Chapter 5

Power domination in some classes of graphs

The power domination number of various classes of graphs has been determined using a two-step process: Finding an upper bound and a lower bound. The upper bound is usually obtained by providing a pattern to construct a set, together with a proof.

Some results of this chapter are included in the following papers.
that constructed set is a PDS. The lower bound is usually found by exploiting the structural properties of the particular class of graphs. Not many exact values of $\gamma_{P,k}$ for special graph classes are known. In this chapter, we determine the power domination number of 3-regular Knödel graphs and provide an upper bound for $\gamma_{P}(W_{r+1,2^{r+1}+1}, r \geq 3$. We compute $\gamma_{P,k}$ and $\text{rad}_{P,k}$ of $H^{2}_{p}$. We also study $\gamma_{P,k}$ of $WKP_{(C,L)}$ and this is the first network class with the pyramid structure for which the $k$-power domination number is studied.

## 5.1 Knödel graphs

In this section, we study the power domination number of Knödel graphs.

It is clear from Definition 1.2.20 that $W_{\Delta,2\nu}$ is bipartite. Also, $W_{\Delta,2\nu}$ is connected if and only if $\Delta \geq 2$, since in that case it suffices to alternate edges in dimension 0 and 1 to get a Hamiltonian cycle.

From Observation 1.4.9 (b), we get that $\gamma_{P,k}(W_{\Delta,2\nu}) = 1$ for
\[ \Delta \geq 2 \text{ and } k \geq \Delta - 1. \] Therefore it is interesting to study 
k-power domination number of \( W_{\Delta,2\nu} \) for \( k \leq \Delta - 2 \).

For \( \Delta = 1, W_{1,2\nu} \) consists of \( \nu \) disjoint copies of \( K_2 \) and 
therefore \( \gamma_P(W_{1,2\nu}) = \nu \). For \( \nu \in \mathbb{N}_2 \) and \( \Delta = 2, W_{2,2\nu} \) is a cycle 
on \( 2\nu \) vertices and clearly \( \gamma_P(W_{2,2\nu}) = 1 \). We have the following 
theorem for the case \( \Delta = 3 \), if \( \nu \in \mathbb{N}_4 \).

**Theorem 5.1.1.** For \( \nu \in \mathbb{N}_4 \), \( \gamma_P(W_{3,2\nu}) = 2 \).

*Proof.\* We prove that the set \( S = \{(1,0),(2,2)\} \) is a PDS of 
\( W_{3,2\nu} \). Then the set of dominated vertices is given by \( \mathcal{P}_1^0(S) = \{(i,j): i \in [2], j \in [3]_0 \} \cup \{(1, \nu - 1), (2,3)\} \). For \( \nu = 4 \), \( S \) is a 
dominating set of \( W_{3,8} \) and for \( \nu = 5, 6 \), we can easily observe 
that all vertices of \( W_{3,2\nu} \) get monitored after the first propagation 
step and therefore \( S \) is a PDS. Let \( \nu \in \mathbb{N}_7 \). Depending on 
whether \( \nu \) is odd or even, we write \( \nu = 2m - 1 \) or \( \nu = 2m, \) 
\( m \in \mathbb{N}_4 \), respectively. Then for \( i \in [m-3], \)

\[
\mathcal{P}_1^i(S) = \left( \{(1,j): j \in [i+3]_0 \} \cup \{(1, \nu - j): j \in [i+2]\} \right)
\cup \left( \{(2,j): j \in [i+5]_0 \} \cup \{(2, \nu - j): j \in [i]\} \right).
\]

We get that \( \mathcal{P}_1^{m-3}(S) = V(W_{3,2\nu}) \), if \( \nu \) is odd, and \( \mathcal{P}_1^{m-2}(S) = \)
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\[ P_1^{m-3}(S) \cup \{(1, m), (2, m + 2)\} = V(W_{3,2\nu}), \text{ if } \nu \text{ is even.} \]

Hence, in both cases we see that every vertex of \( W_{3,2\nu} \) gets monitored after step \( \left\lfloor \frac{\nu}{2} \right\rfloor - 2 \) and therefore \( S \) is a PDS of \( W_{3,2\nu} \).

To prove that \( \gamma_P(W_{3,2\nu}) \geq 2 \), let us assume that \( \{v\} \) is a PDS of \( W_{3,2\nu} \). Then, since \( W_{3,2\nu} \) is bipartite, after the domination step, each of the neighbours of \( v \) has exactly two unmonitored neighbours which prevents the further propagation. Hence \( \gamma_P(W_{3,2\nu}) = 2 \).

We now focus on the family of Knödel graphs \( W_{r+1,2r+1} \). In the next theorem, we prove that the power domination number of \( W_{r+1,2r+1} \) is at most \( 2r^2 - 2 \). For that, we construct a PDS of cardinality \( 2r^2 - 2 \) in \( W_{r+1,2r+1} \). One can easily check that \( S' = \{(1, 1), (2, 6)\} \) is a PDS of \( W_{4,16} \). It is proved in [30] that \( W_{r+1,2r+1} \) can be constructed by taking two copies of \( W_{r,2r} \) and linking the vertices of each copy by a certain perfect matching. Therefore, in order to construct a PDS for \( W_{5,32} \), we take two copies of the set \( S' \), each from a copy of \( W_{4,16} \) that lies in \( W_{5,32} \) and then prove that the new set is a PDS of \( W_{5,32} \). We now extend the same idea to construct a PDS of \( W_{r+1,2r+1} \) for larger values of \( r \). In the proof of the following theorem, we first
produce a set $S$ and then give the set of vertices that are dominated given by $P_0(S)$. After that we give the elements in the set $P_1(S)$ and $P_2(S)$, the sets of vertices that get monitored at the first and second propagation step, respectively. We obtain that the entire graph get monitored in two propagation steps and thus $S$ is a PDS of $W_{r+1,2r+1}$.

**Theorem 5.1.2.** For $r \in \mathbb{N}_3$, $\gamma_p(W_{r+1,2r+1}) \leq 2^{r-2}$.

**Proof.** Let $\nu = 2^r$ and $S = \{(1, 2^{r-3} + j), (2, 7 \cdot 2^{r-3} - 1 + j) : j \in [2^{r-3}]_0\}$. Then

$$P_0^0(S) = S \cup \{(1, 7 \cdot 2^{r-3} + j - 2^\ell \mod \nu), (2, 2^{r-3} + j + 2^\ell - 1 \mod \nu) : j \in [2^{r-3}]_0, \ell = r - 3, r - 2, r - 1, r\}.$$ 

For $r = 3$, the vertex $(1, 2j + 1)$ monitors $(2, 2j + 1)$ for every $j \in [3]$ and the vertex $(2, 2j)$ monitors $(1, 2j)$ for every $j \in [3]_0$.

