n^2 + 2n) = \frac{n^4}{4} + \frac{3n^3}{2} + \frac{11n^2}{4} + \frac{3n}{2} \).

**Example 4.2.5.** Show that 
\[
1 \cdot 2^2 + 2 \cdot 3^2 + 3 \cdot 4^2 + \cdots + n \cdot (n + 1)^2 = \frac{n^4}{4} + \frac{7n^3}{6} + \frac{7n^2}{4} + \frac{5n}{6} \\
\text{and } \sum n(n + 2)(n + 4) = \frac{n^4}{4} + \frac{5n^3}{2} + \frac{29n^2}{4} + 10n.
\]

**4.3 Differentiation Technique.**

It is the reverse process of integral technique.

"If we know the formula \( \sum g(n) = f(n) \) where \( g(n), f(n) \) be polynomials in \( n \) of degree \( p + 1, p + 2 \) respectively, then differentiating both sides with respect to \( n \) and removing constant term on the right hand side, we get \( \sum g'(n) = f'(n) - f'(0) \)."

Here polynomial is L.H.S under summation is \( p \) th degree etc.

\[
\text{If } g(n) = n^{p+1} \text{ i.e. } \sum n^{p+1} = f(n) \quad (4.3.1) \\
\text{Then } \sum n^p = \frac{f'(n) - f'(0)}{p+1} \quad (4.3.2)
\]

We find formulae for \( \sum 1, \sum n, \sum n^2, \cdots, \sum n^p \), if we know formula for \( \sum n^{p+1} \) where integer \( p \geq 0 \).

Comparing equation (4.2.8) with equation (4.3.1) and using equation (4.3.2)
i.e. using differentiation technique to equation equation (4.2.8), we get,

\[ \sum n^6 = \frac{n^7 + \frac{7}{2}n^6 + \frac{7}{2}n^5 - \frac{7}{6}n^3 + \frac{n}{6} - 0}{7} = \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42}, \text{ which is equation (4.2.7)} \]

Using differentiating technique,

\[ \sum n^5 = \frac{n^6 + 3n^5 + \frac{5}{2}n^4 - \frac{1}{2}n^2 + \frac{1}{12} - \frac{1}{12}}{6} = \frac{n^6}{6} + \frac{n^5}{2} + \frac{5}{6}n^4 - \frac{1}{12}n^2, \text{ which is equation (4.2.6)} \]

Again using differentiation technique successively,

\[ \sum n^4 = \frac{n^5 + \frac{5}{2}n^4 + \frac{5}{3}n^3 - \frac{n}{6} - 0}{5} = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}, \text{ which is equation (4.2.5)} \]

\[ \sum n^3 = \frac{n^4 + 2n^3 + n^2 - \frac{1}{30} - \left(-\frac{1}{30}\right)}{4} = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}, \text{ which is equation (4.2.4)} \]

\[ \sum n^2 = \frac{n^3 + \frac{3}{2}n^2 + \frac{n}{2} - 0}{3} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}, \text{ which is equation (4.2.3)} \]

\[ \sum n = \frac{n^2 + n + \frac{1}{6} - \frac{1}{6}}{2} = \frac{n^2}{2} + \frac{n}{2}, \text{ which is equation (4.2.2)} \]
\[ \sum_1^n = \frac{n + \frac{1}{2} - \frac{1}{2}}{1} = n, \quad \text{which is equation (4.2.1)} \]

**Example 4.3.1.** If it is given that
\[ \sum (2n - 1)^2 = \frac{4n^3}{3} - \frac{n}{3} \quad (4.3.3) \]
then we can determine \( \sum (2n - 1) \).

Applying differentiating technique to equation (4.3.3), we get
\[ \sum 4(2n - 1) = 4n^2 - \frac{1}{3} - \left( -\frac{1}{3} \right). \]
Hence \( \sum (2n - 1) = n^2 \), which is equation (4.2.12).

