CHAPTER 4

HYBRID FAMILIES OF SPECIAL POLYNOMIALS
ASSOCIATED WITH APPELL SEQUENCES

4.1 Introduction

In 1982, H. M. Srivastava [138] considered several characterizations for the well known Appell polynomials and for their basic analogues. One aspect of such study is to find recurrence relations and differential equations for the Appell polynomials. The interest in Appell polynomials and their applications in different fields has significantly increased. The recent applications of Appell polynomials in probability theory and statistics are considered in [9, 133]. The generalized Appell polynomials as tools for approximating 3D-mappings were introduced for the first time in [113] in combination with Clifford analysis methods. The representation theoretic results like those of [105] provide new examples of applications of Appell polynomials and gave evidence to the central role of Appell polynomials as orthogonal polynomials. Representation theory is also the tool for their applications in quantum physics as explained in [152].

The hybrid polynomials are obtained by combining two polynomial families using their quasi-monomial properties and certain operational rules. These polynomials can also be framed within the context of monomiality principle. The recurrence relations, differential equations and other results of the hybrid special polynomials can be used to solve
the existing as well as new emerging problems in certain branches of science. To establish the determinant forms for the hybrid special polynomials is a new and recent investigation [96, 97] which can be helpful for computation purposes. These polynomials can be studied from different points of view, for example, to establish orthogonality of these polynomials may be taken as an important future aspect. To study the hybrid numbers corresponding to the hybrid families from combinatorial aspect can also be taken as future possibility. To find generalizations of the positive linear operators involving hybrid special functions for applications in approximation theory may also be explored.

Recently, the Laguerre-Gould-Hopper polynomials (LGHP) \( L_n^{(m,r)}(x, y, z) \) [86] are introduced by combining \( 2VgLP_m L_n(x, y) \) and the Gould-Hopper polynomials \( H_n^{(r)}(x, y) \), which are defined by means of the generating function

\[
C_0(-xt^m) \exp(yt + zt^r) = \sum_{n=0}^{\infty} L_n^{(m,r)}(x, y, z) \frac{t^n}{n!}.
\]  

(4.1.1)

The LGHP \( L_n^{(m,r)}(x, y, z) \) are shown to be quasi-monomial under the action of the following multiplicative and derivative operators [86]:

\[
\hat{M}_{LH} := y + mD_x^{-1} \frac{\partial^{n-1}}{\partial y^{m-1}} + rz \frac{\partial^{r-1}}{\partial y^{r-1}}
\]  

(4.1.2)

and

\[
\hat{P}_{LH} := \frac{\partial}{\partial y},
\]  

(4.1.3)

respectively.

The hybrid special polynomials of three variables are important from the point of view of applications. Also, these polynomials allow the derivation of a number of useful identities in a fairly straightforward way and help in introducing new families of special polynomials. Most of the multi-variable special polynomials and their generalizations have been suggested by physical problems. They provided new means of analysis for the solution of large classes of partial differential equations often encountered in physical problems.

For suitable values of the indices and variables, the LGHP \( L_n^{(m,r)}(x, y, z) \) reduce to a number of other known special polynomials. These special cases are mentioned in Table 4.1.1.
Hybrid families of special polynomials associated with Appell sequences

<table>
<thead>
<tr>
<th>S. No.</th>
<th>Values of the indices and variables</th>
<th>Relation between the LGHP $L H_n^{(m, r)}(x, y, z)$ and its special case</th>
<th>Name of the known polynomials Generating functions of the known polynomials</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.</td>
<td>$m = 1, r = 2, x \to -x$</td>
<td>$L H_n^{(1, 2)}(-x, y, z) = L H_n(x, y, z)$</td>
<td>3-Variable Laguerre-Hermite</td>
</tr>
<tr>
<td>II.</td>
<td>$m = 1, r = 2, z = -\frac{1}{2}, x \to -x$</td>
<td>$L H_n^{(1, 2)}(-x, y, -\frac{1}{2}) = L H_n^*(x, y)$</td>
<td>2-Variable Laguerre-Hermite</td>
</tr>
<tr>
<td>III.</td>
<td>$m = 1, r = 2, y = 1, z \to y, x \to -x$</td>
<td>$L H_n^{(1, 2)}(-1, y, y) = \varphi_n(x, y)$</td>
<td>Hermite type</td>
</tr>
<tr>
<td>IV.</td>
<td>$x = 0$</td>
<td>$L H_n^{(m, r)}(0, y, z) = H_n^{(r)}(y, z)$</td>
<td>Gould-Hopper</td>
</tr>
<tr>
<td>V.</td>
<td>$x = 0$</td>
<td>$L H_n^{(m, r)}(x, y, 0) = m L_n(x, y)$</td>
<td>2-Variable generalized Laguerre</td>
</tr>
<tr>
<td>VI.</td>
<td>$r = m; z = 0, y \to -D_x^{-1}$;</td>
<td>$L H_n^{(m, m)}(0, y, 0) = [m] L_n(x, y)$</td>
<td>2-Variable generalized Laguerre type</td>
</tr>
<tr>
<td>VII.</td>
<td>$r = m - 1; x = 0, y \to z, z \to y$</td>
<td>$L H_n^{(m, r)}(0, 0, y) = U^{(m)}(x, y)$</td>
<td>Generalized Chebyshev</td>
</tr>
<tr>
<td>VIII.</td>
<td>$m = 1; z = 0, x \to -x$</td>
<td>$L H_n^{(1, r)}(-x, y, 0) = L_n(x, y)$</td>
<td>Laguerre</td>
</tr>
<tr>
<td>IX.</td>
<td>$m = 1; y \to -D_x^{-1}$;</td>
<td>$n^2 L H_n^{(1, r)}(y, -D_x^{-1}, 0) = R_n(x, y)$</td>
<td>2-Variable Legendre</td>
</tr>
<tr>
<td>X.</td>
<td>$x = 0, y \to x, z \to y, y \to -D_y^{-1}$;</td>
<td>$L H_n^{(m, r)}(0, 0, y) = ^{(r)}(x, y)$</td>
<td>2-Variable truncated of order $x$</td>
</tr>
<tr>
<td>XI.</td>
<td>$r = 2; x = 0$</td>
<td>$L H_n^{(m, 2)}(0, y, z) = H_n(x, y, z)$</td>
<td>2-Variable Hermite-Kampf de Foccart</td>
</tr>
<tr>
<td>XII.</td>
<td>$r = 2; x = 0, y \to D_x^{-1}, z \to y$</td>
<td>$L H_n^{(m, 2)}(0, D_x^{-1}, y) = G_n(x, y)$</td>
<td>Hermite type</td>
</tr>
<tr>
<td>XIII.</td>
<td>$m = 2; z = 0, x \to x^2-1, y \to x$</td>
<td>$L H_n^{(2, r)}(x^2-1, x, 0) = P_n(x)$</td>
<td>Legendre</td>
</tr>
<tr>
<td>XIV.</td>
<td>$x \to y \partial_y, y \to x$</td>
<td>$L H_n^{(m, r)}(y \partial_y, x, z)$</td>
<td>3-Variable generalized Hermite</td>
</tr>
<tr>
<td>XV.</td>
<td>$m = 2, r = 3, x \to z \partial_z, y \to z \partial_y$</td>
<td>$L H_n^{(2, 3)}(z \partial_z, x, y)$</td>
<td>Bell-type</td>
</tr>
</tbody>
</table>
In this chapter, the Laguerre-Gould-Hopper based Appell polynomial families are introduced by using the concepts and the methods associated with monomiality principle. In Section 4.2, the Laguerre-Gould-Hopper-Appell polynomials (LGHAP) \( L_{H(m,r)}A_n(x, y, z) \) are introduced and framed within the context of the monomiality principle formalism. Some operational representations are also derived. In Section 4.3, results for some members belonging the LGHAP family are obtained. In Section 4.4, the determinant definition of the LGHAP \( L_{H(m,r)}A_n(x, y, z) \) is established. In Section 4.5, the Laguerre-Gould-Hopper-Sheffer polynomials (LGHSP) \( L_{H(m,r)}s_n(x, y, z) \) are introduced and their properties are established.

### 4.2 Laguerre-Gould-Hopper-Appell Polynomials

To introduce the Laguerre-Gould-Hopper based Appell polynomials (LGHAP) denoted by \( L_{H(m,r)}A_n(x, y, z) \), the following result is proved:

**Theorem 4.2.1.** For the Laguerre-Gould-Hopper-Appell polynomials \( L_{H(m,r)}A_n(x, y, z) \), the following generating function holds true:

\[
A(t)C_0(-xt^m) \exp(yt + zt^r) = \sum_{n=0}^{\infty} L_{H(m,r)}A_n(x, y, z) \frac{t^n}{n!}; \tag{4.2.1}
\]

**Proof.** Replacement of \( x \) in the l.h.s. and r.h.s of equation (1.2.2) by the multiplicative operator \( \hat{M}_{\text{LGHP}} \) of the LGHP \( L_{H_n}^{(m,r)}(x, y, z) \) gives

\[
A(t) \exp(\hat{M}_{\text{LGHP}}t) = \sum_{n=0}^{\infty} A_n(\hat{M}_{\text{LGHP}}) \frac{t^n}{n!}; \tag{4.2.2}
\]

Using the expression of \( \hat{M}_{\text{LGHP}} \) given in equation (4.1.2) and then decoupling the exponential operator in the l.h.s. of the resultant equation by using the Crofton-type identity (1.4.4), it follows that

\[
A(t) \exp \left( z \frac{\partial^r}{\partial y^r} \right) \exp \left( \left( y + mD_x^{-1} \frac{\partial^{m-1}}{\partial y^{m-1}} \right) t \right) = \sum_{n=0}^{\infty} A_n \left( y + mD_x^{-1} \frac{\partial^{m-1}}{\partial y^{m-1}} + rz \frac{\partial^{r-1}}{\partial y^{r-1}} \right) \frac{t^n}{n!},
\]

which on further use of identity (1.4.4) in the l.h.s. becomes
\[ A(t) \exp \left( z \frac{\partial^p}{\partial y^p} \right) \exp \left( D_x^{-1} \frac{\partial^m}{\partial y^m} \right) \exp(yt) = \sum_{n=0}^{\infty} A_n \left( y + m D_x^{-1} \frac{\partial^m}{\partial y^m} + rz \frac{\partial^r}{\partial y^r} \right) t^n n!. \] (4.2.3)

Expanding the second exponential in the l.h.s. of equation (4.2.3) and then using definition (1.3.6) so that it becomes

\[ A(t)C_0 (-xt^m) \exp \left( z \frac{\partial^p}{\partial y^p} \right) \exp(yt) = \sum_{n=0}^{\infty} A_n \left( y + m D_x^{-1} \frac{\partial^m}{\partial y^m} + rz \frac{\partial^r}{\partial y^r} \right) t^n n!. \] (4.2.4)

Again, expanding the first exponential in the l.h.s. of equation (4.2.4) and denoting the resultant LGHAP in the r.h.s. by \( \hat{L}_{H}^{(m,r)} A_n(x, y, z) \), that is

\[ \hat{L}_{H}^{(m,r)} A_n(x, y, z) = A_n(\hat{M}_{LH}) = A_n \left( y + m D_x^{-1} \frac{\partial^m}{\partial y^m} + rz \frac{\partial^r}{\partial y^r} \right), \] (4.2.5)

assertion (4.2.1) is proved.

