CHAPTER 7

AN INVENTORY MODEL WITH NON-INSTANTANEOUS DETERIORATION, TRADE CREDIT, PARTIAL BACKLOGGING AND INFLATION FOR A TWO-WAREHOUSE SYSTEM

7.1 Introduction

In some practical situations, when suppliers offer price discounts for bulk purchases or the products are seasonal, the retailer may purchase more goods than can be stored in his/her own warehouse (OW). Therefore, a rented warehouse (RW) is used to store the excess units over the fixed capacity $W_1$ of the own warehouse. Usually, the rented warehouse may charge higher unit holding cost than the own warehouse due to additional cost of maintenance, material handling, etc. Moreover, the RW generally provides better preserving facility than the OW resulting in a lower deterioration rate for the goods. To reduce the inventory costs, it will be economical to consume the goods of RW at the earliest. Consequently, the firm stores goods in OW before RW, but clears the stocks in RW before OW.

Here it is assumed that the supplier permits the retailer certain fixed period of time (permissible delay in time) for settling the account and does not charge any interest from the retailer on the amount owed during this period. We have incorporated this concept in the proposed model.
The effect of inflation and time value of money were disregarded in the model. It has happened most because of the belief that the inflation and the time value of money would not influence the inventory policy to any significant degree. However, in the last several years, most of the countries have suffered from large-scale inflation and sharp decline in the purchasing power of money. As a result, while determining the optimal inventory policy, the effects of inflation and time value of money cannot be ignored.

In the present chapter, a two warehouse deterministic inventory model for non-instantaneous deteriorating items with permissible delay in payments is proposed in which the inflation and time value of money are considered. Also, in this model the demand is deterministic function which includes selling price and advertisement cost. Shortages are partially backlogged and the backlogging rate is variable which is dependent on the length of the waiting time for next replenishment. We have shown the effect due to changes in various parameters by taking suitable numerical examples and sensitivity analysis. To the author’s best of knowledge, such type of inventory model for non-instantaneous deteriorating items with partial backlogging and permissible delay in payments under inflation, by considering two-warehouse, has not yet been considered.

The rest of the chapter is organized as follows: In the next section problem description of this chapter is given. In section 7.3, the notations and assumptions are used throughout this chapter. In section 7.4, we have formulated a mathematical model to evaluate the optimal replenishment policy. Section 7.5 presents solution procedure to find the optimal length of time and order quantity. In section 7.6, number of numerical examples is
given to illustrate the model. Sensitivity analysis with respect to major parameters of the system is carried out in section 7.7. This is followed by conclusion in section 7.8.

7.2 Problem description

In this chapter, a two-warehouse EOQ model for non-instantaneous deteriorating items with permissible delay in payments under the effect of inflation and time value of money is proposed in which the demand is considered as a deterministic function of selling price and advertisement cost. Also in this model, shortages are allowed and partially backlogged. The backlogging rate is dependent on the waiting time for the next replenishment. The objective is to find the optimal time intervals and the optimal order quantity that would minimize the total inventory cost. Furthermore, numerical examples are provided to illustrate the proposed model, sensitivity analysis of the optimal solutions with respect to major parameters is carried out and some managerial inferences are obtained.

7.3 Notations and assumptions

7.3.1 Notations

The following added notations are used to develop the mathematical model of the chapter.

- \( T \) The length of the order cycle.
- \( \alpha \) The deterioration rate in OW, \( 0 \leq \alpha < 1 \).
- \( \beta \) The deterioration rate in RW, \( 0 \leq \beta < 1, \alpha > \beta \).
- \( C_{hr} \) The holding cost (excluding interest charges) per item in RW.
\( C_{ho} \) The holding cost (excluding interest charges) per item in OW,
\( C_{hr} > C_{ho} \).

\( TC \) The total cost of the system.

### 7.3.2 Assumptions
To develop the mathematical model, the following assumptions are being made.

1. A single item is considered over the fixed period \( T \).
2. There is no replacement or repair of deteriorated items takes place in a given cycle.
3. The lead time is zero.
4. Deterioration takes place after the life time of items. That is, during some fixed period, the product has no deterioration. After that, it will deteriorate with constant rate.
5. The replenishment takes place at an infinite rate.
6. The effects of inflation and time value of money are considered.
7. The demand rate \( D \) is a deterministic function of selling price, \( s \) and advertisement cost, \( A_c \) per unit item. i.e.) \( D(A_c, s) = as^{-m} + bA_c^n \), \( a, b > 0 \). \( m \) is the index of price elasticity, \( b \) and \( n \) are constants which indicate the effect of advertising. The larger value of \( b \) implies more investment in advertising.
8. Shortages are allowed and during stock out period, the backlogging rate is variable and is dependent on the length of the waiting time for next replenishment. So that the backlogging rate for negative
inventory is, \( B(t) = \frac{1}{1 + \delta(T - t)} \), where \( \delta \) is backlogging parameter \( 0 < \delta < 1 \) and \( (T - t) \) is waiting time \( (t_1 \leq t \leq T) \). This is practical because as the waiting time of customers for receiving their goods increases, the backlogging rate decreases. Many inventory modelers developed their inventory models considering backlogging rate to be exponential. But in real life situation backlogging rate should not be as high as exponential. Therefore, taking backlogging rate as \( B(t) = \frac{1}{1 + \delta(T - t)} \), seems to be better representation than that of exponential waiting time dependent rate. And the remaining fraction \( (1 - B(t)) \) is lost.