### 4.4 Forward Difference Technique.

Newton’s forward difference interpolation formula [55]:
\[ f(a + nh) = f(a) + n\Delta f(a) + \frac{n(n-1)}{2!} \Delta^2 f(a) + \frac{n(n-1)(n-2)}{3!} \Delta^3 f(a) + \cdots \]

Taking \( a = 0 \) and \( h = 1 \), we obtain
\[ f(n) = f(0) + n\Delta f(0) + \frac{n(n-1)}{2!} \Delta^2 f(0) + \cdots \quad (4.4.1) \]

We can use equation (4.4.1) to obtain the value of \( \sum_{x=1}^n x^p \) where \( p \) is a nonnegative integer. Value of \( \sum_{x=1}^n x^p \) is expressed as polynomial expression in \( n \) of degree \( p + 1 \).

For this we need \( p + 2 \) values of \( f(x) \) at \( x = 0, 1, 2, \ldots, p + 1 \).
Table 4.1: Forward Difference table for \( f(n) = 1 + 2 + 3 + \cdots + n \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( \Delta f(x) )</th>
<th>( \Delta^2 f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### 4.4.1 To find \( 1 + 2 + 3 + \cdots n = f(n) \), Quadratic in \( n \):

By equation (4.4.1),

\[
f(n) = f(0) + n\Delta f(0) + \frac{n(n-1)}{2}\Delta^2 f(0)
\]

\[
= 0 + n + \frac{n(n-1)}{2}
\]

\[
= \frac{n(n+1)}{2}
\]

### 4.4.2 To find \( 1^2 + 2^2 + 3^2 + \cdots + n^2 = f(n) \), third degree polynomial in \( n \).

Table 4.2: Forward Difference table for \( f(n) = 1^2 + 2^2 + 3^2 + \cdots + n^2 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( \Delta f(x) )</th>
<th>( \Delta^2 f(x) )</th>
<th>( \Delta^3 f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2^2 = 4</td>
<td>9</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1^2 + 2^2 + 3^2 = 14</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

By equation (4.4.1),

\[
f(n) = f(0) + n\Delta f(0) + \frac{n(n-1)}{2}\Delta^2 f(0) + \frac{n(n-1)(n-2)}{6}\Delta^3 f(0)
\]
\[
= 0 + n + \frac{3n(n-1)}{2} + \frac{2n(n-1)(n-2)}{6}
\]
\[
= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} = \frac{n(n+1)(2n+1)}{6}
\]

4.4.3 To find \(1^3 + 2^3 + 3^3 + \cdots + n^3 = f(n)\), fourth degree polynomial in \(n\).

<table>
<thead>
<tr>
<th>(x)</th>
<th>(f(x))</th>
<th>(\Delta f(x))</th>
<th>(\Delta^2 f(x))</th>
<th>(\Delta^3 f(x))</th>
<th>(\Delta^4 f(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>7</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>1^3 = 1</td>
<td>8</td>
<td></td>
<td>19</td>
<td>18</td>
</tr>
<tr>
<td>2</td>
<td>1^3 + 2^3 = 9</td>
<td>27</td>
<td>37</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1^3 + 2^3 + 3^3 = 36</td>
<td>64</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1^3 + 2^3 + 3^3 + 4^4 = 100</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

By equation (4.4.1), we have

\[
f(n) = f(0) + n\Delta f(0) + \frac{n(n-1)}{2}\Delta^2 f(0) + \frac{n(n-1)(n-2)}{6}\Delta^3 f(0) + \frac{n(n-1)(n-2)(n-3)}{24}\Delta^4 f(0)
\]
\[
= 0 + n + \frac{7n(n-1)}{2} + \frac{12n(n-1)(n-2)}{6} + \frac{6n(n-1)(n-2)(n-3)}{24}
\]
\[
= \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} = \left[ \frac{n(n+1)^2}{2} \right]
\]