Thus we get $P_1^1(S) = V(W_{4,16})$. Assume now that $r \in \mathbb{N}_4$. Then, for each $j$ and $\ell$, where $j \in [2^{r-4}]_0$, $\ell = r - 2, r - 1, r$, the vertices in the set $\{(1, 7 \cdot 2^{r-3} + j - 2^\ell \mod \nu)\}$ monitor the vertices in the set $\{(2, 8 \cdot 2^{r-3} + j - 2^\ell - 1 \mod \nu)\}$ by propagation. Also,
for each \( j \) and \( \ell \), where \( 2^{r-4} \leq j \leq 2^{r-3} - 1 \), \( \ell = r - 2, r - 1, r \), the vertices in the set \( \{(2, 2^{r-3} + j + 2^\ell - 1 \mod \nu)\} \) monitor the vertices in the set \( \{(1, j + 2^\ell \mod \nu)\} \) by propagation.

Hence the set of vertices monitored at step 1 is given by

\[
P_1^1(S) = \{(1, j + 2^\ell \mod \nu) : 2^{r-4} \leq j \leq 2^{r-3} - 1, \ell = r - 2, r - 1, r\}
\]

\[
\cup \{(2, 8 \cdot 2^{r-3} + j - 2^\ell - 1 \mod \nu) : j \in [2^{r-4}]_0, \ell = r - 2, r - 1, r\}
\]

\[
\cup P_0^1(S).
\]

Again following the propagation rule, for each \( j \) and \( \ell \), where \( 2^{r-4} \leq j \leq 2^{r-3} - 1 \), \( \ell = r - 2, r - 1, r \), the vertices in the set \( \{(1, 7 \cdot 2^{r-3} + j - 2^\ell \mod \nu)\} \) monitor the vertices in the set \( \{(2, 8 \cdot 2^{r-3} + j - 2^\ell - 1 \mod \nu)\} \). And, for each \( j \) and \( \ell \), where \( j \in [2^{r-4}]_0, \ell = r - 2, r - 1, r \), the vertices in the set \( \{(2, 2^{r-3} + j + 2^\ell - 1 \mod \nu)\} \) monitor the vertices in the set \( \{(1, j + 2^\ell \mod \nu)\} \) by propagation. Hence the set of vertices
monitored at step 2 is given by

\[ P_2^2(S) = \{(1, j + 2^\ell \pmod{\nu}) : j \in [2^{r-4}], \ell = r - 2, r - 1, r\} \]

\[ \cup \{(2, 8 \cdot 2^{r-3} + j - 2^\ell - 1 \pmod{\nu}) : 2^{r-3} \leq j \leq 2^{r-3} - 1, \ell = r - 2, r - 1, r\} \]

\[ \cup P_1^1(S) \]

\[ = V(W_{r+1,2r+1}). \]

Therefore every vertex of \( W_{r+1,2r+1} \) gets monitored after step 2 and hence \( S \) is a PDS of \( W_{r+1,2r+1} \) and \( \gamma_P(W_{r+1,2r+1}) \leq |S| = 2^{r-2}. \]

For \( r = 3 \), any singleton set \( \{v\}, v \in W_{4,16} \) cannot itself power dominate the entire graph, as each of the neighbours of \( v \) will have exactly three unmonitored neighbours after the domination step. Hence the bound in Theorem 5.1.2 is sharp for \( r = 3 \). We further illustrate Theorem 5.1.2 for the graph \( W_{5,32} \).

The vertices of the set \( S \) as defined in the theorem are coloured black in Figure 5.1. In Figure 5.2, \( P_1^0(S) \), the set of dominated vertices, are coloured black and the remaining vertices white. The black vertices in Figure 5.3 and Figure 5.4 represent the
vertices in the set $P_1(S)$ and $P_2(S)$, respectively. The directed edges in the figures indicate the direction in which the propagation occurs at each step. For instance, the directed edge $[(2, 2), (1, 1)]$ in Figure 5.3 indicates that $(2, 2)$ monitors $(1, 1)$ in the first propagation step. We observe that all the vertices get monitored by step 2 and therefore $S$ is a PDS of $W_{5,32}$. 
5.2 Hanoi graphs

We get from Definition 1.2.21 that $H^n_1$ is the graph $K_1$ for any $n \in \mathbb{N}_0$. For $n \in \mathbb{N}_1$, $H^n_2$ is the disjoint union of $2^{n-1}$ copies of $K_2$, i.e. $H^n_2 \cong W_{1,2^{n}}$.

In this section, we study the behaviour of power domination in $H^p_2$. The cases $p \in [2]$ are trivial with $\gamma_{P,k}(H^2_1) = \gamma_{P,k}(K_1) = 1$ and $\gamma_{P,k}(H^2_2) = 2 = \gamma_{P,k}(W_{1,4})$, respectively, for all $k$.

Recall that for $p \in \mathbb{N}_3$ and $n = 2$,

$V(H^2_p) = \{s_2s_1 : s_1, s_2 \in [p]_0\}$ and

$E(H^2_p) = \{\{r_i, r_j\}, \{i\ell, j\ell\} : r, i, j \in [p]_0, i \neq j, \ell \in [p]_0 \setminus \{i, j\}\}$.

Vertices of the form $ss$ are called the extreme vertices of $H^2_p$. Note that the extreme vertices are of degree $p - 1$ and all the other vertices are of degree $2p - 3$ in $H^2_p$. It is easy to observe that $\gamma(H^2_p) = p$. Indeed, any set containing a vertex from each
of the $p$ cliques in $H^2_p$ forms a dominating set of $H^2_p$. Since each of the $p$ cliques contains an extreme vertex, any dominating set of $H^2_p$ must contain at least $p$ vertices and hence $\gamma(H^2_p) = p$.

For $p = 3$, $H^3_3$ is isomorphic to the Sierpiński graph, $S^3_3$, see [43, p.143 ff]. It is proved (refer Theorem 1.4.12) that

$$
\gamma_{p,k}(S^3_3) = \begin{cases} 
1, & n = 1 \text{ or } k \in \mathbb{N}_2; \\
2, & n = 2 \text{ and } k = 1; \\
3^{n-2}, & n \in \mathbb{N}_3 \text{ and } k = 1.
\end{cases}
$$

Therefore $\gamma_{p,1}(H^2_3) = 2$ and $\gamma_{p,k}(H^2_3) = 1$ for $k \in \mathbb{N}_2$.