9. The OW has limited capacity of \( W_1 \) units and the RW has unlimited capacity. For economic reasons, the items of RW are consumed first and next the items of OW.

10. The retailer can accumulate revenue and earn interest after his/her customer pays for the amount of purchasing cost to the retailer until the end of the trade credit period offered by the supplier.

### 7.4 Formulation of the model

The inventory system is developed as follows: \( S \) units of items arrive at the inventory system at the beginning of each cycle. \( W_1 \) units are kept in OW and the rest is stored in RW. The items of OW are consumed only after consuming the goods kept in RW. In the RW, during the time interval \([0, \mu_2]\), the inventory level is decreasing only due to demand rate and the
inventory level is dropping to zero owing to demand and deterioration during the time interval \([\mu_2, t_r]\). In OW, during the time interval \([0, \mu_1]\), there is no change in the inventory level. However, the inventory \(W_1\) decreases during \([\mu_1, t_r]\) due to deterioration only, but during \([t_r, t_0]\), the inventory is depleted due to both demand and deterioration. By the time \(t_0\), both warehouses are empty. Finally, a shortage occurs due to demand and partial backlogging during the time interval \([t_0, T]\). The behaviour of the inventory model is demonstrated in Figure 7.1.

Based on the above description, during the time interval \([0, \mu_2]\), the inventory level in RW is decreasing only due to demand rate and the differential equation representing the inventory status is given by

Figure 7.1: Pictorial representation of the inventory system
\[ \frac{dq_r(t)}{dt} = -D, \quad 0 \leq t \leq \mu_2 \]  

(123)

With the condition \( q_r(0) = W_2 \), the solution of Equation (123) is

\[ q_r(t) = W_2 - Dt \quad 0 \leq t \leq \mu_2 \]  

(124)

In the second interval \([\mu_2, t_r]\) in RW, the inventory level decreases due to demand and deterioration. Thus, the differential equation below represents the inventory status:

\[ \frac{dq_r(t)}{dt} + \beta q_r(t) = -D, \quad \mu_2 \leq t \leq t_r \]  

(125)

With the condition \( q_r(t_r) = 0 \), we get the solution of Equation (125) is

\[ q_r(t) = \frac{D}{\beta} \left[ e^{\beta(t_r - t)} - 1 \right], \quad \mu_2 \leq t \leq t_r \]  

(126)

Put \( t = \mu_2 \) in Equations (124) and (126) we find the value of \( W_2 \) as

\[ W_2 = D \left[ \mu_2 + \frac{e^{\beta(t_r - \mu_2)} - 1}{\beta} \right], \]  

(127)

Substituting Equation (127) in Equation (124) we get

\[ q_r(t) = D \left[ \mu_2 - t + \frac{e^{\beta(t_r - \mu_2)} - 1}{\beta} \right], \quad 0 \leq t \leq \mu_2 \]  

(128)

In OW, during the time interval \([0, \mu_1]\), there is no change in the inventory level and during \([\mu_1, t_r]\) the inventory \( W_1 \) decreases due to deterioration only. Therefore, the rate of change in the inventory is given by

\[ \frac{dq_o(t)}{dt} = 0, \quad 0 \leq t \leq \mu_1 \]  

(129)

\[ \frac{dq_o(t)}{dt} + \alpha q_o(t) = 0, \quad \mu_1 \leq t \leq t_r \]  

(130)
With the conditions \( q_o(0) = W_1 \) and \( q_o(\mu_1) = W_1 \), the solutions of Equations (129) and (130) are
\[
q_o(t) = W_1, \quad 0 \leq t \leq \mu_1 \tag{131}
\]
\[
q_o(t) = W_1 e^{\alpha(\mu_1 - t)}, \quad \mu_1 \leq t \leq \tau_r \tag{132}
\]
In the interval \([\tau_r, t_0]\) in OW, the inventory level decreases due to demand and deterioration. Thus, the differential equation is
\[
\frac{dq_o(t)}{dt} + \alpha q_o(t) = -D, \quad \tau_r \leq t \leq t_0 \tag{133}
\]
With the condition \( q_o(t_0) = 0 \), we get the solution of Equation (133) is
\[
q_o(t) = \frac{D}{\alpha} \left[ e^{\alpha(t_0 - t)} - 1 \right], \quad \tau_r \leq t \leq t_0 \tag{134}
\]
Put \( t = \tau_r \) in Equations (132) and (134), we get
\[
\tau_r = t_0 - \mu_1 - \frac{1}{\alpha} \ln \left( \frac{W_1 \alpha}{D} \right), \tag{135}
\]
During the interval \([t_0, T]\), shortage occurred and the demand is partially backlogged. That is, the inventory level at time \( t \) is governed by the following differential equation:
\[
\frac{dq_s(t)}{dt} = -\frac{D}{1 + \delta(T - t)}, \quad t_0 \leq t \leq T \tag{136}
\]
With the condition \( q_s(t_0) = 0 \), the solution of Equation (136) is
\[
q_s(t) = D(t_0 - t) \left[ 1 - \delta(T - t) + \frac{\delta}{2} (t_0 + t) \right], \quad t_0 \leq t \leq T \tag{137}
\]
The maximum backordered inventory \( BI \) is obtained at \( t = T \), then from Equation (137)
\[ BI = -q_s(T) = D(T - t_0) \left[ 1 + \frac{\delta}{2}(t_0 - T) \right] \]  

Now, the maximum inventory level per cycle \( S = W_1 + W_2 \).  