Next, to show that the LGHAP \( \hat{L}_{H}^{(m,r)} A_n(x, y, z) \) satisfy the monomiality property, the following result is proved:

**Theorem 4.2.2.** The Laguerre-Gould-Hopper-Appell polynomials \( \hat{L}_{H}^{(m,r)} A_n(x, y, z) \) are quasi-monomial with respect to the following multiplicative and derivative operators:

\[ \hat{M}_{LHA} = y + m D_x^{-1} \frac{\partial^m}{\partial y^m} + rz \frac{\partial^r}{\partial y^r} + \frac{A'(\partial y)}{A(\partial y)} \] (4.2.6)

and

\[ \hat{P}_{LHA} = \partial y, \] (4.2.7)

respectively, where \( \partial y := \frac{\partial}{\partial y} \).

**Proof.** Consider the following identity:

\[ \partial_y \{ \exp(yt + zt^r) \} = t \exp(yt + zt^r). \] (4.2.8)
Differentiating equation (4.2.2) partially with respect to $t$ and in view of relation (4.2.5), it follows that
\[
\left( \hat{M}_{LH} + \frac{A'(t)}{A(t)} \right) A(t) \exp (\hat{M}_{LH} t) = \sum_{n=0}^{\infty} L_{H(m,r)} A_{n+1}(x, y, z) \frac{t^n}{n!},
\]
which on using equations (1.4.10) and (4.1.1) gives
\[
\left( \hat{M}_{LH} + \frac{A'(t)}{A(t)} \right) A(t) C_0 (-xt^m) \exp (yt + zt^r) = \sum_{n=0}^{\infty} L_{H(m,r)} A_{n+1}(x, y, z) \frac{t^n}{n!}. \quad (4.2.9)
\]

In view of relation (4.2.8), the above equation becomes
\[
\left( \hat{M}_{LH} + \frac{A'(t)}{A(t)} \right) \left\{ A(t) C_0 (-xt^m) \exp (yt + zt^r) \right\} = \sum_{n=0}^{\infty} L_{H(m,r)} A_{n+1}(x, y, z) \frac{t^n}{n!}, \quad (4.2.10)
\]
which on using generating function (4.2.1) becomes
\[
\left( \hat{M}_{LH} + \frac{A'(t)}{A(t)} \right) \left\{ \sum_{n=0}^{\infty} L_{H(m,r)} A_n(x, y, z) \frac{t^n}{n!} \right\} = \sum_{n=0}^{\infty} L_{H(m,r)} A_{n+1}(x, y, z) \frac{t^n}{n!}. \quad (4.2.11)
\]

Adjusting the summation in the l.h.s. of equation (4.2.11) and then equating the coefficients of like powers of $t$ in both sides, it follows that
\[
\left( \hat{M}_{LH} + \frac{A'(t)}{A(t)} \right) \left\{ \sum_{n=0}^{\infty} L_{H(m,r)} A_n(x, y, z) \frac{t^n}{n!} \right\} = L_{H(m,r)} A_{n+1}(x, y, z), \quad (4.2.12)
\]
which, in view of equation (1.4.5) indicates that the corresponding multiplicative operator for $L_{H(m,r)} A_n(x, y, z)$ is given as:
\[
\hat{M}_{LHA} = \left( \hat{M}_{LH} + \frac{A'(t)}{A(t)} \right).
\]

Finally, using equation (4.1.2) in the r.h.s of above expression, assertion (4.2.6) follows.

A consequence of identity (4.2.8) is
\[
\partial_y \left\{ A(t) C_0 (-xt^m) \exp (yt + zt^r) \right\} = t A(t) C_0 (-xt^m) \exp (yt + zt^r),
\]
which on using generating function (4.2.1) becomes
\[ \partial_y \left\{ \sum_{n=0}^{\infty} L_{H^{(m,r)}} A_n(x, y, z) \frac{t^n}{n!} \right\} = \sum_{n=1}^{\infty} L_{H^{(m,r)}} A_{n-1}(x, y, z) \frac{t^n}{(n-1)!}. \]

On adjusting the summation in the l.h.s. of the above equation and then equating the coefficients of like powers of \( t \), it gives
\[ \partial_y \{ L_{H^{(m,r)}} A_n(x, y, z) \} = n L_{H^{(m,r)}} A_{n-1}(x, y, z), \quad n \geq 1, \]
(4.2.13)
which in view of equation (1.4.6) yields assertion (4.2.7).

**Theorem 4.2.3.** The Laguerre-Gould-Hopper-Appell polynomials \( L_{H^{(m,r)}} A_n(x, y, z) \) satisfy the following differential equation:
\[
\left( y \partial_y + m D_x^{-1} \frac{\partial^n}{\partial y^m} + rz \frac{\partial^r}{\partial y^r} + \frac{A'(\partial_y)}{A(\partial_y)} \partial_y - n \right) L_{H^{(m,r)}} A_n(x, y, z) = 0. \]
(4.2.14)

**Proof.** Using equations (4.2.6) and (4.2.7) in the monomiality equation (1.4.8) for the LGHAP \( L_{H^{(m,r)}} A_n(x, y, z) \), assertion (4.2.14) is proved.

**Remark 4.2.1.** In view of equations (1.4.10) and (4.2.5), equation (4.2.2) can be written as
\[
A(t) \sum_{n=0}^{\infty} L_{H_n^{(m,r)}}(x, y, z) \frac{t^n}{n!} = \sum_{n=0}^{\infty} L_{H^{(m,r)}} A_n(x, y, z) \frac{t^n}{n!}, \]
(4.2.15)
which on using expansion (1.2.3) of \( A(t) \) in the l.h.s. and then equating the coefficients of like powers of \( t \) on both sides of the resultant equation, yields the following series definition for the LGHAP \( L_{H^{(m,r)}} A_n(x, y, z) \):
\[
L_{H^{(m,r)}} A_n(x, y, z) = \sum_{k=0}^{n} \binom{n}{k} L_{H_n^{(m,r)}}(x, y, z) \alpha_k, \]
(4.2.16)
where \( \alpha_k \) is given by equation (1.2.3).

Special cases of the LGHP \( L_{H_n^{(m,r)}}(x, y, z) \) are mentioned in Table 4.1.1. Now, for the same choice of the variables and indices the LGHAP \( L_{H^{(m,r)}} A_n(x, y, z) \) reduce to the corresponding hybrid special polynomials. These known and new special polynomials related to the Appell sequences are listed in the following table:
### Table 4.2.1. Special cases of the LGHAP $L_H^{(m,r)} A_n(x, y, z)$

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Values of the indices and variables</th>
<th>Relation between the LGHAP $L_H^{(m,r)} A_n(x, y, z)$ and its special case</th>
<th>Name of the hybrid special polynomials</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.</td>
<td>$m = 1, r = 2, x \rightarrow -x$</td>
<td>$L_H^{(1,2)} A_n(-x, y, z) = L_H A_n(x, y, z)$</td>
<td>3-Variable Laguerre-Hermite-Appell polynomials (3VHLAP)</td>
</tr>
<tr>
<td>II.</td>
<td>$m = 1, r = 2, z = \frac{1}{2}, x \rightarrow -x$</td>
<td>$L_H^{(1,2)} A_n(-x, y, -\frac{1}{2}) = L_H \ast A_n(x, y)$</td>
<td>2-Variable Laguerre-Hermite-Appell polynomials (2VHLAP)</td>
</tr>
<tr>
<td>III.</td>
<td>$m = 1, r = 2, y = 1, z \rightarrow y, x \rightarrow -x$</td>
<td>$L_H^{(1,2)} A_n(-x, 1, y) = \varphi A_n(x, y)$</td>
<td>2-Variable Laguerre-Hermite type Appell polynomials (LHTAP)</td>
</tr>
<tr>
<td>IV.</td>
<td>$x = 0$</td>
<td>$L_H^{(m,r)} A_n(0, y, z) = H^{(r)} A_n(y, z)$</td>
<td>Gould-Hopper-Appell polynomials (GHAP) [94]</td>
</tr>
<tr>
<td>V.</td>
<td>$z = 0$</td>
<td>$L_H^{(m,r)} A_n(x, y, 0) = m L A_n(x, y)$</td>
<td>2-Variable generalized Laguerre-Appell polynomials (2VGLAP)</td>
</tr>
<tr>
<td>VI.</td>
<td>$r = m, x = 0, y \rightarrow -D^{-1}_x, z \rightarrow y$</td>
<td>$L_H^{(m,m)} A_n(0, -D^{-1}_x, y) = [m] L A_n(x, y)$</td>
<td>2-Variable generalized Laguerre type Appell polynomials (2VGLTAP)</td>
</tr>
<tr>
<td>VII.</td>
<td>$r = m - 1, x = 0, y \rightarrow x, z \rightarrow y$</td>
<td>$L_H^{(m,m-1)} A_n(0, x, y) = \zeta^{(m)} A_n(x, y)$</td>
<td>Generalized Chebyshev-Appell polynomials (GCCAP)</td>
</tr>
<tr>
<td>VIII.</td>
<td>$m = 1, z = 0, x \rightarrow -x$</td>
<td>$L_H^{(1,1)} A_n(-x, y, 0) = L A_n(x, y)$</td>
<td>2-Variable Laguerre-Appell polynomials (2VLAAP) [99]</td>
</tr>
<tr>
<td>IX.</td>
<td>$m = 1, z = 0, x \rightarrow y, y \rightarrow -D^{-1}_x$</td>
<td>$L_H^{(1,1)} A_n(y, -D^{-1}_x, 0) = \frac{R A_n(x, y)}{m!}$</td>
<td>2-Variable Legendre-Appell polynomials (2VLAGP)</td>
</tr>
<tr>
<td>X.</td>
<td>$x = 0, y \rightarrow x, z \rightarrow y\partial_y y$</td>
<td>$L_H^{(m,r)} A_n(0, x, y\partial_y y) = \zeta^{(r)} A_n(x, y)$</td>
<td>2-Variable truncated exponential-Appell polynomials (2VTAP)</td>
</tr>
<tr>
<td>XI.</td>
<td>$r = 2, x = 0$</td>
<td>$L_H^{(m,2)} A_n(0, y, z) = H A_n(y, z)$</td>
<td>2-Variable Hermite Kampé de Fériet-Appell polynomials (2VHKdFAP)</td>
</tr>
<tr>
<td>XII.</td>
<td>$r = 2, x = 0, y \rightarrow D^{-1}_x, z \rightarrow y$</td>
<td>$L_H^{(m,2)} A_n(0, y, z) = G A_n(x, y)$</td>
<td>Hermite type Appell polynomials (HTAP)</td>
</tr>
<tr>
<td>XIII.</td>
<td>$m = 2, z = 0, x \rightarrow \left(\frac{z^2 - 1}{4}\right), y \rightarrow \frac{x}{2}$</td>
<td>$L_H^{(2,r)} A_n(\frac{z^2 - 1}{4}, x, 0) = p A_n(x)$</td>
<td>Legendre-Appell polynomials (LAP)</td>
</tr>
<tr>
<td>XIV.</td>
<td>$x \rightarrow y\partial_y y, y \rightarrow x$</td>
<td>$L_H^{(m,r)} A_n(y\partial_y y, x, z) = H^{(r,m)} A_n(x, y, z)$</td>
<td>3-Variable generalized Hermite-Appell polynomials (3VGHAP)</td>
</tr>
<tr>
<td>XV.</td>
<td>$m = 2, r = 3, x \rightarrow 2z, y \rightarrow x, z \rightarrow y$</td>
<td>$L_H^{(2,3)} A_n(z\partial_z z, x, y) = H^{(3,2)} A_n(x, y, z)$</td>
<td>Bell type Appell polynomials (BTAP)</td>
</tr>
</tbody>
</table>
Remark 4.2.2. In view of the special cases mentioned in Table 4.2.1, the corresponding results for the special polynomials related to the Appell sequences can be obtained.