There are perfect codes for all Hanoi graphs isomorphic to Sierpiński graphs and also for $H^2_p$ [43]. But, for $p \in \mathbb{N}_4$, the Hanoi graphs do not contain perfect codes for $n \in \mathbb{N}_3$, as found out by Q. Stierstorfer [67]. The domination number of these graphs is not known. Therefore we concentrate on $n = 2$. (For $n = 1$, $H^1_p \cong K_p \cong S^1_p$.)

**Theorem 5.2.1.** Let $p \in \mathbb{N}_4$. Then

$$
\gamma_{p,k}(H^2_p) = \begin{cases} 
1, & k \in \mathbb{N}_{p-2}; \\
p - k - 1, & k \in [p-3].
\end{cases}
$$
Proof. Case 1: $k \in \mathbb{N}_{p-2}$.

Let $v$ be an arbitrary vertex in $H^2_p$. Let $K^i_p$ denote the subgraph induced by the vertices $\{ij : j \in [p]_0\}$. Assume that $v \in K^i_p$ for some $i$. Let $S = \{v\}$. Then $V(K^i_p) \subseteq \mathcal{P}^0_k(S)$. Since each vertex in $K^i_p$ other than the vertex $ii$ has $p-2$ neighbours outside $K^i_p$, for any $j \neq i$, $V(K^j_p) \setminus \{jj, ji\} \subseteq \mathcal{P}^1_k(S)$. Hence any vertex $j\ell$ in $K^j_p$, $\ell \neq i,j$, will have two unmonitored neighbours, namely $jj$ and $ji$. Since $k \geq p - 2 \geq 2$, these vertices will get monitored by propagation, i.e. $V(K^j_p) \subseteq \mathcal{P}^2_k(S)$. Since this is true for any $j \neq i$, $S$ is a $k$-PDS of $H^2_p$.

Case 2: $k \in [p-3]$. 
We first prove that $\gamma_{P,k}(H^2_p) \leq p - k - 1$. Let $S$ be the set of vertices $\{i(i-1): i \in [p-k-2]\} \cup \{0(p-k-2)\}$ (For $k = 1$ and $p = 4$, the vertices of $S$ are coloured black in Figure 5.5.) Then $\mathcal{P}^0_k(S) = \{V(K^i_p): i \in [p-k-1]_0\} \cup \{ij: p-k-1 \leq i \leq p-1, j \in [p-k-2]_0\} \cup \{i(p-k-2): p-k-1 \leq i \leq p-1\}$. Let $Y$ be the set of vertices $\{ij: i \in [p-k-1]_0, p-k-1 \leq j \leq p-1\}$. Then any vertex $v = i'j'$ in $Y$ has exactly $k$ unmonitored neighbours given by $\{\ell j': p-k-1 \leq \ell \leq p-1, \ell \neq j'\}$ which will get monitored by propagation. Therefore, the remaining set of unmonitored vertices is given by $\{jj: V(K^j_p) \cap S = \emptyset\}$, which will then get monitored by propagation by its neighbours in $K^j_p$. Thus $S$ is a $k$-PDS of $H^2_p$, which implies $\gamma_{P,k}(H^2_p) \leq p - k - 1$.

We next prove that $\gamma_{P,k}(H^2_p) \geq p - k - 1$. Let $S$ be a $k$-PDS of $H^2_p$. Suppose on the contrary that $\gamma_{P,k}(H^2_p) \leq p - k - 2$. Assume first that $S$ has exactly one vertex in $p$-cliques $K^i_p$ for $i \in \{i_1, \ldots, i_{p-k-2}\}$. Let $\{i_1j_1, \ldots, i_{p-k-2}j_{p-k-2}\}$ be the set of $p-k-2$ vertices in $S$. Then $S \cap V(K^i_p) = \emptyset$ for any $i' \in I = [p]_0 \{i_1, \ldots, i_{p-k-2}\}$. Let $X = \{i'j_1, \ldots, i'j_{p-k-2}\}$. Then $\mathcal{P}^0_k(S) \cap V(K^i_p) \subseteq X$. This holds for any $i' \in I$. Let $J' = [p]_0 \{j_1, \ldots, j_{p-k-2}\}$. Then the set of vertices $\{i'j': i' \in I', j' \in J'\}$ has an empty intersection with $\mathcal{P}^0_k(S)$. Since every vertex
in $H^2_p$ has either no or more than $k$ neighbours in this set, no vertex from this set can get monitored later on, a contradiction. Assume next that $|S| < p - k - 2$ or that $S$ intersects some $K^i_p$ in more than one vertex. Then we can conclude analogously that not all vertices of $K^i_p$ will be monitored and hence $\gamma_{P,k}(H^2_p) \geq p - k - 1$.

It is obtained in Theorem 1.4.12 that for $p \in \mathbb{N}_4$,

$$
\gamma_{P,k}(S^2_p) = \begin{cases} 
1, & k \in \mathbb{N}_{p-1}; \\
p - k, & k \in [p-2].
\end{cases}
$$

We can observe that for $p \in \mathbb{N}_4$, $\gamma_{P,k}(S^2_p) - \gamma_{P,k}(H^2_p) = 1$ if and only if $k \in [p-2]$ and for $k \in \mathbb{N}_{p-1}$, the two values coincide.

We now compute the $k$-propagation radius of $H^2_p$. For $p = 3$, it is proved that $\text{rad}_{P,1}(H^2_3) = 2$ and $\text{rad}_{P,k}(H^2_3) = 3$ for $k \in \mathbb{N}_2$ (refer Theorem 1.4.14). The following theorem indicates that the graph $H^2_p$ can be monitored in 3 steps.