Thus, the order size during total time interval \([0, T]\) is,  

\[ Q = S + BI = W_1 + D \left\{ \mu_2 + \frac{e^{\beta(t_r - \mu_2)}}{\beta} - 1 \right\} + (T - t_0) \left[ 1 + \frac{\delta}{2}(t_0 - T) \right] \]  

Now, we want to find the different inventory costs with the effect of inflation as:

Ordering cost \( = A \).  

The holding cost \( HC_r \) for the RW during the period \([0, T]\) is  

\[ HC_r = C_{hr} \left[ \mu_2 q_r(t) e^{-Rt} dt + \int_{t_r}^{t_r} q_r(t) e^{-Rt} dt \right] \]  

\[ = C_{hr} \left\{ \frac{D}{R} \left\{ \left( \frac{e^{\beta(t_r - \mu_2)}}{\beta} - 1 \right) \left( 1 - e^{-R\mu_2} \right) + \mu_2 \right\} \right. \]  

\[ + \left. \frac{D}{\beta} \left( e^{-Rt_r} \left( \frac{\beta}{R(\beta + R)} \right) + e^{-R\mu_2} \left( \frac{e^{\beta(t_r - \mu_2)}}{\beta + R} - \frac{1}{R} \right) \right) \right\} \]  

The holding cost \( HC_o \) for the OW during the period \([0, T]\) is  

\[ HC_o = C_{ho} \left[ \mu_1 q_o(t) e^{-Rt} dt + \int_{t_r}^{t_r} q_o(t) e^{-Rt} dt + \int_{t_r}^{t} q_o(t) e^{-Rt} dt \right] \]  

\[ = C_{ho} \left\{ \frac{W_1}{R} \left( 1 - e^{-R\mu_1} \right) + \frac{W_1 e^{\alpha \mu_1}}{\alpha + R} \left( e^{-(\alpha + R)\mu_1} - e^{-(\alpha + R)t_r} \right) \right. \]  

\[ + \left. \frac{D}{\alpha} \left( e^{-Rt_0} \left( \frac{\alpha}{R(\alpha + R)} \right) + e^{-Rt_r} \left( \frac{e^{\alpha(t_0 - t_r)}}{\alpha + R} - \frac{1}{R} \right) \right) \right\} \]
The deteriorating cost $DC_r$ for RW during the period $[0, T]$ is

$$DC_r = C_2 \int_{t^*}^{t_{t_0}} \beta q_r(t) e^{-R t} \, dt$$

$$= DC_2 \left[ e^{-R t_{t_0}} \left( \frac{\beta}{R(\beta + R)} \right) + e^{-R t_{t_0}} \left( \frac{e^{(\beta t_{t_0} - \mu_2)}}{\beta + R} - \frac{1}{R} \right) \right]$$  \hspace{1cm} (144)$$

The deteriorating cost $DC_o$ for OW during the period $[0, T]$ is

$$DC_o = C_2 \alpha \left[ \int_{t^*}^{t_{t_0}} q_o(t) e^{-R t} \, dt + \int_{t_{t_0}}^{t_t} q_o(t) e^{-R t} \, dt \right]$$

$$= C_2 \alpha \left[ \frac{W_1 e^\alpha }{\alpha + R} \left( e^{-(\alpha + R) t_{t_0}} - e^{-(\alpha + R) t_t} \right) \right.$$  

$$+ \frac{D}{\alpha} \left\{ e^{-R t_{t_0}} \left( \frac{\alpha}{R(\alpha + R)} \right) + e^{-R t_t} \left( \frac{e^{\alpha (t_0 - t_t)}}{\alpha + R} - \frac{1}{R} \right) \right\} \right]$$  \hspace{1cm} (145)$$

Total shortage cost $SC$ during the period $[0, T]$ is given by

$$SC = -C_3 \int_{t_0}^{T} q_s(t) e^{-R t} \, dt$$

$$= -C_3 D \left\{ e^{-R T} \left[ t_0 \left( \delta T - \frac{\delta}{2} t_0 - 1 \right) + T \left( 1 - \frac{\delta}{2} \frac{T}{R} \right) + \frac{1}{R} \left( 1 + \frac{\delta}{T} \right) \right] \right.$$  

$$+ \frac{e^{-R t_{t_0}}}{R} \left[ \delta \left( T - t_0 - \frac{1}{R} \right) - 1 \right] \right\}$$  \hspace{1cm} (146)$$

The lost sales cost $LC$ during the period $[0, T]$ is

$$LC = C_4 \int_{t_0}^{T} \left( 1 - \frac{1}{1 + \delta(T-t)} \right) \, dt$$

152
\[ \text{Eqn. 147} \]

\[ \begin{aligned} &\frac{-C_4 D \delta}{R^2} \left[ e^{-R \tau} + e^{-R t_0} \left( R(T - t_0) - 1 \right) \right] \\
\end{aligned} \]

**7.4.1 The interest payable opportunity cost**

There are five cases depicted as Figure 7.2.