Next, to establish the operational representation for the LGHAP $L_{H(m,r)}A_n(x, y, z)$, the following results are proved:

**Theorem 4.2.4.** The following operational representation between the Laguerre-Gould-Hopper-Appell polynomials $L_{H(m,r)}A_n(x, y, z)$ and the Appell polynomials $A_n(x)$ holds true:

$$L_{H(m,r)}A_n(x, y, z) = \exp \left( D_x^{-1} \frac{\partial^m}{\partial y^m} + z \frac{\partial^r}{\partial y^r} \right) A_n(y).$$  \hspace{1cm} (4.2.17)

**Proof.** In view of equation (4.2.5), the proof is direct use of identity (1.4.4).  \hspace{1cm} ■

**Theorem 4.2.5.** The following operational representation between the Laguerre-Gould-Hopper-Appell polynomials $L_{H(m,r)}A_n(x, y, z)$ and the 2-variable generalized Laguerre-Appell polynomials $m_LA_n(x, y)$ holds true:

$$L_{H(m,r)}A_n(x, y, z) = \exp \left( z \frac{\partial^r}{\partial y^r} \right) m_LA_n(x, y).$$  \hspace{1cm} (4.2.18)

**Proof.** From equation (4.2.1), it follows that

$$\frac{\partial^r}{\partial y^r} L_{H(m,r)}A_n(x, y, z) = \frac{\partial}{\partial z} L_{H(m,r)}A_n(x, y, z).$$  \hspace{1cm} (4.2.19)

Again, since (Table 4.2.1 (V))

$$L_{H(m,r)}A_n(x, y, 0) = m_LA_n(x, y).$$  \hspace{1cm} (4.2.20)

Therefore, solving equation (4.2.19) subject to initial condition (4.2.20), assertion (4.2.18) is obtained.  \hspace{1cm} ■

**Theorem 4.2.6.** The following operational representation between the Laguerre-Gould-Hopper-Appell polynomials $L_{H(m,r)}A_n(x, y, z)$ and the Gould-Hopper-Appell polynomials $H^{(r)}A_n(y, z)$ holds true:

$$L_{H(m,r)}A_n(x, y, z) = \exp \left( D_x^{-1} \frac{\partial^m}{\partial y^m} \right) H^{(r)}A_n(y, z).$$  \hspace{1cm} (4.2.21)
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Proof. From equations (1.3.6) and (4.2.1), it follows that

$$\frac{\partial^m}{\partial y^m} L_H^{(m,r)} A_n(x, y, z) = \frac{\partial}{\partial D^{-1}_x} L_H^{(m,r)} A_n(x, y, z)$$  \hspace{1cm} (4.2.22)

where ([57]; p. 32 (8)):

$$\frac{\partial}{\partial D^{-1}_x} := \frac{\partial}{\partial x} x \frac{\partial}{\partial x}.$$

Again, since (Table 4.2.1 (IV))

$$L_H^{(m,r)} A_n(0, y, z) = H_r A_n(y, z).$$  \hspace{1cm} (4.2.23)

Therefore, solving equation (4.2.22) subject to initial condition (4.2.23), assertion (4.2.21) is proved.

The Appell polynomials have been studied because of their remarkable applications not only in different branches of mathematics but also in theoretical physics and chemistry. These polynomials contains important sequences such as Bernoulli, Euler, Genocchi, Miller-Lee polynomials etc.

Next, the series definition for the Laguerre-Gould-Hopper-Appell polynomials $$L_H^{(m,r)} A_n(x, y, z)$$ is obtained in terms of the Appell polynomials.

Use of definition (1.2.2) in equation (4.2.1) gives

$$\exp(zt^r) C_0(-xt^m) \sum_{n=0}^{\infty} A_n(y) \frac{t^n}{n!} = \sum_{n=0}^{\infty} L_H^{(m,r)} A_n(x, y, z) \frac{t^n}{n!},$$  \hspace{1cm} (4.2.24)

which on using definition (1.3.5) in the l.h.s. and expanding the exponential becomes

$$\sum_{n,l,k=0}^{\infty} x^k z^l A_n(y) t^{n+rl+mk} \frac{(k!)^2 l! n!}{(k!)^2 l! n!} = \sum_{n=0}^{\infty} L_H^{(m,r)} A_n(x, y, z) \frac{t^n}{n!}.$$

Equating the coefficients of like powers of $$t$$ in the above equation, the following series definition for the Laguerre-Gould-Hopper-Appell polynomials $$L_H^{(m,r)} A_n(x, y, z)$$ is obtained:

$$L_H^{(m,r)} A_n(x, y, z) = n! \sum_{l=0}^{\left\lfloor \frac{n}{r} \right\rfloor} \sum_{k=0}^{\left\lfloor \frac{n}{m} \right\rfloor} \frac{A_{n-rl-mk}(y) z^l x^k}{l!(k!)^2 (n - rl - mk)!}. $$  \hspace{1cm} (4.2.25)

In the next section, the results for some members belonging to the Laguerre-Gould-Hopper-Appell polynomial family $$L_H^{(m,r)} A_n(x, y, z)$$ are obtained. The surface plots for these polynomials are also drawn.
4.3 Examples

In order to obtain the results for certain hybrid members of the Laguerre-Gould-Hopper-Appell polynomials family, the following examples are considered:

Example 4.3.1. Since for $A(t) = \frac{t}{(e^t - 1)}$, the Appell polynomials $A_n(x)$ become the Bernoulli polynomials $B_n(x)$ (Table 1.2.1 (I)). Therefore, for the same choice of $A(t)$, the LGHAP reduce to the Laguerre-Gould-Hopper-Bernoulli polynomials (LGHBP)

$L_{H(m,r)}B_n(x, y, z)$.

Thus, by putting the above value of $A(t)$ in equations (4.2.1), (4.2.6) and (4.2.7) and (4.2.25), the generating function, multiplicative and derivative operators and the series definition respectively for the LGHBP $L_{H(m,r)}B_n(x, y, z)$ are obtained as:

$$\frac{t}{(e^t - 1)} C_0(-xt^m) \exp(yt + zt^r) = \sum_{n=0}^{\infty} L_{H(m,r)}B_n(x, y, z) \frac{t^n}{n!}, \quad (4.3.1)$$

$$\hat{M} = y + mD^{-1} \frac{\partial^{m-1}}{\partial y^{m-1}} + rz \frac{\partial^r-1}{\partial y^r-1} + \frac{e^{\partial_y}(1 - \partial_y) - 1}{\partial_y(e^{\partial_y} - 1)} \quad (4.3.2)$$

and

$$\hat{P} = \partial_y \quad (4.3.3)$$

and

$$L_{H(m,r)}B_n(x, y, z) = n! \sum_{l=0}^{\lfloor \frac{n}{r} \rfloor} \sum_{k=0}^{\lfloor \frac{m}{l} \rfloor} \frac{B_{n-rl-mk}(y) z^l x^k}{l!(k!)^2(n - r l - mk)!!}. \quad (4.3.4)$$

Further, to draw the surface plot of the LGHBP $L_{H(m,r)}B_n(x, y, z)$, the values of the first six Bernoulli polynomials $B_n(x)$ are required. These values are given in the following table:

Table 4.3.1. First six Bernoulli polynomials

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_n(x)$</td>
<td>1</td>
<td>$x - \frac{1}{2}$</td>
<td>$x^2 - x + \frac{1}{6}$</td>
<td>$x^3 - \frac{3}{2}x^2 + \frac{x}{4}$</td>
<td>$x^4 - 2x^3 + x^2 - \frac{1}{12}$</td>
<td>$x^5 - \frac{5}{2}x^4 + \frac{5}{4}x^3 - \frac{x}{6}$</td>
</tr>
</tbody>
</table>
Next, taking \( n = 4 \), \( m = 3 \) and \( r = 5 \) in definition (4.3.4), to obtain

\[
L_{H(3,5)}B_4(x, y, z) = B_4(y) + 24B_1(y)x,
\]

which on using the particular values of \( B_n(x) \) from Table 4.3.1 gives

\[
L_{H(3,5)}B_4(x, y, z) = y^4 - 2y^3 + y^2 + 24xy - 12x - \frac{1}{30}.
\]

By making use of equation (4.3.6), the following surface plot of \( L_{H(3,5)}B_4(x, y, z) \) is drawn:

![Surface plot of \( L_{H(3,5)}B_4(x, y, z) \)](image)

**Figure 4.3.1**

**Example 4.3.2.** Since for \( A(t) = \frac{2}{(e^t+1)} \), the Appell polynomials \( A_n(x) \) become the Euler polynomials \( E_n(x) \) (Table 1.2.1 (II)). Therefore, for the same choice of \( A(t) \) the LGHAP reduce to the Laguerre-Gould-Hopper-Euler polynomials (LGHEP) \( L_{H(m,r)}E_n(x, y, z) \).

Thus, by taking the above value of \( A(t) \) in equations (4.2.1), (4.2.6), (4.2.7) and (4.2.25), the following results for the LGHEP \( L_{H(m,r)}E_n(x, y, z) \) are obtained:
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\[
\frac{2}{(e^t + 1)} C_0(-xt^m) \exp(yt + zt^r) = \sum_{n=0}^{\infty} L_{H(m,r)} E_n(x, y, z) \frac{t^n}{n!},
\]

(4.3.7)

\[
\hat{M} = y + m D_x^{-1} \frac{\partial^{m-1}}{\partial y^{m-1}} + rz \frac{\partial^{r-1}}{\partial y^{r-1}} - \frac{e^{\partial_y}}{(e^{\partial_y} + 1)},
\]

(4.3.8)

\[
\hat{P} = \partial_y
\]

(4.3.9)

and

\[
L_{H(m,r)} E_n(x, y, z) = n! \sum_{l=0}^{[\frac{n}{r}]} \sum_{k=0}^{[\frac{n}{m}]} \frac{E_{n-rl-mk}(y) z^l x^k}{l!(k!)^2 (n - rl - mk)!}.
\]

(4.3.10)

Further, to draw the surface plot of the LGHEP \( L_{H(m,r)} E_n(x, y, z) \) the values of the first six Euler polynomials \( E_n(x) \) are required. These values are given in the following table:

<table>
<thead>
<tr>
<th>( n ) ( E_n(x) )</th>
<th>( 0 )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( 4 )</th>
<th>( 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_0(x) )</td>
<td>1</td>
<td>( x - \frac{1}{2} )</td>
<td>( x^2 - x )</td>
<td>( x^3 - \frac{3}{2} x^2 + \frac{1}{6} )</td>
<td>( x^4 - 2x^3 + \frac{3}{2} x )</td>
<td>( x^5 - \frac{5}{2} x^4 + \frac{5}{6} x^2 - \frac{1}{2} )</td>
</tr>
</tbody>
</table>

Next, taking \( n = 4 \), \( m = 3 \) and \( r = 5 \) in definition (4.3.10), to obtain

\[
L_{H(3,5)} E_4(x, y, z) = E_4(y) + 24 E_1(y) x,
\]

(4.3.11)

which on using the particular values of \( E_n(x) \) from Table 4.3.2 gives

\[
L_{H(3,5)} E_4(x, y, z) = y^4 - 2y^3 + 24xy + \frac{2}{3}y - 12x.
\]

(4.3.12)

In view of equation (4.3.12), the following surface plot of \( L_{H(3,5)} E_4(x, y, z) \) is drawn:
Example 4.3.3. Since for \( A(t) = \frac{1}{(1-t)^{s+1}} \), the Appell polynomials \( A_n(x) \) become the Miller-Lee polynomials \( G^{(s)}_n(x) \) (Table 1.2.1 (XIII)) . Therefore, for the same choice of \( A(t) \) the LGHAP reduce to the Laguerre-Gould-Hopper-Miller-Lee polynomials (LGHMLP) \( L^{(m,r)}_{H(s)} G^{(s)}_n(x, y, z) \).