4.4.4 To find \(1^4 + 2^4 + 3^4 + \cdots + n^4 = f(n)\), fifth degree polynomial in \(n\).

By equation (4.4.1),

\[
f(n) = f(0) + n\Delta f(0) + \frac{n(n-1)}{2}\Delta^2 f(0) + \cdots + \frac{n(n-1)(n-1)(n-3)(n-4)}{120}\Delta^5 f(0)
\]
Table 4.4: Forward Difference table for \( f(n) = 1^4 + 2^4 + 3^4 + \cdots + n^4 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( \Delta f(x) )</th>
<th>( \Delta^2 f(x) )</th>
<th>( \Delta^3 f(x) )</th>
<th>( \Delta^4 f(x) )</th>
<th>( \Delta^5 f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>15</td>
<td>50</td>
<td>60</td>
<td>24</td>
</tr>
<tr>
<td>1</td>
<td>1^4</td>
<td>16</td>
<td>65</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1^4 + 2^4</td>
<td>81</td>
<td>110</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1^4 + 2^4 + 3^4</td>
<td>256</td>
<td>194</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1^4 + 2^4 + 3^4 + 4^4</td>
<td>625</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1^4 + 2^4 + 3^4 + 4^4 + 5^4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\begin{align*}
= 0 + n + \frac{15n(n - 1)}{2} + \frac{15n(n - 1)}{2} + \frac{15n(n - 1)(n - 2)}{6} \\
+ \frac{60n(n - 1)(n - 2)(n - 3)}{24} + \frac{24n(n - 1)(n - 2)(n - 3)(n - 4)}{120} \\
= \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}
\end{align*}
\]

4.4.5 To find \( 1^5 + 2^5 + 3^5 + \cdots + n^5 = f(n) \), sixth degree polynomial in \( n \).

By equation (4.4.1) and table 4.5,

\[
f(n) = f(0) + n\Delta f(0) + \frac{n(n - 1)}{2} \Delta^2 f(0) + \cdots + \frac{n(n - 1)(n - 3)(n - 4)(n - 5)}{720} \Delta^6 f(0)
\]

\[
= 0 + n + \frac{31n(n - 1)}{2} + \frac{180n(n - 1)(n - 2)}{6} + \frac{390n(n - 1)(n - 2)(n - 3)}{24} \\
+ \frac{360n(n - 1)(n - 2)(n - 3)(n - 4)}{120} + \frac{120n(n - 1)(n - 2)(n - 3)(n - 4)(n - 5)}{720} \\
= \frac{n^6}{6} + \frac{n^5}{2} + \frac{5}{12}n^4 - \frac{1}{12}n^2
\]
Table 4.5: Forward Difference table for \( f(n) = 1^5 + 2^5 + 3^5 + \cdots + n^5 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( \Delta f(x) )</th>
<th>( \Delta^2 f(x) )</th>
<th>( \Delta^3 f(x) )</th>
<th>( \Delta^4 f(x) )</th>
<th>( \Delta^5 f(x) )</th>
<th>( \Delta^6 f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0 = 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1^5 = 1</td>
<td>1</td>
<td>31</td>
<td>180</td>
<td>360</td>
<td>120</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1^4 + 2^5 = 33</td>
<td>32</td>
<td>211</td>
<td>570</td>
<td>750</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1^5 + 2^5 + 3^5 = 276</td>
<td>243</td>
<td>781</td>
<td>1320</td>
<td>480</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1^5 + 2^5 + 3^5 + 4^5 = 1300</td>
<td>1024</td>
<td>2101</td>
<td>2550</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1^5 + 2^5 + 3^5 + 4^5 + 5^5 = 1425</td>
<td>3125</td>
<td>4651</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>1^5 + 2^5 + 3^5 + 4^5 + 5^5 + 6^5 = 12201</td>
<td>7776</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Example 4.4.1. Operating both sides of formula, say \( \sum g(n) = f(n) \) under difference operator, we obtain a lower degree formula. For example suppose we have,

\[
\sum n^2 = \frac{n(n+1)(2n+1)}{6}
\]

Then \( \Delta \sum n^2 = \Delta \left[ \frac{1}{6}(2n^2 + 3n^2 + n) \right] \)

i.e. \( \sum [n^2 - (n-1)^2] = \frac{1}{6}[(2n^2 + 3n^2 + n) - (2[n-1]^3 + 3[n-1]^2 + [n-1])] \)

\( \Rightarrow \sum (2n - 1) = n^2 \), which is equation (4.2.12).