![Figure 7.2: Different cases of trade credit period \( M \)](image)

**7.4.1.1 Case 1: \( 0 < M \leq \mu_1 \)**

In this case, the length of delay in payment \( (M) \) is absolutely less than the length with no deterioration \( (\mu_1) \) in OW. Since the interest is payable during the time \( (t_0 - M) \), the interest payable in any cycle \([0, T]\) is
\[
IC = pI_c \left[ \int_M^{\mu_1} q_o(t) e^{-Rt} dt + \int_{\mu_1}^{l_r} q_o(t) e^{-Rt} dt + \int_{l_r}^{t_e} q_o(t) e^{-Rt} dt \right]
\]

\[
+ \int_M^{\mu_2} q_r(t) e^{-Rt} dt + \int_{\mu_2}^{l_r} q_r(t) e^{-Rt} dt
\]

\[
IC = pI_c \left\{ \frac{W_1}{R} \left( e^{-RM} - e^{-R\mu_1} \right) + \frac{W_1 e^{\alpha \mu_1}}{\alpha + R} \left( e^{-\alpha R \mu_1} - e^{-(\alpha + R) l_r} \right) \right.
\]

\[
+ \frac{D}{\alpha} \left\{ e^{-Rl_0} \left( \frac{\alpha}{R(\alpha + R)} \right) + e^{-Rl_r} \left( e^\alpha (t_0 - t_r) \left( \frac{\alpha}{\alpha + R} - \frac{1}{R} \right) \right) \right. 
\]

\[
+ \frac{D}{R} \left\{ \left( \frac{1}{R} - \frac{e^{\beta (t_r - \mu_2)}}{\beta} \right) \left( e^{-R\mu_2} - e^{-RM} \right) + e^{-RM} \left( \mu_2 - M \right) \right. 
\]

\[
+ \frac{D}{\beta} \left\{ e^{-Rl_r} \left( \frac{\beta}{R(\beta + R)} \right) + e^{-R\mu_2} \left( \frac{e^{\beta (t_r - \mu_2)}}{\beta + R} - \frac{1}{R} \right) \right. \right\} 
\]

(148)

7.4.1.2 Case 2: \( \mu_1 < M \leq \mu_2 \)

In this case the interest payable is

\[
IC = pI_c \left[ \int_M^{t_r} q_o(t) e^{-Rt} dt + \int_{t_r}^{l_q} q_o(t) e^{-Rt} dt + \int_M^{\mu_2} q_r(t) e^{-Rt} dt + \int_{\mu_2}^{t_e} q_r(t) e^{-Rt} dt \right]
\]

\[
IC = pI_c \left\{ \frac{W_1 e^{\alpha M}}{\alpha + R} \left( e^{-\alpha R M} - e^{-(\alpha + R) l_r} \right) \right. 
\]

\[
+ \frac{D}{\alpha} \left\{ e^{-Rl_0} \left( \frac{\alpha}{R(\alpha + R)} \right) + e^{-Rl_r} \left( e^\alpha (t_0 - t_r) \left( \frac{\alpha}{\alpha + R} - \frac{1}{R} \right) \right) \right. 
\]

\[
+ \frac{D}{R} \left\{ \left( \frac{1}{R} - \frac{e^{\beta (t_r - \mu_2)}}{\beta} \right) \left( e^{-R\mu_2} - e^{-RM} \right) + e^{-RM} \left( \mu_2 - M \right) \right. \right\} 
\]

154
\[ + \frac{D}{\beta} \left\{ e^{-Rt_r} \left( \frac{\beta}{R(\beta + R)} \right) + e^{-R\mu_2} \left( \frac{e^{R(t_r - \mu_2)}}{\beta + R} - \frac{1}{R} \right) \right\} \]  

(149)

### 7.4.1.3 Case 3: \( \mu_2 < M \leq t_r \)

In this case, the period of delay in payment (\( M \)) is more than the period with no deterioration (\( \mu_2 \)) in RW but less than the period with positive inventory (\( t_0 \)). Then, the interest payable in any cycle \([0, T]\) is:

\[ IC = pI_c \left[ \int_{M}^{t_r} q_o(t) e^{-Rt} \, dt + \int_{t_r}^{t_0} q_o(t) e^{-Rt} \, dt \right] \]

\[ IC = pI_c \left\{ \frac{W_1 e^{\alpha M}}{\alpha + R} \left( e^{-(\alpha + R)M} - e^{-(\alpha + R)t_r} \right) \right. \]

\[ + \frac{D}{\alpha} \left\{ e^{-Rt_0} \left( \frac{\alpha}{R(\alpha + R)} \right) + e^{-Rt_r} \left( \frac{e^{\alpha(t_0 - t_r)}}{\alpha + R} - \frac{1}{R} \right) \right\} \]

\[ + \frac{D}{\beta} \left\{ e^{-Rt_r} \left( \frac{\beta}{R(\beta + R)} \right) + e^{-RM} \left( \frac{e^{\beta(t_r - M)}}{\beta + R} - \frac{1}{R} \right) \right\} \]  