Thus, by taking the above value of \( A(t) \) in equations (4.2.1), (4.2.6), (4.2.7) and (4.2.25), the following results for the LGHMLP \( L^{(m,r)}_{H(s)} G^{(s)}_n(x, y, z) \) are obtained:

\[
\frac{1}{(1-t)^{s+1}} C_0(-xt^m) \exp(yt + zt^r) = \sum_{n=0}^{\infty} L^{(m,r)}_{H(s)} G^{(s)}_n(x, y, z) t^n, \tag{4.3.13}
\]

\[
\hat{M} = y + m D_x^{-1} \frac{\partial^{m-1}}{\partial y^{m-1}} + rz \frac{\partial^{r-1}}{\partial y^{r-1}} + (s + 1) \frac{1}{1 - \partial_y}, \tag{4.3.14}
\]

\[
\hat{P} = \partial_y \tag{4.3.15}
\]

and

\[
L^{(m,r)}_{H(s)} G^{(s)}_n(x, y, z) = n! \sum_{l=0}^{[\frac{n}{r}]} \sum_{k=0}^{[\frac{n}{m}]} \frac{G^{(s)}_{n-rl-mk}(y) z^l x^k}{l!(k!)^2(n - rl - mk)!}. \tag{4.3.16}
\]
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It should be noted that for \( s = 0 \) and \( s = \beta - 1 \), the Miller-Lee polynomials reduce to the truncated exponential polynomials \( e_n(x) \) and the modified Laguerre polynomials \( f_n^{(\beta)}(x) \) respectively. Therefore, for \( s = 0 \) the LGHMLP \( L_{H(m,r)}e_n(x, y, z) \) reduce to the Laguerre-Gould-Hopper-truncated exponential polynomials (LGHTEP) \( L_{H(m,r)}e_n(x, y, z) \) and for \( s = \beta - 1 \) the LGHMLP \( L_{H(m,r)}G_n^{(s)}(x, y, z) \) reduce to the Laguerre-Gould-Hopper-modified Laguerre polynomials (LGHmLP) \( L_{H(m,r)}f_n^{(\beta)}(x, y, z) \). Thus, by taking same values of \( s \) in equations (4.3.13)-(4.3.16), the corresponding results for the LGHTEP \( L_{H(m,r)}e_n(x, y, z) \) and LGHmLP \( L_{H(m,r)}f_n^{(\beta)}(x, y, z) \) can be obtained.

The LGHTEP \( L_{H(m,r)}e_n(x, y, z) \) are defined by the following series:

\[
L_{H(m,r)}e_n(x, y, z) = n! \sum_{l=0}^{[\frac{n}{m}]} \sum_{k=0}^{[\frac{n}{n}]} \frac{e_{n-rl-mk}(y)z^l x^k}{l!(k!)^2(n - rl - mk)!}.
\]  

(4.3.17)

Further, to draw the surface plot of LGHTEP, the values of the first six truncated exponential polynomials \( e_n(x) \) are required. These values are given in the following table:

**Table 4.3.3. First six truncated exponential polynomials**

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e_n(x) )</td>
<td>( 1 )</td>
<td>( x + 1 )</td>
<td>( \frac{1}{2}x^2 + x + 1 )</td>
<td>( \frac{1}{6}x^3 + \frac{1}{2}x^2 + x + 1 )</td>
<td>( \frac{1}{24}x^4 + \frac{1}{6}x^3 + \frac{1}{2}x^2 + x + 1 )</td>
<td>( \frac{1}{120}x^5 + \frac{1}{24}x^4 + \frac{1}{6}x^3 + \frac{1}{2}x^2 + x + 1 )</td>
</tr>
</tbody>
</table>

Next, taking \( n = 4 \), \( m = 3 \) and \( r = 5 \) in definition (4.3.17), to obtain

\[
L_{H(3,5)}e_4(x, y, z) = e_4(y) + 24e_1(y)x,
\]  

(4.3.18)

which on using particular values of \( e_n(x) \) from Table 4.3.3 gives

\[
L_{H(3,5)}e_4(x, y, z) = \frac{1}{24}y^4 + \frac{1}{6}y^3 + \frac{1}{2}y^2 + 24xy + y + 24x + 1.
\]  

(4.3.19)

In view of equation (4.3.19), the following surface plot of \( L_{H(3,5)}e_4(x, y, z) \) is drawn.
Example 4.3.4. Since for $A(t) = \frac{2t}{(e^t+1)}$ the Appell polynomials $A_n(x)$ become the Genocchi polynomials $G_n(x)$ (Table 1.2.1 (XIV)) . Therefore, for the same choice of $A(t)$ the LGHAP reduce to the Laguerre-Gould-Hopper-Genocchi polynomials LGHGP $L_{H(m,r)}G_n(x,y,z)$.

Thus, by taking the above value of $A(t)$ in equations (4.2.1), (4.2.6), (4.2.7) and (4.2.25), the following results for the LGHGP $L_{H(m,r)}G_n(x,y,z)$ are obtained:

\[
\frac{2t}{(e^t+1)}C_0(-xt^m) \exp(yt + zt^r) = \sum_{n=0}^{\infty} L_{H(m,r)}G_n(x,y,z) \frac{t^n}{n!},
\]

(4.3.20)

\[
\hat{M} = y + mD_x^{-1} \frac{\partial^{m-1}}{\partial y^{m-1}} + rz \frac{\partial^{r-1}}{\partial y^{r-1}} + \frac{e^{\hat{y}}(1 - \partial_y) + 1}{(e^{\hat{y}} + 1)},
\]

(4.3.21)

\[
\hat{P} = \partial_y
\]

(4.3.22)

and

\[
L_{H(m,r)}G_n(x,y,z) = n! \sum_{l=0}^{[\frac{r}{l}]} \sum_{k=0}^{[\frac{m}{k}]} \frac{G_{n-rl-mk}(y)z^lx^k}{l!(k!)^2(n-rl-mk)!}.
\]

(4.3.23)

Further, to draw the surface plot of the LGHGP $L_{H(m,r)}G_n(x,y,z)$, the values of the first six Genocchi polynomials $G_n(x)$ are required. These values are given in the following table:
Hybrid families of special polynomials associated with Appell sequences

Table 4.3.4. First six Genocchi polynomials

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_n(x)$</td>
<td>0</td>
<td>1</td>
<td>$2x - 1$</td>
<td>$3x^2 - 3x$</td>
<td>$4x^3 - 6x^2 + 1$</td>
<td>$5x^4 - 10x^3 + 5x$</td>
</tr>
</tbody>
</table>

Next, taking $n = 4$, $m = 3$ and $r = 5$ in definition (4.3.23), to obtain

$$L_{H^{(3,5)}}G_4(x, y, z) = G_4(y) + 24G_1(y)x,$$

which on using the particular values of $G_n(x)$ from Table 4.3.4 gives

$$L_{H^{(3,5)}}G_4(x, y, z) = 4y^3 - 6y^2 + 24x + 1.$$  \tag{4.3.25}

In view of equation (4.3.25), the following surface plot of $L_{H^{(3,5)}}G_4(x, y, z)$ is drawn:

![Surface plot of $L_{H^{(3,5)}}G_4(x, y, z)$](image)

**Figure 4.3.4**

Similarly, for other members of the Appell family listed in Table 1.2.1, there corresponds a new special polynomial belonging to the LGHA family. The generating function and other properties of these hybrid special polynomials can be obtained from the results derived in Section 4.2.
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These results along with the name and notation of the resultant hybrid special polynomial belonging to the LGHAP family are listed in the following table:

Table 4.3.5. Certain results for the members belonging to the LGHAP family

<table>
<thead>
<tr>
<th>S. No.</th>
<th>Name/Notation of the resultant special polynomial</th>
<th>Generating functions</th>
<th>Multiplicative and derivative operators</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.</td>
<td>( \text{LHG}(m, r) ) ( B_l^{(m)}(x, y, z) ) ( = ) Laguerre-Gould-Hopper generalized Bernoulli polynomials</td>
<td>[ \sum_{n=0}^{\infty} L_{H}(m, r) B_n^{(m)}(x, y, z) z^n ]</td>
<td>[ \frac{\partial}{\partial y} ]</td>
</tr>
<tr>
<td>II.</td>
<td>( \text{LHG}(m, r) ) ( E_\alpha^{(m)}(x, y, z) ) ( = ) Laguerre-Gould-Hopper generalized Euler polynomials</td>
<td>[ \sum_{n=0}^{\infty} L_{H}(m, r) E_\alpha^{(m)}(x, y, z) x^n ]</td>
<td>[ \frac{\partial}{\partial y} ]</td>
</tr>
<tr>
<td>III.</td>
<td>( \text{LHG}(m, r) ) ( B_\lambda^{(m)}(x, y, z; \lambda) ) ( = ) Laguerre-Gould-Hopper Apostol-Bernoulli polynomials of order ( \alpha )</td>
<td>[ \sum_{n=0}^{\infty} L_{H}(m, r) B_\lambda^{(m)}(x, y, z; \lambda) ]</td>
<td>[ \frac{\partial}{\partial y} ]</td>
</tr>
<tr>
<td>IV.</td>
<td>( \text{LHG}(m, r) ) ( B_\lambda^{(m)}(x, y, z; \lambda) ) ( = ) Laguerre-Gould-Hopper Apostol-Bernoulli polynomials</td>
<td>[ \sum_{n=0}^{\infty} L_{H}(m, r) B_\lambda^{(m)}(x, y, z; \lambda) ]</td>
<td>[ \frac{\partial}{\partial y} ]</td>
</tr>
<tr>
<td>V.</td>
<td>( \text{LHG}(m, r) ) ( E_\alpha^{(m)}(x, y, z; \lambda) ) ( = ) Laguerre-Gould-Hopper Apostol-Euler polynomials of order ( \alpha )</td>
<td>[ \sum_{n=0}^{\infty} L_{H}(m, r) E_\alpha^{(m)}(x, y, z; \lambda) ]</td>
<td>[ \frac{\partial}{\partial y} ]</td>
</tr>
<tr>
<td>VI.</td>
<td>( \text{LHG}(m, r) ) ( E_\alpha^{(m)}(x, y, z; \lambda) ) ( = ) Laguerre-Gould-Hopper Apostol-Euler polynomials</td>
<td>[ \sum_{n=0}^{\infty} L_{H}(m, r) E_\alpha^{(m)}(x, y, z; \lambda) ]</td>
<td>[ \frac{\partial}{\partial y} ]</td>
</tr>
</tbody>
</table>

In the next section, the determinant definitions for the LGHAP \( L_{H}(m, r) A_n(x, y, z) \) and some of its members are established.