**Theorem 5.2.2.** For $p \in \mathbb{N}_4$, $\text{rad}_{P,k}(H^2_p) = 3$.

**Proof.** For $k \in \mathbb{N}_{p-2}$, $\gamma_{P,k}(H^2_p) = 1$ and let $S = \{ij\}$ be a $k$-PDS of $H^2_p$. If $i \neq j$, we prove that the the vertices $ji$ and $jj$ do
not belong to $\mathcal{P}_k^1(S)$. Clearly, $ji, jj \notin \mathcal{P}_k^0(S)$. Also none of the
neighbours of $ji$ and $jj$ belongs to $\mathcal{P}_k^0(S)$. Therefore, $ji$ and $jj$
cannot be monitored in step 1. For $i = j$, we can similarly prove
that the vertices $\ell i$ and $\ell \ell$, for $\ell \neq i$, do not belong to $\mathcal{P}_k^1(S)$
and hence $\text{rad}_{P,k}(H_p^2) \geq 3$. To prove the upper bound, consider
the set $S = \{ii\}$. Then,

$$\mathcal{P}_k^0(S) = V(K_p^i),$$
$$\mathcal{P}_k^1(S) = \mathcal{P}_k^0(S) \cup \bigcup \{V(K_p^\ell) \setminus \{\ell i, \ell \ell\} : \ell \in [p] \setminus \{i\}\},$$
$$\mathcal{P}_k^2(S) = \mathcal{P}_k^1(S) \cup \{\ell i, \ell \ell : \ell \in [p] \setminus \{i\}\} = V(H_p^2).$$

Hence $\text{rad}_{P,k}(H_p^2) \leq \text{rad}_{P,k}(G, S) = 3$.

Suppose that $k \in [p - 3]$ and let $S$ be a minimum $k$-PDS of
$H_p^2$. Then $\gamma_{P,k}(H_p^2) = p - k - 1$ and thus there exist at least $k + 1$
p-cliques $K_p^i$ not containing any vertex of $S$. Let $K_p^{i'}$ be an arbitrary
such clique. We prove that the vertex $i'i'$ is not in $\mathcal{P}_k^1(S)$.

Clearly, the vertex $i'i'$ does not belong to $\mathcal{P}_k^0(S)$. Moreover,
$$|V(K_p^{i'}) \cap \mathcal{P}_k^0(S)| \leq p - k - 1$$
and therefore $|V(K_p^{i'}) \setminus \mathcal{P}_k^0(S)| \geq k + 1$. Hence any neighbour of $i'i'$ has more than $k$
unmonitored vertices preventing any propagation to this vertex on that step.
Thus $i'j'$ is not in $\mathcal{P}^1_k(S)$. To prove the upper bound, consider the set $S = \{i(i - 1) \colon i \in [p - k - 2]\} \cup \{0(p - k - 2)\}$. Then,

\[
\mathcal{P}^0_k(S) = \{V(K^i_p) : i \in [p - k - 1]_0\} \\
\cup \{ij: p - k - 1 \leq i \leq p - 1, j \in [p - k - 1]_0\},
\]

\[
\mathcal{P}^1_k(S) = \mathcal{P}^0_k(S) \cup \{ij: p - k - 1 \leq i, j \leq p - 1, i \neq j\},
\]

\[
\mathcal{P}^2_k(S) = \mathcal{P}^1_k(S) \cup \{ii: p - k - 1 \leq i \leq p - 1\} = V(H^2_p).
\]

\]

\]

5.3 WK-Pyramid networks

In this section, we determine the $k$-power domination number of $WKP_{(C,L)}$. We also obtain the $k$-propagation radius of $WKP_{(C,L)}$ in some cases.

Observe from Definition 1.2.22 that WK-Recursive mesh, $WK_{(C,L)}$, has $C^L$ vertices and $\frac{C}{2}(C^L - 1)$ edges. Vertices in $WK_{(C,L)}$ which are of the form $(\bar{a} \ldots \bar{a})$ are called extreme vertices of $WK_{(C,L)}$. Clearly, $WK_{(C,L)}$ contains $C$ extreme vertices of degree $C - 1$ and all the other vertices are of degree $C$. We
have $WK_{(1,L)} \cong K_1$ ($L \geq 1$), $WK_{(2,L)} \cong P_{2^L}$ ($L \geq 1$) and $WK_{(C,1)} \cong K_C$ ($C \geq 1$).

A vertex of $WK_P(C,L)$ with the addressing scheme $(r, (a_r a_{r-1} \ldots a_1))$ is called a vertex at level $r$. The part $(a_r a_{r-1} \ldots a_1)$ of the address determines the address of a vertex within the WK-recursive mesh at level $r$. All vertices in level $r > 0$ of $WK_P(C,L)$ induce a WK-recursive mesh $WK_{(C,r)}$. Hence $|V(WK_P(C,L))| = \sum_{i=0}^{L} C^i = \frac{C^{L+1} - 1}{C - 1}$. Note that $WK_P(C,1) \cong K_{C+1}$ ($C \geq 1$), $WK_P(1,L) \cong P_{L+1}$ ($L \geq 1$). Vertices of the form $(r, (\bar{a} \ldots \bar{d}))$ are called the extreme vertices of $WK_P(C,L)$. The vertex $(0, (1))$ has degree $C$ and at any level except the $L^{th}$ level, the extreme vertices are of degree $2C$ and the other vertices are of degree $2C + 1$. In the $L^{th}$ level, the extreme vertices have degree $C$ and the other vertices have degree $C + 1$.

We shall use the following notations in the rest of the chapter.

Let $V_1$ and $V_2$ denote the set of vertices of $WK_P(C,2)$ in levels 1 and 2, respectively. Let $Q_i$ denote a $C$-clique induced by the set of vertices $\{(2, (ij)): j \in [C]_0\}$ for some $i$.

For $C, L \in \mathbb{N}_3$, let $w \in [C]_0^{L-2}$. Denote $V_w^{C,L} = \{(L, (wij)) \in$
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$WKP_{(C,L)}: i, j \in [C]_0$ and $G^{C,L}_w = \langle V^{C,L}_w \rangle$, i.e. $G^{C,L}_w$ is the induced subgraph in level $L$ of $WKP_{(C,L)}$. In fact, $G^{C,L}_w$ is isomorphic to $WK_{(C,2)}$ for any $w \in [C]_{L-2}$ and any $L \in \mathbb{N}_3$ (Figure 5.6). We first consider the easier case as stated in the following theorem.

**Theorem 5.3.1.** Let $C, L \in \mathbb{N}_1$. If $C = 1$ or $L = 1$ or $k \geq C$, then $\gamma_{P,k}(WKP_{(C,L)}) = 1$.

**Proof.** Recall that $WKP_{(C,1)} \cong K_{C+1} (C \geq 1)$ and that $WKP_{(1,L)} \cong P_{L+1} (L \geq 1)$. Hence $\gamma_{P,k}(G) = 1$ for these graphs $G$.