(150)

### 7.4.1.4 Case 4: \( t_r < M \leq t_0 \)

In this case the interest payable is:

\[ IC = pI_c \left[ \int_{M}^{t_0} q_o(t) e^{-Rt} \, dt \right] \]

\[ = pI_c D \left\{ e^{-Rt_0} \left( \frac{\alpha}{R(\alpha + R)} \right) + e^{-RM} \left( \frac{e^{\alpha(t_0 - M)}}{\alpha + R} - \frac{1}{R} \right) \right\} \]  

(151)
7.4.1.5 Case 5: $M > t_0$

In this case no interest is payable during the period for the item kept in stock.

7.4.2 The opportunity interest earned

There are two cases as follows:

7.4.2.1 Case 1: $M \leq t_0$

In this case the interest earned in the cycle period $[0, T]$ is

$$IE = \int_0^M Dt e^{-Rt} dt = \frac{sL_e D}{R^2} \left[ 1 - e^{-RM} \right] (1 + RM)$$ (152)

7.4.2.2 Case 2: $M > t_0$

In this case, the period of delay in payment ($M$) is more than the period with positive inventory ($t_0$). Also, the retailer earns interest on the sales revenue up to the permissible delay period. Interest earned for the time period $[0, T]$ is

$$IE = \int_0^{t_0} D t e^{-Rt} dt + (M - t_0) \int_0^{t_0} D e^{-Rt} dt$$

$$= \frac{sL_e D}{R^2} \left[ 1 - e^{-Rt_0} (1 + RM) + R(M - t_0) \right]$$ (153)

Therefore, the total average cost of the system per unit time is given by
\[ TC = \begin{cases} 
TC_1, & 0 < M \leq \mu_1 \\
TC_2, & \mu_1 < M \leq \mu_2 \\
TC_3, & \mu_2 < M \leq t_r \\
TC_4, & t_r < M \leq t_0 \\
TC_5, & M > t_0 
\end{cases} \]  

(154)

That is,

Total average cost per cycle = (ordering cost + inventory holding cost in RW + inventory holding cost in OW + deterioration cost in RW + deterioration cost in OW + shortage cost + lost sales cost + interest payable (charged)− interest earned) / \( T \).

i.e., \( TC = \frac{1}{T} \left[ A + HC_r + HC_o + DC_r + DC_o + SC + LC + IC - IE \right] \)

where

\[
TC_1 = \left\{ A + \frac{C_{h_r}D}{R} \left\{ \left( \frac{e^{\beta(t_r-\mu_2)}}{\beta} - \frac{1}{R} \right) \left[ 1 - e^{-R\mu_2} \right] + \mu_2 \right\} 
\right. \\
+ \left. \left\{ e^{-Rt_r} \left( 1 + \frac{\beta}{R(\beta + R)} \right) + e^{-R\mu_2} \left( \frac{e^{\beta(t_r-\mu_2)}}{\beta + R} - \frac{1}{R} \right) \right\} 
\right. \\
\times \left. \left( C_2 + \frac{C_{h_r} + pI_c}{\beta} + \frac{C_{h_o} W_1}{R} \left[ 1 - e^{-R\mu_4} \right] + \left( pI_c + C_{h_o} + C_2 \alpha \right) \right) 
\right. \\
\times \left. \left( W_e e^{\alpha \mu_4} \left( e^{-\left( \alpha + R \right) \mu_4} - e^{-\left( \alpha + R \right) t_r} \right) \right) 
\right. \\
+ \left. \left( \frac{D}{\alpha} \left( e^{-Rt_0} \left( \frac{\alpha}{R(\alpha + R)} \right) + e^{-Rt_r} \left( \frac{e^{\alpha(t_0-t_r)}}{\alpha + R} - \frac{1}{R} \right) \right) \right) 
\right. \\
- \left. \frac{sI_c D}{R^2} \left[ 1 - e^{-RM} \left( 1 + RM \right) \right] \right. 
\]