4.4 Determinant Definitions

The determinant approach is equivalent to the corresponding approach based on operational methods. However, the simplicity of this approach allows non-specialists to use its applications as it is suitable for computation.
Hybrid families of special polynomials associated with Appell sequences

We recall that the Appell polynomials have the following determinant definition [33, p.1533]:

\[ A_0(x) = \frac{1}{\beta_0}, \quad \beta_0 = \frac{1}{A_0}, \quad (4.4.1) \]

\[ A_n(x) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix}
1 & x & x^2 & \cdots & x^{n-1} & x^n \\
\beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\
0 & \beta_0 & (\binom{2}{1})\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\
0 & 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1
\end{vmatrix}, \quad (4.4.2) \]

\[ \beta_n = -\frac{1}{A_0} \left( \sum_{k=1}^{n} \binom{n}{k} A_k \beta_{n-k} \right), \quad n = 1, 2, 3, \cdots, \]

where \( \beta_0, \beta_1, \cdots, \beta_n \in \mathbb{R}, \quad \beta_0 \neq 0. \)

In order to give the determinant definition of the LGHAP \( _{LH}^{(m,r)} A_n(x, y, z) \), the following result is proved:

**Theorem 4.4.1.** The Laguerre-Gould-Hopper-Appell polynomials \( _{LH}^{(m,r)} A_n(x, y, z) \) of degree \( n \) are defined by

\[ _{LH}^{(m,r)} A_0(x, y, z) = \frac{1}{\beta_0}, \quad \beta_0 = \frac{1}{\alpha_0}, \quad (4.4.3) \]

\[ _{LH}^{(m,r)} A_n(x, y, z) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix}
1 & _{LH}^{(m,r)} H_1(x, y, z) & _{LH}^{(m,r)} H_2(x, y, z) & \cdots & _{LH}^{(m,r)} H_{n-1}(x, y, z) & _{LH}^{(m,r)} H_n(x, y, z) \\
\beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\
0 & \beta_0 & (\binom{2}{1})\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\
0 & 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1
\end{vmatrix}, \quad (4.4.4) \]
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\[ \beta_n = -\frac{1}{\alpha_0} \left( \sum_{k=1}^{n} \binom{n}{k} A_k \beta_{n-k} \right), \quad n = 1, 2, 3, \cdots, \]

where \( \beta_0, \beta_1, \cdots, \beta_n \in \mathbb{R}, \beta_0 \neq 0 \) and \( L_n^{(m,r)}(x,y,z) \) \( (n = 0, 1, \cdots) \) are the Laguerre-Gould-Hopper polynomials defined by equation (4.1.1).

**Proof.** Taking \( n = 0 \) in series definition (4.2.16) and then using equation (4.4.1) in the resultant equation, assertion (4.4.3) follows.

Expansion of the determinant in equation (4.4.2) with respect to the first row gives

\[
A_n(x) = \frac{(-1)^n x}{(\beta_0)^{n+1}} \begin{vmatrix}
\beta_0 & \beta_1 & \cdots & \beta_{n-1} & \beta_n \\
\beta_0 \binom{2}{1} \beta_1 & \cdots & (n-1) \binom{n-1}{1} \beta_{n-2} & n \binom{n}{1} \beta_{n-1} \\
0 & \beta_0 & \cdots & (n-1) \binom{n-1}{2} \beta_{n-3} & n \binom{n}{2} \beta_{n-2} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \beta_0 & n \binom{n}{n-1} \beta_1
\end{vmatrix}
- \frac{(-1)^n x}{(\beta_0)^{n+1}}
\]

\[
\begin{vmatrix}
\beta_0 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\
0 & \binom{2}{1} \beta_1 & \cdots & (n-1) \binom{n-1}{1} \beta_{n-2} & n \binom{n}{1} \beta_{n-1} \\
0 & \beta_0 & \cdots & (n-1) \binom{n-1}{2} \beta_{n-3} & n \binom{n}{2} \beta_{n-2} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \beta_0 & n \binom{n}{n-1} \beta_1
\end{vmatrix} + \frac{(-1)^n x^2}{(\beta_0)^{n+1}}
\]
Hybrid families of special polynomials associated with Appell sequences

\[ + \cdot \cdot \cdot + \left( -1 \right)^{2n+1} x^{n-1} \left( \beta_0 \right)^{n+1} \]

| \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_n | \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} \\
0 & \beta_0 \binom{1}{1} \beta_1 & \cdots & \binom{n}{1} \beta_{n-1} & 0 & \beta_0 \binom{1}{1} \beta_1 & \cdots & \binom{n-1}{1} \beta_{n-2} \\
0 & 0 & \beta_0 & \cdots & \binom{n}{2} \beta_{n-2} & 0 & 0 & \beta_0 & \cdots & \binom{n-1}{2} \beta_{n-3} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \binom{n}{n-1} \beta_1 | 0 & 0 & 0 & \cdots & \beta_0 \]

(4.4.5)

Since each minor in equation (4.4.5) is independent of \( x \), therefore replacing \( x \) by \( \hat{M}_{LH} \) in equation (4.4.5) and then using the monomiality principle equation \( L H^{(m,r)}(x, y, z) = M_{LH}^n \{1\} (n = 1, 2, 3, \cdots) \) in the r.h.s. of the resultant equation, it follows that

\[ A_n(\hat{M}_{LH}) = \left( -1 \right)^n \left( \beta_0 \right)^{n+1} \]

| \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n | \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\
\beta_0 & \binom{2}{1} \beta_1 & \cdots & \binom{n-1}{1} \beta_{n-2} & \binom{n}{1} \beta_{n-1} & 0 & \beta_0 & \cdots & \binom{n-1}{2} \beta_{n-3} & \binom{n}{2} \beta_{n-2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \beta_0 & \binom{n}{n-1} \beta_1 | 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1} \beta_1 \\

\left( -1 \right)^n \left( \beta_0 \right)^{n+1} \left( L H^{(m,r)}(x, y, z) \right) - \frac{\left( -1 \right)^n \left( \beta_0 \right)^{n+1} \left( L H^{(m,r)}(x, y, z) \right)}{(\beta_0)^{n+1}} \]

\[ + \left( -1 \right)^n \frac{L H^{(m,r)}_2(x, y, z)}{(\beta_0)^{n+1}} \]
which on using operational rule (4.2.5) in the l.h.s. and combining the terms in the r.h.s.
proves assertion (4.4.4).

By giving suitable values to the variables and indices and in view of Tables 4.1.1
and 4.3.5, the determinant definitions for the members belonging to the Laguerre-Gould-
Hopper-Appell family can also be obtained.

The Bernoulli and Euler polynomials are two important members of the Appell fam-
ily. The Bernoulli polynomials are employed in the integral representation of differentiable
periodic functions and play an important role in the approximation of such functions by
means of polynomials. The Euler polynomials are strictly connected with the Bernoulli
ones and enter in the Taylor’s expansion in a neighborhood of the origin of the trigonomet-
ric and hyperbolic secant functions. The Bernoulli and Euler numbers have deep connec-
tions with number theory and occur in combinatorics.
Recall the following determinant definitions of the Bernoulli and Euler polynomials [33]:

\[
B_0(x) = 1, \quad B_n(x) = (-1)^n \begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} & x^n \\ 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} & \frac{1}{n+1} \\ 0 & 1 & (2)^{\frac{1}{2}} & \cdots & \left(\frac{n-1}{1}\right) & \left(\frac{n}{1}\right) \\ 0 & 0 & 1 & \cdots & \left(\frac{n-1}{2}\right) & \left(\frac{n}{2}\right) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \left(\frac{n}{n-1}\right) \end{vmatrix}, \quad n = 1, 2, \cdots \tag{4.4.8}
\]

and

\[
E_0(x) = 1, \quad E_n(x) = (-1)^n \begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} & x^n \\ 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & (2)^{\frac{1}{2}} & \cdots & \frac{1}{2} & \left(\frac{n-1}{2}\right) & \left(\frac{n}{2}\right) \\ 0 & 0 & 1 & \cdots & \frac{1}{2} & \left(\frac{n-1}{2}\right) & \left(\frac{n}{2}\right) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \frac{1}{2} \end{vmatrix}, \quad n = 1, 2, \cdots \tag{4.4.10}
\]

The solution of the linear interpolation problem can be expressed using the determinant forms of these polynomials.

The determinant definitions of the Laguerre-Gould-Hopper-Bernoulli polynomials \( L_{H(m,r)}^B(x, y, z) \) and Laguerre-Gould-Hopper-Euler polynomials \( L_{H(m,r)}^E(x, y, z) \) are obtained in the form of the following results:

**Theorem 4.4.2.** The Laguerre-Gould-Hopper-Bernoulli polynomials \( L_{H(m,r)}^B(x, y, z) \) and Laguerre-Gould-Hopper-Euler polynomials \( L_{H(m,r)}^E(x, y, z) \) of degree \( n \) are defined by

\[
L_{H(m,r)}^B(x, y, z) = 1, \quad L_{H(m,r)}^E(x, y, z) \tag{4.4.11}
\]
\[ L_{H(m,r)} B_n(x, y, z) = (-1)^n \]

\[
\begin{vmatrix}
1 & lH_1^{(m,r)}(x, y, z) & lH_2^{(m,r)}(x, y, z) & \cdots & lH_{n-1}^{(m,r)}(x, y, z) & lH_n^{(m,r)}(x, y, z) \\
1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} & \frac{1}{n+1} \\
0 & 1 & \left(\frac{2}{1}\right) & \frac{1}{2} & \cdots & \frac{1}{n-1} \\
0 & 0 & 1 & \cdots & \frac{1}{n-2} & \frac{1}{n-1} \\
0 & 0 & 0 & \cdots & 1 & \frac{1}{n} \\
\end{vmatrix}
\]

\[ n = 1, 2, 3 \cdots \quad (4.4.12) \]

**Proof.** In view of equations (4.4.7) and (4.4.8) and proceeding on the same lines as for the derivation of the determinant form of the LGHAP \( L_{H(m,r)} A_n(x, y, z) \), assertions (4.4.11) and (4.4.12) are proved. \[ \blacksquare \]

**Theorem 4.4.3.** The Laguerre-Gould-Hopper-Euler polynomials \( L_{H(m,r)} E_n(x, y, z) \) of degree \( n \) are defined by

\[ lH_{H(m,r)} E_0(x, y, z) = 1, \]

\[ lH_{H(m,r)} E_n(x, y, z) = (-1)^n \]

\[
\begin{vmatrix}
1 & lH_1^{(m,r)}(x, y, z) & lH_2^{(m,r)}(x, y, z) & \cdots & lH_{n-1}^{(m,r)}(x, y, z) & lH_n^{(m,r)}(x, y, z) \\
1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\
0 & 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\
0 & 0 & 1 & \cdots & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & \cdots & 1 & \frac{1}{2} \\
\end{vmatrix}
\]

\[ n = 1, 2, 3 \cdots \quad (4.4.14) \]
Proof. In view of equations (4.4.9) and (4.4.10) and proceeding on the same lines as for the derivation of the determinant form of the LGHAP \( L_H^{(m,r)} A_n(x, y, z) \), assertions (4.4.13) and (4.4.14) are proved. ■

In the next section, the Laguerre-Gould-Hopper based Sheffer polynomials are introduced and some of their properties are also derived.