If $k \geq C$, then take $S = \{(0,(1))\}$. It monitors the vertices in level 1. Since each vertex in level $r$ has exactly $C$ neighbours in its successive level $r+1$, once the level $r$ is monitored, the vertices in level $r + 1$ get monitored by propagation. This propagation goes on till level $L$ and hence $S$ is a $k$-PDS of $WKP_{(C,L)}$. \qed
We have determined the value of $\gamma_{P,k}(WKP_{(C,L)})$ when $k \geq C$. Now, we consider the remaining case $k \leq C - 1$. We begin with the computation of $\gamma_{P,k}$ for $L = 2$ and will prove in Theorem 5.3.4 that $\gamma_{P,k}(WKP_{(C,2)}) = C - k$ for $C \geq 2, k \leq C - 1$.

We first obtain the following upper bound. For that, we produce a set $S$ of cardinality $C - k$ and prove that $S$ monitors the whole graph in two propagation steps.

**Lemma 5.3.2.** For $C \in \mathbb{N}_3$ and $k \in [C - 1]$, 
$\gamma_{P,k}(WKP_{(C,2)}) \leq C - k$.

*Proof.* Let $S = \{(1, (i)): k \leq i \leq C - 1\}$. (For $k = 1$ and $C = 5$, the vertices in $S$ are coloured black in Figure 5.7.)

Then $P^0_k(S) = \{(1, (j)): j \in [C]_0\} \cup \{(2, (ij)): k \leq i \leq C - 1, j \in [C]_0\} \cup \{(0, (1))\}$,

$P^1_k(S) = P^0_k(S) \cup \{(2, (ij)): i \in [k]_0, k \leq j \leq C - 1\}$ and

$P^2_k(S) = P^1(S) \cup \{(2, (ij)): i, j \in [k]_0\} = V(WKP_{(C,2)})$.

Hence $S$ is a $k$-PDS, which implies $\gamma_{P,k}(WKP_{(C,2)}) \leq |S| = C - k$. \hfill $\square$

**Lemma 5.3.3.** For $C \in \mathbb{N}_3$ and $k \in [C - 2]$, 
$\gamma_{P,k}(WKP_{(C,2)}) \geq C - k$. 

Proof. Let $S$ be a minimum $k$-PDS of $WKP_{(C,2)}$. We may assume that $S \subseteq V^1 \cup V^2$.

Claim: $|S \cap (V^1 \cup V^2)| \geq C - k$.  

Suppose on the contrary that $|S \cap (V^1 \cup V^2)| \leq C - k - 1$. We consider the case when $S$ contains vertices from both $V^1$ and $V^2$. Assume first that $|S \cap (V^1 \cup V^2)| = C - k - 1$ and that $S$ contains a vertex $(1,(i')) \in V^1$ and the remaining $C - k - 2$
vertices from the $C$-cliques $Q_{i_1}, \ldots, Q_{i_{C-k-2}}$, where $i' \neq i_\ell$, $\ell \in [C - k - 2]$ such that each of these $C$-cliques contains exactly one vertex in $S$. Let $Q_\ell$ be an arbitrary clique that does not contain any vertex of $S$, where $\ell \neq i'$. Let $X = \{(2, (\ell i'))\} \cup \{(2, (\ell i_1)), \ldots, (2, (\ell i_{C-k-2}))\}$. Then $P_k^1(S) \cap V(Q_\ell) = X$. This holds for every $l \in I = [C]_0 \setminus \{i', i_1, \ldots, i_{C-k-2}\}$. Thus the set of vertices $J = \{(2, (\ell \ell')): \ell \in I, \ell' \in I\}$ has an empty intersection with $P_k^1(S)$. Since every vertex in $WKP_{(C,2)} - J$ has either 0 or $k + 1$ neighbours in $J$, no vertex from this set $J$ may get monitored later on, which is a contradiction. Assume next that $|S \cap (V^1 \cup V^2)| < C - k - 1$ or that $S$ intersects some $C$-clique $Q_i$ in more than one vertex. Then we can analogously conclude that not all vertices of $Q_\ell$ will be monitored. Now, the case when $S \cap V^1 = \emptyset$ or $S \cap V^2 = \emptyset$ can be proved in a similar manner. Hence the claim.

Therefore, $\gamma_{P,k}(WKP_{(C,2)}) = |S| = |S \cap (V^1 \cup V^2)| \geq C - k$.

From Lemmas 5.3.2 and 5.3.3, we can easily deduce the following theorem.

**Theorem 5.3.4.** For $C \in \mathbb{N}_2$ and $k \in [C - 1]$,
\[ \gamma_{P,k}(WKP_{(C,2)}) = C - k. \]

**Proof.** Clearly, \( \gamma_{P,1}(WKP_{(2,2)}) = 1 \). Let \( C \geq 3 \). For \( k = C - 1 \), any vertex in level 1 forms a \( k \)-PDS of \( WKP_{(C,2)} \). For \( k \in [C - 2] \), the result follows from Lemmas 5.3.2 and 5.3.3.

Thus we compute \( \gamma_{P,k}(WKP_{(C,2)}) \) for all values of \( k \) and \( C \).

We now consider the case \( C \in \mathbb{N}_3, L \in \mathbb{N}_3 \) and \( k \in [C - 2] \) and prove an upper bound in the following lemma. We construct a set \( S \subseteq V(WKP_{(C,L)}) \) that monitors the whole graph. The idea is to construct \( S \) in such a way that it initially monitors all the vertices of level \( L \) and \( L - 1 \). For that, we use the hamiltonian property of its subgraphs. Since the graph possesses a pyramid structure, each vertex in a level has exactly one neighbour in its preceding level. Therefore once the levels \( L \) and \( L - 1 \) get monitored, the preceding levels can be monitored by propagation.

**Lemma 5.3.5.** For \( C \in \mathbb{N}_3, L \in \mathbb{N}_3 \) and \( k \in [C - 2] \),

\[ \gamma_{P,k}(WKP_{(C,L)}) \leq (C - k - 1)C^{L-2}. \]

**Proof.** In \( WKP_{(C,L)} \), the vertices in the \( L \)th level induce \( WK_{(C,L)} \) which is hamiltonian [45, 51]. Also, by contracting each of the
subgraphs $G_{w}^{C,L}$ into a single vertex, the graph induced by the vertices in level $L$ is isomorphic to $WK_{(C,L-2)}$. Hence, in level $L$ of $WKP_{(C,L)}$, we can arrange the subgraphs of the form $G_{w}^{C,L}$ into a cycle such that there exists exactly one edge between the consecutive subgraphs. We now construct a set $S$ in such a way that corresponding to each subgraph $G_{w}^{C,L}$ in level $L$, the set $S$ contains one vertex from the neighbour set of $G_{w}^{C,L}$ in level $L-1$ (which induces a clique) and $C - k - 2$ additional vertices from $G_{w}^{C,L}$.