157
\[-\frac{C_3D}{R} \left\{ e^{-RT} \left[ t_0 \left( \delta T - \frac{\delta}{2} t_0 - 1 \right) + T \left( 1 - \frac{\delta}{2} T \right) + \frac{1}{R} \left( 1 + \frac{\delta}{R} \right) \right] \right. \\
+ \frac{e^{-Rt_0}}{R} \left\{ \delta \left( t - t_0 - \frac{1}{R} - 1 \right) \right\} - \frac{C_4D\delta}{R^2} \left\{ e^{-RT} + e^{-Rt_0} \left( R(T - t_0) - 1 \right) \right\} \\
\left. + plc \left\{ \frac{D}{R} \left( \frac{1}{R} - \frac{e^{\beta(\gamma - \mu_2)} - 1}{\beta} \right) \left( e^{-Rt_2} - e^{-RM} \right) \right\} \\
\left. + e^{-RM} \left( \mu_2 - M \right) \right\| + \frac{W_1}{R} \left( e^{-RM} - e^{-R\mu_1} \right) \right\} \right] / T \] (155)

\[ TC_2 = \left\{ A + \frac{C_{ht}D}{R} \left\{ \left( \frac{e^{\beta(t_2 - \mu_2)} - 1}{\beta} \right) \left( 1 - e^{-Rt_2} \right) + \mu_2 \right\} \right. \\
+ \left\{ e^{-Rt_2} \left( \frac{\beta}{R(\beta + R)} \right) + e^{-Rt_2} \left( \frac{e^{\beta(t_2 - \mu_2)} - 1}{\beta} \right) \right\} \left( \frac{C_2 + \frac{C_{ht} + plc}{\beta}}{D} \right) \\
+ \frac{C_{ho}W_1}{R} \left( 1 - e^{-R\mu_1} \right) + \left( C_{ho} + C_2\alpha \right) \left\{ \frac{W_1 e^{\alpha\mu_1}}{\alpha + R} \left( e^{-R(t_2 + \alpha R)\mu_1} - e^{-R(t_2 + \alpha R)\mu_1} \right) \right\} \\
+ \frac{D}{\alpha} \left( e^{-Rt_0} \left( \left( \frac{\alpha}{R(\alpha + R)} \right) + e^{-Rt_0} \left( \frac{e^{\alpha(t_0 - t_2)}}{\alpha + R} - \frac{1}{R} \right) \right) \right\} \\
- \frac{slc}{R^2} \left[ 1 - e^{-RM} \left( 1 + RM \right) \right] \\
\left. - \frac{C_3D}{R} \left\{ e^{-RT} \left[ t_0 \left( \delta T - \frac{\delta}{2} t_0 - 1 \right) + T \left( 1 - \frac{\delta}{2} T \right) + \frac{1}{R} \left( 1 + \frac{\delta}{R} \right) \right] \right. \\
\left. + \frac{e^{-Rt_0}}{R} \left\{ \delta \left( t - t_0 - \frac{1}{R} - 1 \right) \right\} - \frac{C_4D\delta}{R^2} \left\{ e^{-RT} + e^{-Rt_0} \left( R(T - t_0) - 1 \right) \right\} \right] \right] \]
\[ + p l_c \left[ \frac{W_t e^{\alpha M}}{\alpha + R} \left( e^{-(\alpha + R)t_r} - e^{-(\alpha + R)T} \right) \right] \\
+ \frac{D}{\alpha} \left[ e^{-Rt_0} \left( \frac{\alpha}{R(\alpha + R)} \right) + e^{-Rt_r} \left( \frac{e^{\alpha(t_0 - t_r)}}{\alpha + R - \frac{1}{R}} \right) \right] \\
+ \frac{D}{R} \left[ \frac{1}{R} - \frac{e^{\beta(t_0 - t_r)}}{\beta} \right] \left( e^{-Rt_2} - e^{-RM} \right) + e^{-RM} \left( t_2 - M \right) \right] \right] / T(156) \\
TC_3 = \left\{ A + \frac{C_{h_t} D}{R} \left[ \left( \frac{e^{\beta(t_0 - t_r)}}{\beta} - \frac{1}{R} \right) \left( 1 - e^{-Rt_2} \right) + t_2 \right] \right\} \\
+ \left\{ e^{-Rt_r} \left( \frac{\beta}{R(\beta + R)} \right) + e^{-Rt_2} \left( \frac{e^{\beta(t_0 - t_r)}}{\beta + R - \frac{1}{R}} \right) \right\} D \left[ C_2 + \frac{C_{h_t}}{\beta} \right] \\
+ \frac{C_{h_0} W_t}{R} \left( 1 - e^{-Rt_2} \right) + (C_{h_0} + C_2) \left( \frac{W_t e^{\alpha t_1}}{\alpha + R} \left( e^{-(\alpha + R)T} - e^{-(\alpha + R)t_r} \right) \right) \\
+ \frac{D}{\alpha} \left[ e^{-Rt_0} \left( \frac{\alpha}{R(\alpha + R)} \right) + e^{-Rt_r} \left( \frac{e^{\alpha(t_0 - t_r)}}{\alpha + R - \frac{1}{R}} \right) \right] \right\} \\
- \frac{s l_c D}{R^2} \left[ 1 - e^{-RM} \left( 1 + RM \right) \right] \\
- \frac{C_3 D}{R} \left[ e^{-RT} \left[ t_0 \left( \delta T - \frac{\delta}{2} t_0 - 1 \right) + T \left( 1 - \frac{\delta}{2} T \right) + \frac{1}{R} \left( 1 + \frac{\delta}{R} \right) \right] \
+ e^{-Rt_0} \left( \delta \left( T - t_0 - \frac{1}{R} \right) - 1 \right) \right] - \frac{C_4 D \delta}{R^2} \left[ e^{-RT} + e^{-Rt_0} \left( R(T - t_0) - 1 \right) \right] \\
+ p l_c \left[ \frac{W_t e^{\alpha M}}{\alpha + R} \left( e^{-(\alpha + R)t_r} - e^{-(\alpha + R)T} \right) \right] \\
\]
\[ + \frac{D}{\alpha} \left\{ e^{-Rt_0} \left( \frac{\alpha}{R(\alpha + R)} \right) + e^{-Rt_r} \left( \frac{e^{\alpha(t_0 - t_r)}}{\alpha + R} - \frac{1}{R} \right) \right\} + \frac{D}{\beta} \left\{ e^{-Rt_r} \left( \frac{\beta}{R(\beta + R)} \right) + e^{-RM} \left( \frac{e^{\beta(t_r - M)}}{\beta + R} - \frac{1}{R} \right) \right\} \right\} \bigg/ T \] (157)