### 4.5 Concluding Remarks

The Appell family generated by (1.2.2) is obviously rather restrictive; it does not allow the treatment of some other polynomial sets on the Laguerre or the Bessel polynomials within the context of the operational formalism. Recently, Dattoli et. al. [52] have shown that the extension of Appell family to Sheffer family [124] allows such a possibility. Motivated by this approach, the Laguerre-Gould-Hopper-Appell polynomials \( L_H^{(m,r)} A_n(x, y, z) \) are extended to the Laguerre-Gould-Hopper-Sheffer polynomials (LGHSP) \( L_H^{(m,r)} s_n(x, y, z) \).

Proceeding on the same lines as for the LGHAP, certain properties for LGHSP \( L_H^{(m,r)} s_n(x, y, z) \) are derived in the form of the following results:

**Theorem 4.5.1.** For the Laguerre-Gould-Hopper-Sheffer polynomials \( L_H^{(m,r)} s_n(x, y, z) \), the following generating function holds true:

\[
A(t) C_0 \left(-x \left(\mathcal{H}(t)\right)^m\right) \exp \left(y \mathcal{H}(t) + z \left(\mathcal{H}(t)\right)^r\right) = \sum_{n=0}^{\infty} L_H^{(m,r)} s_n(x, y, z) \frac{t^n}{n!}. \tag{4.5.1}
\]

**Theorem 4.5.2.** The Laguerre-Gould-Hopper-Sheffer polynomials \( L_H^{(m,r)} s_n(x, y, z) \) are quasi-monomial under the action of the following multiplicative and derivative operators:

\[
\hat{M}_{LHs} = \left(y + mD_x^{-1} \frac{\partial^{m-1}}{\partial y^{m-1}} + rz \frac{\partial^{r-1}}{\partial y^{r-1}}\right) \mathcal{H}' \left(\mathcal{H}^{-1}(\partial_y)\right) + \frac{A' \left(\mathcal{H}^{-1}(\partial_y)\right)}{A \left(\mathcal{H}^{-1}(\partial_y)\right)} \tag{4.5.2}
\]

and

\[
\hat{P}_{LHs} = \mathcal{H}^{-1}(\partial_y), \tag{4.5.3}
\]

respectively, where \( \partial_y := \frac{\partial}{\partial y} \).
Chapter 4

**Theorem 4.5.3.** The Laguerre-Gould-Hopper-Sheffer polynomials \( L_{H^{(m,r)}} s_n(x, y, z) \) satisfy the following differential equation:

\[
\left( \left( \left( y + m D_x^{-1} \frac{\partial^{m-1}}{\partial y^{m-1}} + rz \frac{\partial^{r-1}}{\partial y^{r-1}} \right) H' (H^{-1} (\partial_y)) + \frac{A' (H^{-1} (\partial_y))}{A (H^{-1} (\partial_y))} \right) H^{-1} (\partial_y) 
\right) - n \right) L_{H^{(m,r)}} s_n(x, y, z) = 0. \tag{4.5.4}
\]

In Table 4.1.1, special cases of the LGHP \( L_{H^{(m,r)}} s_n(x, y, z) \) are mentioned. In order to obtain the results for the corresponding new or known special polynomials related to the Sheffer sequences, the following examples are considered:

**Example 4.5.1.** Since for \( m = 1, r = 2, x \to -x \), the LGHP \( L_{H^{(m,r)}} s_n(x, y, z) \) reduce to the 3-variable Laguerre-Hermite polynomials (3VLHP) \( L_{H^{(m,r)}} s_n(x, y, z) \) (Table 4.1.1 (I)). Therefore, for the same choice of \( m, r \) and \( x \), the LGHSP \( L_{H^{(m,r)}} s_n(x, y, z) \) reduce to the 3-variable Laguerre-Hermite-Sheffer polynomials (3VLHSP) \( L_{H^{(m,r)}} s_n(x, y, z) \). Thus, by making these substitutions in Theorems 4.5.1-4.5.3, the following results for the 3VLHSP \( L_{H^{(m,r)}} s_n(x, y, z) \) are obtained:

**Table 4.5.1. Results for the 3VLHSP \( L_{H^{(m,r)}} s_n(x, y, z) \)**

<table>
<thead>
<tr>
<th>S. No.</th>
<th>Results</th>
<th>Mathematical expressions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Generating function</td>
<td>( A(t) C_0 \left( x \left( H(t) \right) \right) \exp \left( y H(t) + z \left( H(t) \right)^2 \right) = \sum_{n=0}^{\infty} L_{H^{(m,r)}} s_n(x, y, z) \frac{t^n}{n!} )</td>
</tr>
<tr>
<td>2.</td>
<td>Multiplicative and derivative operators</td>
<td>( M = \left( y - D_x^{-1} + 2z \frac{\partial}{\partial y} \right) H' (H^{-1} (\partial_y)) + \frac{A' (H^{-1} (\partial_y))}{A (H^{-1} (\partial_y))} )</td>
</tr>
<tr>
<td></td>
<td>P = f (( \partial_y )) = H^{-1} (( \partial_y ))</td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td>Differential equation</td>
<td>( \left( \left( y - D_x^{-1} + 2z \frac{\partial}{\partial y} \right) H' (H^{-1} (\partial_y)) + \frac{A' (H^{-1} (\partial_y))}{A (H^{-1} (\partial_y))} \right) H^{-1} (\partial_y) - n \right) L_{H^{(m,r)}} s_n(x, y, z) = 0 )</td>
</tr>
</tbody>
</table>

**Example 4.5.2.** Since for \( m = 1, r = 2, x \to -x, z = -\frac{1}{2} \), the LGHP \( L_{H^{(m,r)}} s_n(x, y, z) \) reduce to the 2-variable Laguerre-Hermite polynomials (2VLHP) \( L_{H^*} s_n(x, y) \) (Table 4.1.1 (II)). Therefore, for the same choice of \( m, r, x \) and \( z \), the LGHSP \( L_{H^{(m,r)}} s_n(x, y, z) \) reduce to the 2-variable Laguerre-Hermite-Sheffer polynomials (2VLHSP) \( L_{H^*} s_n(x, y) \). Thus, by making these substitutions in Theorems 4.5.1-4.5.3, the following results for the 2VLHSP \( L_{H^*} s_n(x, y) \) are obtained:
Table 4.5.2. Results for the 2VLHSP $LH^s_n(x, y)$

<table>
<thead>
<tr>
<th>S. No.</th>
<th>Results</th>
<th>Mathematical expressions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Generating function</td>
<td>$A(t) C_0 ((x (H(t))) \exp (y H(t) - \frac{1}{2} (H(t))^2) = \sum_{n=0}^{\infty} L_{H^s_n} (x, y) \frac{t^n}{n!}$</td>
</tr>
<tr>
<td>2.</td>
<td>Multiplicative and derivative operators</td>
<td>$\mathcal{M} = (y - D_x^{-1} - \frac{\partial y}{\partial t}) \mathcal{H}^s_n \left( \mathcal{H}^{-1} \mathcal{H}^s_n \mathcal{H}^{-1} \left( \frac{\partial y}{\partial t} \right) \right) + \frac{\mathcal{A} \left( \mathcal{H}^{-1} \mathcal{H}^s_n \mathcal{H}^{-1} \left( \frac{\partial y}{\partial t} \right) \right)}{\mathcal{A} \left( \mathcal{H}^{-1} \left( \frac{\partial y}{\partial t} \right) \right)}$</td>
</tr>
<tr>
<td>3.</td>
<td>Differential equation</td>
<td>$\left( \left( y - D_z^{-1} - \frac{\partial y}{\partial t} \right) \mathcal{H}^s_n \left( \mathcal{H}^{-1} \left( \frac{\partial y}{\partial t} \right) \right) + \frac{\mathcal{A} \left( \mathcal{H}^{-1} \left( \frac{\partial y}{\partial t} \right) \right)}{\mathcal{A} \left( \mathcal{H}^{-1} \left( \frac{\partial y}{\partial t} \right) \right)} \right) \mathcal{H}^{-1} \left( \frac{\partial y}{\partial t} \right) - n \right) L_{H^s_n} (x, y) = 0$</td>
</tr>
</tbody>
</table>

Example 4.5.3. Since for $m = 1$, $r = 2$, $x \to -x$, $y = 1$, $z \to y$, the LGHP $LH^r_n(y, z)$ reduce to the Laguerre-Hermite type polynomials (LHTP) $\varphi_n(x, y)$ (Table 4.1.1 (III)). Therefore, for the same choice of $m$, $r$, $x$, $y$ and $z$, the LGHSP $LH^{(m, r)}_n(x, y, z)$ reduce to the 2-variable Laguerre-Hermite type Sheffer polynomials (LHTSP) $\varphi_{s_n}(x, y)$. Thus, by making these substitutions in Theorems 4.5.1-4.5.3, the following results for the 2VLHTSP $\varphi_{s_n}(x, y)$ are obtained:

Table 4.5.3. Results for the 2VLHTSP $\varphi_{s_n}(x, y)$

<table>
<thead>
<tr>
<th>S. No.</th>
<th>Results</th>
<th>Mathematical expressions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Generating function</td>
<td>$A(t) C_0 ((x (H(t))) \exp (y H(t) + \frac{1}{2} (H(t))^2) = \sum_{n=0}^{\infty} \varphi_{s_n} (x, y) \frac{t^n}{n!}$</td>
</tr>
<tr>
<td>2.</td>
<td>Multiplicative and derivative operators</td>
<td>$\mathcal{M} = (1 - D_x^{-1} - 2y \frac{\partial y}{\partial t} \frac{\partial x}{\partial t}) \mathcal{H}^s_n \left( \mathcal{H}^{-1} \left( -\frac{\partial x}{\partial t} \right) \right) + \frac{\mathcal{A} \left( \mathcal{H}^{-1} \left( \frac{\partial x}{\partial t} \right) \right)}{\mathcal{A} \left( \mathcal{H}^{-1} \left( -\frac{\partial x}{\partial t} \right) \right)}$</td>
</tr>
<tr>
<td>3.</td>
<td>Differential equation</td>
<td>$\left( \left( 1 - D_z^{-1} - 2y \frac{\partial y}{\partial t} \frac{\partial z}{\partial t} \right) \mathcal{H}^s_n \left( \mathcal{H}^{-1} \left( \frac{\partial y}{\partial t} \right) \right) + \frac{\mathcal{A} \left( \mathcal{H}^{-1} \left( \frac{\partial x}{\partial t} \right) \right)}{\mathcal{A} \left( \mathcal{H}^{-1} \left( \frac{\partial z}{\partial t} \right) \right)} \right) \mathcal{H}^{-1} \left( \frac{\partial y}{\partial t} \right) - n \right) \varphi_{s_n} (x, y) = 0$</td>
</tr>
</tbody>
</table>

Example 4.5.4. Since for $x = 0$, the LGHP $LH^{(m, r)}_n(y, z)$ reduce to the Gould Hopper polynomials (GHP) $H^r_n(y, z)$ (Table 4.1.1 (IV)). Therefore, for the same choice of $x$, the LGHSP $LH^{(m, r)}_n(x, y, z)$ reduce to the Gould-Hopper-Sheffer polynomials (GHSP) $H^r_n(x, y, z)$ [97]. Thus, by making these substitutions in Theorems 4.5.1-4.5.3, the corresponding results for the GHSP $H^r_n(x, y, z)$ [97] are obtained.