Let $w', w'' \in [C]_{0}^{L-2}$. Let $G_{w'}^{C,L}, G_{w''}^{C,L}$ and $G_{w'''}^{C,L}$ be consecutive subgraphs in the selected hamiltonian order. Let $xx'$ be the edge between $G_{w}^{C,L}$ and $G_{w'}^{C,L}$, where $x \in G_{w}^{C,L}, x' \in G_{w'}^{C,L}$ and let $y'y''$ be the edge between $G_{w'}^{C,L}$ and $G_{w''}^{C,L}$, where $y' \in G_{w'}^{C,L}, y'' \in G_{w''}^{C,L}$. Let $H$ and $Q$ be the $C$-cliques in $G_{w'}^{C,L}$ that contain the vertices $x'$ and $y'$, respectively. Denote $x = (L, (wii))$ and $y' = (L, (w'jj))$ for some $i$ and $j$, $i \neq j$. We now construct a set $S$ as explained above. We first choose the elements of $S$ corresponding to the subgraph $G_{w'}^{C,L}$. Let $S$ contain the vertex $(L - 1, (w'j))$, which is the neighbour of $y'$ in the $(L - 1)^{th}$ level. Then $C - k - 2$ additional vertices from $G_{w'}^{C,L}$ are added to $S$ in such a way that no two vertices lying in the same $C$-clique in $G_{w'}^{C,L}$ and no
one lying in the $C$-cliques, $H$ and $Q$ (i.e. $S \cap V(H) = \emptyset$ and $S \cap V(Q) = \emptyset$). Now, do this in parallel for all the corresponding subgraphs. In particular, the vertex $(L - 1, (w_i))$ in the $(L - 1)^{\text{th}}$ level corresponding to the vertex $x$ is put into $S$, when considering $G_{w'}^{C,L}$. Thus $C - k$ vertices of $H$ lie in $P_k^1(S)$: one of these vertices is $x'$, the other $C - k - 1$ are those vertices of $H$ that have a neighbour in the $C$-cliques in $G_{w'}^{C,L}$ that contain $C - k - 2$ vertices of $S$ and that have a neighbour in the $C$-clique $Q$ in $G_{w'}^{C,L}$. Also, the neighbour of $H$ in the $(L - 1)^{\text{th}}$ level belongs to $P_k^0(S)$, since $(L - 1, (w_i)) \in S$. Hence the remaining $k$ vertices of $H$ lie in $P_k^2(S)$ and it is straightforward to check that all the vertices of $G_{w'}^{C,L}$ lie in $P_k^\infty(S)$. In a similar way, every vertex in the $L^{\text{th}}$ level is monitored. We know that, for any $w$, the neighbours of $G_{w'}^{C,L}$ in the $(L - 1)^{\text{th}}$ level induce a $C$-clique. By the construction of $S$, each $C$-clique in the $(L - 1)^{\text{th}}$ level contains a vertex in $S$. Thus we get that all the vertices in levels $L - 1$ and $L - 2$ belong to $P_k^0(S)$. Now, since each vertex in level $L - 2$ has exactly one neighbour in its preceding level, vertices in the $(L - 3)^{\text{rd}}$ level are monitored by propagation. This propagation continues to the preceding levels and hence the whole graph gets monitored. Thus we conclude that $S$ is a
$k$-PDS. Since each subgraph $G_{w}^{C,L}$ contains $C - k - 1$ vertices of $S$, $|S| \leq (C - k - 1)C^{L-2}$.

An illustration of Lemma 5.3.5 is included in the last section of this chapter.

**Lemma 5.3.6.** For $C \in \mathbb{N}_3$, $L \in \mathbb{N}_3$ and $k \in [C - 2]$, 
$$\gamma_{P,k}(WKP(C,L)) \geq (C - k - 1)C^{L-2}.$$ 

**Proof.** Let $S$ be a minimum $k$-PDS of $WKP(C,L)$ and $w \in [C]^{L-2}_0$. Denote $V^{C,L-1}_w = \{(L - 1, (wi)) : (i, (wij)) \in WKP(C,L), i \in [C]_0\}$.

**Claim:** $|S \cap (V^{C,L}_w \cup V^{C,L-1}_w)| \geq C - k - 1.$

Suppose on the contrary that $|S \cap (V^{C,L}_w \cup V^{C,L-1}_w)| \leq C - k - 2$. Consider the case when $S \cap V^{C,L-1}_w = \emptyset$. Then $|S \cap V^{C,L}_w| \leq C - k - 2$. Assume first that $|S \cap V^{C,L}_w| = C - k - 2$. Let $H_i$ be a $C$-clique in $G^{C,L}_w$, i.e. $H_i$ is induced by the set of vertices $\{(L, (wi)) : (i, (wij)) \in WKP(C,L), j \in [C]_0\}$ for some $i$. Assume that $S$ has exactly one vertex in $C$-cliques $H_i$ for $i \in \{i_1, \ldots, i_{C-k-2}\}$. Then $S \cap V(H_{i'}) = \emptyset$ holds for other $k + 2$ coordinates $i'$. Let $H_{i'}$ be an arbitrary such clique in $G^{C,L}_w$ that does not contain any vertex of $S$. Let $X = \{(L, (wli_1)), (L, (wli_{C-k-2}))\} \cup \ldots \cup \{(L, (wli_{C-k-2}))\}$.
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\{ (L, (w\ell)) \}. Then \( \mathcal{P}_k^1(S) \cap V(H_\ell) \subseteq X \). This holds for every \( \ell \in I = [C]_0 \setminus \{i_1, \ldots, i_{C-k-2}\} \). Thus the set of vertices 
\{ (L, (w\ell')) : \ell \in I, \ell' \in I, \ell \neq \ell' \} has an empty intersection with \( \mathcal{P}_k^1(S) \). Since every vertex in \( WP(C, L) \) has either 0 or \( k + 1 \) neighbours in this set, no vertex from this set may get monitored later on, a contradiction. Assume next that \( |S \cap V^{C,L}_w| < C - k - 2 \) or that \( S \) intersects some \( C \)-clique \( H_i \) in more than one vertex. Then we can analogously conclude that not all vertices of \( H_\ell \) will be monitored. Thus the case that \( S \cap V^{C,L-1}_w = \emptyset \) is not possible.