\[ TC_4 = A + \frac{C_{h,2} D}{R} \left\{ \left( \frac{e^{\beta(t_r - \mu_2)}}{\beta} - \frac{1}{R} \right) \left( 1 - e^{-R\mu_2} \right) + \mu_2 \right\} \]

\[ + \frac{C_{h,0} W_1}{R} \left( 1 - e^{-R\mu_1} \right) + \left( C_{h,0} + C_2 \alpha \right) \left( \frac{W_1 e^{\alpha \mu_1}}{\alpha + R} \left( e^{-(\alpha + R)\mu_1} - e^{-(\alpha + R)t_r} \right) \right) \]

\[ + \frac{D}{\alpha} \left\{ e^{-Rt_0} \left( \frac{\alpha}{R(\alpha + R)} \right) + e^{-Rt_r} \left( \frac{e^{\alpha(t_0 - t_r)}}{\alpha + R} - \frac{1}{R} \right) \right\} \]

\[ - \frac{s l e D}{R^2} \left[ 1 - e^{-RM} \left( 1 + RM \right) \right] \]

\[ - \frac{C_3 D}{R} \left\{ e^{-Rt} \left[ t_0 \left( \delta T - \frac{\delta}{2} t_0 - 1 \right) + T \left( 1 - \frac{\delta}{2} T \right) + \frac{1}{R} \left( 1 + \frac{\delta}{R} \right) \right] \right\} \]

\[ + \frac{e^{-Rt_0}}{R} \left[ \delta \left( T - t_0 - \frac{1}{R} \right) - 1 \right] - \frac{C_4 D \delta}{R^2} \left\{ e^{-RT} + e^{-Rt_0} \left( R(T - t_0) - 1 \right) \right\} \]

\[ + \frac{p l e D}{\alpha} \left\{ e^{-Rt_0} \left( \frac{\alpha}{R(\alpha + R)} \right) + e^{-RM} \left( \frac{e^{\alpha(t_0 - M)}}{\alpha + R} - \frac{1}{R} \right) \right\} \bigg/ T \] (158)
\[ TC_5 = \left\{ A + \frac{C_{hr} D}{R} \left\{ \left( \frac{e^{\beta (t_r - \mu_2)} - \frac{1}{\beta}}{\beta} - \frac{1}{R} \right) \left( 1 - e^{-R \mu_2} \right) + \mu_2 \right\} + \left( e^{-R t_r} \left( \frac{\beta}{R (\beta + R)} \right) + e^{-R \mu_2} \left( \frac{e^{\beta (t_r - \mu_2)} - \frac{1}{\beta}}{\beta + R} - \frac{1}{R} \right) \right) D \left( C_2 + \frac{C_{hr}}{\beta} \right) \right. \\
+ \frac{C_{ho} W_i}{R} \left( 1 - e^{-R \mu_1} \right) + \left( C_{ho} + C_2 \alpha \right) \left\{ \frac{W_i e^{\alpha \mu_1}}{\alpha + R} \left( e^{-(\alpha + R) \mu_1} - e^{-(\alpha + R) t_r} \right) \right. \\
+ \frac{D}{\alpha} \left( e^{-R t_0} \left( \frac{\alpha}{R (\alpha + R)} \right) + e^{-R t_r} \left( \frac{e^{\alpha (t_0 - t_r)}}{\alpha + R} - \frac{1}{R} \right) \right) \right\} \\
- \frac{s I_e D}{R^2} \left[ 1 - e^{-R t_0} (1 + RM) + R (M - t_0) \right] \\
- \frac{C_3 D}{R} \left\{ e^{-RT} \left[ t_0 \left( -\delta T - \frac{\delta}{2} t_0 - 1 \right) + T \left( 1 - \frac{\delta T}{2} \right) + \frac{1}{R} \left( 1 + \frac{\delta}{R} \right) \right) \right. \\
+ \frac{e^{-R t_0}}{R} \left[ \delta \left( T - t_0 - \frac{1}{R} \right) - 1 \right] - \frac{C_4 D \delta}{R^2} \left[ e^{-RT} + e^{-R t_0} \left( R (T - t_0) - 1 \right) \right] \right\} / T \] 

(159)

7.5 Solution procedure

In order to find the optimal solution \( t_0^* \) and to minimize the annual total relevant cost, we take the first and second order derivatives of \( TC_i(t_0) \) with respect to \( t_0 \), where \( i = \{1,2,3,4,5\} \). In other words, the necessary and sufficient conditions for minimization of \( TC_i(t_0) \) are respectively

\[
\frac{dTC_i(t_0)}{dt_0} = 0 \quad \text{and} \quad \frac{d^2TC_i(t_0)}{dt_0^2} > 0 \quad \text{where} \quad i = \{1,2,3,4,5\}.
\]
7.5.1 Case 1: $0 < M \leq \mu_1$.