Example 4.5.5. Since for $z = 0$, the LGHP $LH^{(m, r)}_n(x, y, z)$ reduce to the 2-variable generalized Laguerre polynomials (2VLP) $mL_n(x, y)$ (Table 4.1.1 (V)). Therefore, for the same choice of $z$, the LGHSP $LH^{(m, r)}_n(x, y, z)$ reduce to the 2-variable generalized Laguerre-Sheffer polynomials (2VGLSP) $mL_n(x, y)$. Thus, by using these substitutions in Theorems 4.5.1-4.5.3, the following results for the 2VGLSP $mL_n(x, y)$ are obtained:
Chapter 4

Table 4.5.4. Results for the 2VGLSP \( mLs_n(x, y) \)

<table>
<thead>
<tr>
<th>No.</th>
<th>Results</th>
<th>Mathematical expressions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Generating function</td>
<td>( A(t)C_0(−x(H(t))^m)\exp(y(H(t))^m) = \sum_{r=0}^{\infty} mLs_n(x, y) \frac{y^r}{r!} )</td>
</tr>
<tr>
<td>2.</td>
<td>Multiplicative and derivative operators</td>
<td>( \hat{M} = \left( y + mD^{-1}_x \frac{\partial^{m-1}}{\partial y^{m-1}} \right) H^{-1}(−\partial_y) ) + ( \frac{\partial^{m-1}}{\partial y^{m-1}} \left( \frac{\partial}{\partial y} \right) H^{-1}(−\partial_y) ) + ( A\left( H^{-1}(−\partial_y) \right) )</td>
</tr>
<tr>
<td>3.</td>
<td>Differential equation</td>
<td>( \left( \left( y + mD^{-1}_x \frac{\partial^{m-1}}{\partial y^{m-1}} \right) H^{-1}(−\partial_y) \right) = 0 )</td>
</tr>
</tbody>
</table>

Example 4.5.6. Since for \( r = m, \ x = 0, \ y \rightarrow -D^{-1}_x, \ z \rightarrow y \), the LGHP \( LH_{(m)}H_{n,r}(x, y, z) \) reduce to the 2-variable generalized Laguerre type polynomials (2VGLTP) \( [m]L_n(x, y) \) (Table 4.1.1 (VI)). Therefore, for the same choice of \( r, \ x, \ y, \) and \( z \), the LGHSP \( LH_{(m,r)}s_n(x, y, z) \) reduce to the 2-variable generalized Laguerre type Sheffer polynomials (2VGLTSP) \( [m]Ls_n(x, y) \). Thus, by making these substitutions in Theorems 4.5.1-4.5.3, the following results for the 2VGLTSP \( [m]Ls_n(x, y) \) are obtained:

Table 4.5.5. Results for the 2VGLTSP \( [m]Ls_n(x, y) \)

<table>
<thead>
<tr>
<th>No.</th>
<th>Results</th>
<th>Mathematical expressions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Generating function</td>
<td>( A(t)C_0(−x(H(t))^m)\exp(y(H(t))^m) = \sum_{r=0}^{\infty} mLs_n(x, y) \frac{y^r}{r!} )</td>
</tr>
<tr>
<td>2.</td>
<td>Multiplicative and derivative operators</td>
<td>( \hat{M} = \left( y + mD^{-1}_x \frac{\partial^{m-1}}{\partial y^{m-1}} \right) H^{-1}(−\partial_y) ) + ( A\left( H^{-1}(−\partial_y) \right) )</td>
</tr>
<tr>
<td>3.</td>
<td>Differential equation</td>
<td>( \left( \left( y + mD^{-1}_x \frac{\partial^{m-1}}{\partial y^{m-1}} \right) H^{-1}(−\partial_y) \right) = 0 )</td>
</tr>
</tbody>
</table>

Example 4.5.7. Since for \( r = m - 1, \ x = 0, \ y \rightarrow x, \ z \rightarrow y \), the LGHP \( LH_{(m)}H_{n,r}(x, y, z) \) reduce to the generalized Chebyshev polynomials (GCP) \( U_{(m)}^{(m)}(x, y) \) (Table 4.1.1 (VII)). Therefore, for the same choice of \( r, \ x, \ y, \) and \( z \), the LGHSP \( LH_{(m,r)}s_n(x, y, z) \) reduce to the generalized Chebyshev-Sheffer polynomials (GCSP) \( U_{(m)}s_n(x, y) \) [85]. Thus, by making these substitutions in Theorems 4.5.1-4.5.3, the corresponding results for the GCSP \( U_{(m)}s_n(x, y) \) [85] are obtained.

Example 4.5.8. Since for \( m = 1, \ x \rightarrow -x, \ z = 0 \), the LGHP \( LH_{n,r}(x, y, z) \) reduce to the 2-variable Laguerre polynomials (2VLP) \( L_n(x, y) \) (Table 4.1.1 (VIII)). Therefore, for the same choice of \( m, x \) and \( z \), the LGHSP \( LH_{(m,r)}s_n(x, y, z) \) reduce to the 2-variable
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Laguerre-Sheffer polynomials (2VLSP) \( L_{sn}(x,y) \) [92]. Thus, by making these substitutions in Theorems 4.5.1-4.5.3, the corresponding results for the 2VLSP \( L_{sn}(x,y) \) [92] are obtained.

**Example 4.5.9.** Since for \( m = 1, x \rightarrow y, y \rightarrow -D_x^{-1}, z = 0 \), the LGHP \( LH_{n}^{(m,r)}(x,y,z) \) reduce to the 2-variable Legendre polynomials (2VLP) \( R_{n}(x,y) \) (Table 4.1.1 (IX)). Therefore, for the same choice of \( m, x, y \) and \( z \), the LGHSP \( LH_{n}^{(m,r)}s_{n}(x,y,z) \) reduce to the 2-variable Legendre-Sheffer polynomials (2VLeSP) \( R_{n}(x,y) \) [93]. Thus, by making these substitutions in Theorems 4.5.1-4.5.3, the corresponding results for the 2VLeSP \( R_{n}(x,y) \) [93] are obtained.

**Example 4.5.10.** Since for \( x = 0, y \rightarrow x, z \rightarrow y\partial_{y}y \), the LGHP \( LH_{n}^{(m,r)}(x,y,z) \) reduce to the 2-variable truncated polynomials of order \( r \) (2VT) \( e_{n}^{(r)}(x,y) \) (Table 4.1.1 (X)). Therefore, for the same choice of \( x, y \) and \( z \), the LGHSP \( LH_{n}^{(m,r)}s_{n}(x,y,z) \) reduce to the 2-variable truncated exponential-Sheffer polynomials (2VTESP) \( e_{n}^{(r)}s_{n}(x,y) \) [102]. Thus, by making these substitutions in Theorems 4.5.1-4.5.3, the corresponding results for the 2VTESP \( e_{n}^{(r)}s_{n}(x,y) \) [102] are obtained.

**Example 4.5.11.** Since for \( r = 2, x = 0 \), the LGHP \( LH_{n}^{(m,r)}(x,y,z) \) reduce to the 2-variable Hermite Kampé de Fériet polynomials (2VHdFP) \( H_{n}(y,z) \) (Table 4.1.1 (XI)). Therefore, for the same choice of \( r \) and \( x \), the LGHSP \( LH_{n}^{(m,r)}s_{n}(x,y,z) \) reduce to the 2-variable Hermite Kampé de Fériet-Sheffer polynomials (2VHdFSP) \( H_{n}(y,z) \) [90]. Thus, by making these substitutions in Theorems 4.5.1-4.5.3, the corresponding results for the 2VHdFSP \( H_{n}(y,z) \) [90] are obtained.

**Example 4.5.12.** Since for \( r = 2, x = 0, y \rightarrow D_x^{-1}, z \rightarrow y \), the LGHP \( LH_{n}^{(m,r)}(x,y,z) \) reduce to the Hermite type polynomials (HTP) \( G_{n}(x,y) \) (Table 4.1.1 (XII)). Therefore, for the same choice of \( r, x, y \) and \( z \), the LGHSP \( LH_{n}^{(m,r)}s_{n}(x,y,z) \) reduce to the Hermite type Sheffer polynomials (HTSP) \( G_{n}(x,y) \). Thus, by making these substitutions in Theorems 4.5.1-4.5.3, the following results for the HTSP \( G_{n}(x,y) \) are obtained:
Table 4.5.6. Results for the HTSP \( G_{S_n}(x, y) \)

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Results</th>
<th>Mathematical expressions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Generating function</td>
<td>( A(t)C_0 \left( -txH(t) \right) \exp \left( yH(t)^2 \right) = \sum_{s=0}^\infty G_{S_n}(x, y) \frac{t^s}{s!} )</td>
</tr>
<tr>
<td>2.</td>
<td>Multiplicative and derivative operators</td>
<td>( M = (D_z^{-1} + 2y \frac{\partial}{\partial x}) H' \left( -t^2 x \partial x \right) + \frac{\partial}{\partial x} H' \left( -t \partial x \partial x \right) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \hat{P} = H^{-1} (y \partial_x) )</td>
</tr>
</tbody>
</table>
| 3.    | Differential equation                                                  | \( \left( (D_z^{-1} + 2y \frac{\partial}{\partial x}) H' \left( -t^2 x \partial x \right) + \frac{\partial}{\partial x} H' \left( -t \partial x \partial x \right) \right) H^{-1} (\partial x \partial x) - n \) \( G_{S_n}(x, y) = 0 \)

**Example 4.5.13.** Since for \( m = 2, x \to \left( \frac{x^2-1}{4} \right), y \to x, z = 0 \), the LGHP \( L^{(m, r)} H_n (x, y, z) \) reduce to the Legendre polynomials (LeP) \( P_n(x) \) (Table 4.1.1 (XIII)). Therefore for the same substitutions of \( m, x, y \) and \( z \), the LGHSP \( L^{(m, r)} S_n (x, y, z) \) reduce to the Legendre-Sheffer polynomials (LeSP) \( p_n(x) \). Thus, by making these substitutions in Theorems 4.5.1-4.5.3, the following results for the LeSP \( p_n(x) \) are obtained:

Table 4.5.7. Results for the LeSP \( p_n(x) \)

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Results</th>
<th>Mathematical expressions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Generating function</td>
<td>( A(t)C_0 \left( -txH(t) \right) \exp \left( yH(t)^2 \right) = \sum_{s=0}^\infty p_n(x) \frac{t^s}{s!} )</td>
</tr>
<tr>
<td>2.</td>
<td>Multiplicative and derivative operators</td>
<td>( M = (x + 2D_z^{-1} \frac{\partial}{\partial x}) H' \left( -t^2 x \partial x \right) + \frac{\partial}{\partial x} H' \left( -t \partial x \partial x \right) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \hat{P} = H^{-1} \left( x \partial_x \right) )</td>
</tr>
</tbody>
</table>
| 3.    | Differential equation                                                  | \( \left( (x + 2D_z^{-1} \frac{\partial}{\partial x}) H' \left( -t^2 x \partial x \right) + \frac{\partial}{\partial x} H' \left( -t \partial x \partial x \right) \right) H^{-1} (\partial_x) - n \) \( p_n(x) = 0 \)