Now suppose that \( S \cap V^{C,L-1}_w \neq \emptyset \). Assume first that
\[
|S \cap (V^{C,L}_w \cup V^{C,L-1}_w)| = C - k - 2 \]
and that \( S \) contains a vertex \((L-1, (w'\ell')) \in V^{C,L-1}_w\) and the remaining \( C - k - 3 \) vertices from the \( C \)-cliques \( H_{i_1}, \ldots, H_{i_{C-k-3}} \), where \( i' \neq i_\ell, \ell \in [C-k-3] \) such that each of these \( C \)-cliques contains exactly one vertex in \( S \). Let \( H_\ell \) be an arbitrary clique in \( G^{C,L}_w \) that does not contain any vertex of \( S \), where \( \ell \neq i' \). Let \( X = \{(L, (w'\ell')) \} \cup \{(L, (w\ell)) \} \cup \{(L, (w\ell_{i_1})) \}, \ldots, (L, (w\ell_{i_{C-k-3}})) \}. Then \( \mathcal{P}_k^1(S) \cap V(H_\ell) \subseteq X \). This holds for every \( \ell \in I' = [C]_0 \setminus \{i', i_1, \ldots, i_{C-k-3}\} \). Thus the set of vertices
\{ (L, (w\ell')) : \ell \in I', \ell' \in I', \ell \neq \ell' \} has an empty intersection with \( \mathcal{P}_k^1(S) \). Since every vertex in \( WP(C, L) \)
has either 0 or \( k + 1 \) neighbours in this set, no vertex from this set may get monitored later on, which is a contradiction. Assume next that \( |S \cap (V_{w}^{C,L} \cup V_{w}^{C,L-1})| < C - k - 2 \) or that \( S \) intersects some \( C \)-clique \( H_{i} \) in more than one vertex. Then we can analogously conclude that not all vertices of \( H_{\ell} \) will be monitored. Hence the claim \( |S \cap (V_{w}^{C,L} \cup V_{w}^{C,L-1})| \geq C - k - 1 \) is proved. Therefore, \( |S \cap (V(G_{w}^{C,L}) \cup N_{L-1}(G_{w}^{C,L}))| \geq C - k - 1 \), where \( N_{L-1}(G_{w}^{C,L}) \) is the set of neighbours of \( G_{w}^{C,L} \) in the \( (L-1)^{th} \) level. Hence corresponding to each \( G_{w}^{C,L} \) in the \( L^{th} \) level, we get at least \( C - k - 1 \) vertices in \( S \).

Hence \( |S| \geq \sum_{w \in [C]}(C - k - 1) = (C - k - 1)C^{L-2}. \)

The following theorem gives the exact value of \( \gamma_{P,k}(WP(C,L)) \) for \( C \in \mathbb{N}_{3}, L \in \mathbb{N}_{3} \) and \( k \in [C - 2] \).

**Theorem 5.3.7.** For \( C \in \mathbb{N}_{3}, L \in \mathbb{N}_{3} \) and \( k \in [C - 2] \),
\[
\gamma_{P,k}(WP(C,L)) = (C - k - 1)C^{L-2}.
\]

**Proof.** Follows from Lemmas 5.3.5 and 5.3.6. \( \Box \)

Thus we have the following consolidated result:
Let $C, L \in \mathbb{N}_1$. Then

$$\gamma_{P,k}(WK_P(C,L)) = \begin{cases} 
1, & C = 1 \text{ or } L = 1 \text{ or } k \in \mathbb{N}_C; \\
C - k, & L = 2, C \in \mathbb{N}_2, k \in [C - 1]; \\
(C - k - 1)C^{L-2}, & C, L \in \mathbb{N}_3, k \in [C - 2].
\end{cases}$$

For $k = C - 1$, $C \in \mathbb{N}_2$ and $L \in \mathbb{N}_3$, we prove the following upper bound.

**Theorem 5.3.8.** For $C \in \mathbb{N}_2$ and $L \in \mathbb{N}_3$,

$$\gamma_{P,C-1}(WK_P(C,L)) \leq \left\lceil \frac{L+1}{3} \right\rceil.$$  

**Proof.** We consider three cases.

**Case 1:** $L = 3m$, $m \in \mathbb{N}_1$.

$$S = \{ \bigcup_{i=1}^{m} (3i - 1, (0)^{3i-1}) \} \cup \{(0, (1))\}.$$  

Here, $|S| = m + 1$. Also, $\left\lceil \frac{L+1}{3} \right\rceil = \left\lceil \frac{(3m)+1}{3} \right\rceil = m + 1$.

**Case 2:** $L = 3m + 1$, $m \in \mathbb{N}_1$.

$$S = \{ \bigcup_{i=1}^{m} (3i, (0)^{3i}) \} \cup \{(1, (0))\}.$$  

Here, $|S| = m + 1$. Also, $\left\lceil \frac{L+1}{3} \right\rceil = \left\lceil \frac{(3m+1)+1}{3} \right\rceil = m + 1$. 

Case 3: \( L = 3m + 2, \ m \in \mathbb{N}_1. \)

\[
S = \left\{ \bigcup_{i=1}^{m+1} (3i - 2, (0)^{3i-2}) \right\}.
\]

Here, \( |S| = m + 1. \) Also, \( \left\lceil \frac{L+1}{3} \right\rceil = \left\lceil \frac{(3m+2)+1}{3} \right\rceil = m + 1. \)

In each case, \( \mathcal{P}^{C-1}_{C-1}(S) = V(WKP(C,L)) \) and thus \( S \) is a \( k \)-PDS of order \( \left\lceil \frac{L+1}{3} \right\rceil. \) Hence \( \gamma_{P,C-1}(WKP(C,L)) \leq \left\lceil \frac{L+1}{3} \right\rceil. \) \( \square \)

We now determine the \( k \)-propagation radius of \( WKP(C,L) \) for \( C \in \mathbb{N}_1 \) and \( L = 1, 2. \) If \( L = 1, \) the graph is a complete graph and its \( k \)-propagation radius is 1. If \( C = 1, \) \( \text{rad}_{P,k}(WKP_{1,L}) = \text{rad}_{P,k}(P_{L+1}) = \left\lfloor \frac{L+1}{2} \right\rfloor. \)