**Example 4.5.14.** Since for \( x \to y \partial_y, y \to x \), the LGHP \( L^{(m, r)} H_n (x, y, z) \) reduce to the 3-variable generalized Hermite polynomials (3VGHSP) \( H^{(r, m)}_n (x, y, z) \) (Table 4.1.1 (XIV)). Therefore, for the same choice of \( x \) and \( y \), the LGHSP \( L^{(m, r)} S_n (x, y, z) \) reduce to the 3-variable generalized Hermite-Sheffer polynomials (3VGHSP) \( H^{(r, m)} S_n (x, y, z) \). Thus, by making these substitutions in Theorems 4.5.1-4.5.3, the following results for the 3VGHSP \( H^{(r, m)} S_n (x, y, z) \) are obtained:

Table 4.5.8. Results for the 3VGHSP \( H^{(r, m)} S_n (x, y, z) \)

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Results</th>
<th>Mathematical expressions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Generating function</td>
<td>( A(t) \exp \left( xH(t) + y \left( H(t)^m \right) + z \left( H(t)^r \right) \right) )</td>
</tr>
<tr>
<td>2.</td>
<td>Multiplicative and derivative operators</td>
<td>( M = (x + my \frac{\partial}{\partial x}) H' \left( -t \partial x \partial x \right) + \frac{\partial}{\partial x} H' \left( -t \partial x \partial x \right) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \hat{P} = H^{-1} \left( x \partial_x \right) )</td>
</tr>
<tr>
<td>3.</td>
<td>Differential equation</td>
<td>( \left( (x + my \frac{\partial}{\partial x}) H' \left( -t \partial x \partial x \right) + \frac{\partial}{\partial x} H' \left( -t \partial x \partial x \right) \right) H^{-1} (\partial_x) - n ) ( H^{(r, m)} S_n (x, y, z) = 0 )</td>
</tr>
</tbody>
</table>
Example 4.5.15. Since for $m = 2$, $r = 3$, $x \rightarrow z\partial_z$, $y \rightarrow x$, $z \rightarrow y$, the LGHP $LH_n^{(m,r)}(x, y, z)$ reduce to the Bell type polynomials (BTP) $H_n^{(3,2)}(x, y, z)$ (Table 4.1.1 (XV)). Therefore, for the same choice of $m, r, x, y$ and $z$, the LGHSP $LH_n^{(m,r)}Sn(x, y, z)$ reduce to the Bell type Sheffer polynomials (BTSP) $H_n^{(3,2)}Sn(x, y, z)$. Thus, by making these substitutions in Theorems 4.5.1-4.5.3, the following results for the BTSP $H_n^{(3,2)}Sn(x, y, z)$ are obtained:

Table 4.5.9. Results for the BTSP $H_n^{(3,2)}Sn(x, y, z)$

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Results</th>
<th>Mathematical expressions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Generating function</td>
<td>$A(t)\exp\left(xH(t)+y(H(t))^3+y(H(t))^2\right) = \sum_{n=0}^{\infty}H_n^{(3,2)}Sn(x, y, z)\frac{t^n}{n!}$</td>
</tr>
<tr>
<td>2.</td>
<td>Multiplicative and derivative operators</td>
<td>$\hat{M} = \left(x + 3y\frac{\partial}{\partial x} + 2z\frac{\partial}{\partial y}\right)H'(H-1(\partial_x)) + \frac{A'(H-1(\partial_x))}{A(H-1(\partial_x))}$</td>
</tr>
<tr>
<td>3.</td>
<td>Differential equation</td>
<td>$\left(x + 3y\frac{\partial}{\partial x} + 2z\frac{\partial}{\partial y}\right)H'(H-1(\partial_x)) + \frac{A'(H-1(\partial_x))}{A(H-1(\partial_x))}H^{-1}(\partial_x) = 0$</td>
</tr>
</tbody>
</table>

The Sheffer class contains important sequences such as the Hermite, Laguerre, Bernoulli, Poisson-Charlier polynomials etc. These polynomials are important from the viewpoint of applications in physics and number theory. It should be noted that corresponding to each member belonging to the Sheffer family, there exists a new special polynomial belonging to the LGHSP family. The generating function and other properties of these special polynomials can be obtained from the results derived in this section.

Further, an integral representation for the LGHSP $LH_n^{(m,r)}Sn(x, y, z)$ in terms of the 3VGHSP $H_n^{(r,m)}Sn(x, y, z)$ is established by proving the following result:

Theorem 4.5.4. The following integral representation for the Laguerre-Gould-Hopper-Sheffer polynomials $LH_n^{(m,r)}Sn(x, y, z)$ in terms of the 3-variable generalized Hermite-Sheffer polynomials $H_n^{(r,m)}Sn(x, y, z)$ holds true:

$$LH_n^{(m,r)}Sn(x, y, z) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^u u^{-1} H_n^{(r,m)}Sn(y, xu^{-1}, z) \, du.$$ (4.5.5)

Proof. Use of equation (4.1.1) in the l.h.s. of equation (4.5.1) gives

$$\sum_{n=0}^{\infty}LH_n^{(m,r)}Sn(x, y, z) \frac{t^n}{n!} = A(t) \sum_{n=0}^{\infty}LH_n^{(m,r)}(x, y, z) \frac{H(t)^n}{n!},$$ (4.5.6)
which on using the following integral representation of LGHP \( LH_n^{(m,r)}(x, y, z) \) [86]:

\[
LH_n^{(m,r)}(x, y, z) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^{u-1}H_n^{(r,m)}(y, xu^{-1}, z) \, du , \quad (4.5.7)
\]

in the r.h.s. gives

\[
\sum_{n=0}^{\infty} LH_n^{(m,r)}s_n(x, y, z) \frac{t^n}{n!} = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^{u-1} \left( \sum_{n=0}^{\infty} H_n^{(r,m)}(y, xu^{-1}, z) \frac{(H(t))^n}{n!} \right) \, du .
\]

(4.5.8)

Now, making use of the generating function equation of \( H_n^{(r,m)}(x, y, z) \) given in Table 4.1.1 (XIV) in the r.h.s. of the above equation, the following equation is obtained:

\[
\sum_{n=0}^{\infty} LH_n^{(m,r)}s_n(x, y, z) \frac{t^n}{n!} = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^{u-1} \left( A(t) e^{(yH(t)+xu^{-1}(H(t)))'+z(H(t))^m} \right) \, du ,
\]

which in view of generating function of \( H_n^{(r,m)}(x, y, z) \) given in Table 5.8 becomes

\[
\sum_{n=0}^{\infty} LH_n^{(m,r)}s_n(x, y, z) \frac{t^n}{n!} = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \left( \int_{-\infty}^{(0+)} e^{u-1}H_n^{(r,m)}(y, xu^{-1}, z) \, du \right) \frac{t^n}{n!} . \quad (4.5.9)
\]

Finally, equating the coefficients of like powers of \( t \) in both sides of equation (4.5.9), assertion (4.5.5) is obtained.

Next, implicit integral representations for the LGHSP \( LH_n^{(m,r)}s_n(x, y, z) \) are derived by proving the following results:

**Theorem 4.5.5.** The following implicit integral representations for the Laguerre-Gould-Hopper-Sheffer polynomials \( LH_n^{(m,r)}s_n(x, y, z) \) hold true:

\[
LH_n^{(m,r)}s_n(x, y, z) = \int_0^\infty e^{-u} LH_n^{(m,r)}s_n(x, y, uD_z^{-1}) \, du \quad (4.5.10)
\]

and

\[
LH_n^{(m,r)}s_n(x, y, z) = \frac{1}{n!} \int_0^\infty e^{-u} u^n LH_n^{(m+1,r)}s_n \left( \frac{x}{u}, y, z \right) \, du . \quad (4.5.11)
\]

**Proof.** Using the integral representation [86]:

\[
LH_n^{(m,r)}(x, y, z) = \int_0^\infty e^{-u} LH_n^{(m,r)}(x, y, uD_z^{-1}) \, du \quad (4.5.12)
\]
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in the r.h.s. of equation (4.5.6), it follows that

\[ \sum_{n=0}^{\infty} \frac{L_{H}^{(m,r)} s_{n}(x, y, z)}{n!} t^{n} = A(t) \sum_{n=0}^{\infty} \left( \int_{0}^{\infty} e^{-u} \frac{L_{H}^{(m,r)}(x, y, u D_{z}^{-1})}{n!} du \right) \left( \mathcal{H}(t) \right)^{n} / n! , \]

or, equivalently

\[ \sum_{n=0}^{\infty} \frac{L_{H}^{(m,r)} s_{n}(x, y, z)}{n!} t^{n} = \int_{0}^{\infty} e^{-u} \left( A(t) \sum_{n=0}^{\infty} \frac{L_{H}^{(m,r)}(x, y, u D_{z}^{-1})}{n!} \right) \left( \mathcal{H}(t) \right)^{n} / n! \] \hspace{1cm} (4.5.13)

which on again using equation (4.5.6) gives

\[ \sum_{n=0}^{\infty} \frac{L_{H}^{(m,r)} s_{n}(x, y, z)}{n!} t^{n} = \sum_{n=0}^{\infty} \left( \int_{0}^{\infty} e^{-u} L_{H}^{(m,r)}(x, y, u D_{z}^{-1}) du \right) \frac{t^{n}}{n!} . \] \hspace{1cm} (4.5.14)

Finally, equating the coefficients of like powers of \( t \) in both sides of equation (4.5.14), assertion (4.5.10) is obtained.

Similarly, use of integral representation [86]:

\[ L_{H}^{(m,r)}(x, y, z) = \frac{1}{n!} \int_{0}^{\infty} e^{-u} u^{n} L_{H}^{(m+1,r)} \left( \frac{x}{u}, y, z \right) du , \] \hspace{1cm} (4.5.15)

in the r.h.s. of equation (4.5.6) gives

\[ \sum_{n=0}^{\infty} \frac{L_{H}^{(m,r)} s_{n}(x, y, z)}{n!} t^{n} = A(t) \sum_{n=0}^{\infty} \frac{1}{n!} \int_{0}^{\infty} e^{-u} u^{n} L_{H}^{(m+1,r)} \left( \frac{x}{u}, y, z \right) du \left( \mathcal{H}(t) \right)^{n} / n! , \]

or, equivalently

\[ \sum_{n=0}^{\infty} \frac{L_{H}^{(m,r)} s_{n}(x, y, z)}{n!} t^{n} = \frac{1}{n!} \int_{0}^{\infty} e^{-u} u^{n} \left( A(t) \sum_{n=0}^{\infty} \frac{L_{H}^{(m+1,r)}(x, y, z)}{n!} \left( \mathcal{H}(t) \right)^{n} / n! \right) du , \] \hspace{1cm} (4.5.16)

which on again using equation (4.5.6) in the r.h.s and then equating the coefficients of like powers of \( t \) yields assertion (4.5.11).

The hybrid special polynomials related to the Appell and Sheffer sequences introduced in this chapter have applications in various fields of mathematics, physics and engineering. The properties of these hybrid special families lie within the properties of the parent polynomials. The differential, integro-differential and partial differential equations
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for the Hermite-Appell polynomials [101] are derived in [140] by Srivastava et. al. is a recent investigation.

The use of operational techniques in introducing the determinant forms for the hybrid special polynomials is a new study. The approach presented in this chapter is general and can be extended to other families of mixed special polynomials, which are useful in solving various problems arising in certain branches of mathematics, physics and engineering.