**Lemma 5.3.9.** Let \( C \in \mathbb{N}_3, \ k \in [C - 1] \) and \( S \) be a minimum \( k \)-PDS of \( WKP(C,2). \) Then \( S \cap V^1 \neq \emptyset. \)

**Proof.** Suppose that \( S \cap V^1 = \emptyset. \) Consider the case when \( (0, (1)) \notin S. \) Then by Theorem 5.3.4, \( |S \cap V^2| = C - k. \)

Assume first that \( S \) has exactly one vertex in \( C \)-cliques, \( Q_i, \) for \( i \in \{i_1, \ldots, i_{C-k}\}. \) Then \( S \cap V(Q_i) = \emptyset \) for \( k \) coordinates \( i'. \) Let \( Q_\ell \) be an arbitrary such subgraph. Let \( X = \{(2, (\ell i_1)), \ldots, (2, (\ell i_{C-k}))\}. \) Then \( \mathcal{P}_k^1(S) \cap V(Q_\ell) = X \) and \( \mathcal{P}_k^1(S) \cap V^1 = \{(1, i_1), \ldots, (1, i_{C-k})\}. \) This holds for any \( \ell \in \)
\[ J = [C]_0 \setminus \{i_1, \ldots, i_{C-k}\}. \] Therefore the set of vertices \( K = \{(2,(ij)) : i,j \in J\} \cup \{(1,(i)) : i \in J\} \cup \{(0,(1))\} \) has an empty intersection with \( P_k^1(S) \). Since every vertex of \( WK\,P_{(C,2)} - K \) has either 0 or \( k+1 \) neighbours in \( K \), no vertex from this set may get monitored later on, a contradiction. The case when \((0,(1)) \in S\) or that \( S \) intersects some \( Q_i \) in more than one vertex can be proved analogously.

We can now determine the \( k \)-propagation radius of \( WK\,P_{(C,2)} \) using the previous lemma.

**Theorem 5.3.10.** Let \( C \in \mathbb{N}_2 \). Then

\[
\text{rad}_{P,k}(WK\,P_{(C,2)}) = \begin{cases} 
2, & k \geq C; \\
3, & k \in [C-1]. 
\end{cases}
\]

**Proof.** For \( k \geq C \), \( \gamma_{P,k}(WK\,P_{(C,2)}) = 1 \), by Theorem 5.3.1 and observe that \( \gamma(WK\,P_{(C,2)}) > 1 \). Therefore, \( \text{rad}_{P,k}(WK\,P_{(C,2)}) \geq 2 \) (by Proposition 1.4.13). And, for the set \( S = \{(0,(1))\} \), we get that \( P^0_k(S) = S \cup V^1 \) and \( P^1_k(S) = V(WK\,P_{(C,2)}) \). Now let \( k \in [C-1] \). For \( C = 2 \), the result easily follows. Let \( C \geq 3 \). By Theorem 5.3.4, \( \gamma_{P,k}(WK\,P_{(C,2)}) = C-k \) and therefore by Lemma 5.3.9, \( |S \cap V^2| \leq C - k - 1 \) for every minimum
$k$-PDS $S$. Then there exist at least $k + 1$ $C$-cliques, $Q_i$, not containing any vertex of $S$. Let $Q_\nu$ be an arbitrary clique such that $S \cap V(Q_\nu) = \emptyset$ and $(1, (i')) \notin S$. We prove that the vertex $(2, (i'i'))$ is not in $P^1_k(S)$. Clearly, $(2, (i'i')) \notin P^0_k(S)$. Moreover, $|V(Q_\nu) \cap P^0_k(S)| \leq C - k - 1$ and $|V(Q_\nu) \setminus P^0_k(S)| \geq k + 1$. Therefore any neighbour of $(2, (i'i'))$ in $Q_i'$ is adjacent to more than $k$ unmonitored vertices preventing any propagation to this vertex at this step. Also, since $(1, (i'))$ has more than $k$ unmonitored vertices as its neighbours, $(2, (i'i'))$ cannot be monitored by $(1, (i'))$ at this step. Hence $\text{rad}_{P,k}(WKP(C,2)) \geq 3$. Also, by Lemma 5.3.2, $\text{rad}_{P,k}(WKP(C,2)) \leq L$.

**Remark 5.3.1.** For $C, L \in \mathbb{N}_3$, by observing the propagation behaviour described in the proof of Theorem 5.3.1 and Lemma 5.3.5, one can obtain that $\text{rad}_{P,k}(WKP(C,L)) \leq L$ if $k \geq C$ and $\text{rad}_{P,k}(WKP(C,L)) \leq \max\{5, L - 1\}$ if $k \in [C - 2]$.

**Illustration of Lemma 5.3.5**

We illustrate Lemma 5.3.5 for the case $k = 1, C = 5$ and $L = 3$. Figure 5.8 depicts the graph $WKP(5,3)$. We know the vertices in
the third level of $WKP_{(5,3)}$ induce the subgraph $WK_{(5,3)}$ which is hamiltonian. And, the cycle in the subgraph $WK_{(5,3)}$, as defined in Lemma 5.3.5, are drawn as bold edges in the figure. The consecutive subgraphs $G_{w}^{5,3}$, $G_{w'}^{5,3}$, and $G_{w''}^{5,3}$ in the hamiltonian cycle of $WK_{(5,3)}$ and the vertices $x, x', y'$ and $y''$ as chosen in the lemma are also marked in the figure. The vertices of the set $S$ as constructed in the lemma are coloured black in Figure 5.8.

In Figure 5.9, the vertices in the set $P_0^1(S)$, are coloured black and the remaining vertices white. The black vertices in Figure 5.10 and Figure 5.11 represent the vertices in the set $P_1^1(S)$ and $P_2^1(S)$, respectively. The directed edges in the figures indicate the direction in which the propagation occurs at each step.

We can observe that all vertices of $WKP_{(5,3)}$ get monitored by step 2 and $P_2^2(S) = V(WKP_{(5,3)})$. Therefore, $S$ is a 1-PDS of $WKP_{(5,3)}$ and $|S| = 15 = (5 - 1 - 1) \cdot 5^{3-2}$. 
Figure 5.8: The graph $WKP_{(5,3)}$.

Figure 5.9: $\mathcal{P}_1^0(S)$. 

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Figure 5.10: $\mathcal{P}_1^1(S)$.

Figure 5.11: $\mathcal{P}_1^2(S)$. 
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