The necessary and sufficient conditions to minimize $TC_1(t_0)$ are respectively $\frac{dTC_1(t_0)}{dt_0} = 0$ and $\frac{d^2TC_1(t_0)}{dt_0^2} > 0$.

Now, $\frac{dTC_1(t_0)}{dt_0} = 0$ gives the following nonlinear equation in $t_0$.

\[
\frac{D e^{\beta(t_0-\mu_2)}}{R} \left[ C_{hr} (1 - e^{-R\mu_2}) + pl_c \left( e^{-RM} - e^{-R\mu_2} \right) \right] \\
+ \frac{D}{\beta + R} \left( C_2 \beta + C_{hr} + pl_c \right) \left( e^{\beta t_r} e^{-(\beta + R)\mu_2} - e^{-R t_r} \right) + \left( pl_c + C_{h_o} + C_2 \alpha \right) \\
\times \left\{ W_1 e^{\alpha \mu_1} e^{-(\alpha R) t_r} - \frac{D}{(\alpha + R)} \left[ e^{-R t_0} + \frac{1}{\alpha} \left( Re^{-R t_r} e^{\left( \frac{1}{\alpha} - \ln \frac{W_2}{D} \right)} \right) \right] \right\} \\
+ \frac{C_3 D}{R} \left\{ \left( e^{-R t_0} - e^{-RT} \right) (\delta T - \delta t_0 - 1) \right\} + C_4 D \delta e^{-R t_0} (T - t_0) = 0 \tag{160}
\]

and provided $t_0$ satisfies the sufficient condition $\frac{d^2TC_1(t_0)}{dt_0^2} > 0$.

By solving Equation (160) the optimal value of $t_0 = t_0^*$ can be obtained and then from Equations (127), (135), (140) and (155), the optimal value of $W_2 = W_2^*$, $t_r = t_r^*$, $Q = Q^*$ and $TC = TC_1^*$ can be found out respectively.
7.5.2 Case 2: $\mu_1 < M \leq \mu_2$

The necessary and sufficient conditions to minimize $TC_2(t_0)$ are respectively $\frac{dTC_2(t_0)}{dt_0} = 0$ and $\frac{d^2TC_2(t_0)}{dt_0^2} > 0$.

Now, $\frac{dTC_2(t_0)}{dt_0} = 0$ gives the following nonlinear equation in $t_0$.

$$\frac{De^{\beta(t_0 - \mu_2)}}{R} \left[ C_h \left(1 - e^{-R \mu_2}\right) + pl_c \left(e^{-R\mu_2} - e^{-R\mu_2}ight) \right] + \frac{D}{\beta + R} \left(C_2 \beta + C_h \beta + pl_c \right) \left(e^{\beta \mu_2} e^{-(\beta + R) \mu_2} - e^{-R \mu_2}\right) - \left(pl_c + C_{ho} + C_2 \alpha\right)$$

$$\times \frac{D}{(\alpha + R)} \left[ e^{-R \mu_0} + \frac{1}{\alpha} \left(R e^{-R \mu_0} e^{\alpha(\mu_0 + \frac{1}{\alpha} \ln \frac{1}{D}}\right) + C_4 D \delta e^{-R \mu_0} (T - t_0) \right]$$

$$+ \frac{C_3 D}{R} \left(\left[e^{-R \mu_0} - e^{-R T}\right](\delta T - \delta \mu_0 - 1)\right)$$

$$+ W_1 e^{-(\alpha + R) \mu_0} \left[ e^{\alpha \mu_0} \left(C_{ho} + C_2 \alpha\right) + pl_c e^{\alpha M}\right] = 0$$

(161)

and provided $t_0$ satisfies the sufficient condition $\frac{d^2TC_2(t_0)}{dt_0^2} > 0$.

By solving Equation (161) the optimal value of $t_0 = t_0^*$ can be obtained and then from Equations (127), (135), (140) and (156), the optimal value of $W_2 = W_2^*$, $t_r = t_r^*$, $Q = Q^*$ and $TC = TC_2^*$ can be found out respectively.
7.5.3 Case 3: \( \mu_2 < M \leq t_r \)

The necessary and sufficient conditions to minimize \( TC_3(t_0) \) are respectively \( \frac{dTC_3(t_0)}{dt_0} = 0 \) and \( \frac{d^2TC_3(t_0)}{dt_0^2} > 0 \).

Now, \( \frac{dTC_3(t_0)}{dt_0} = 0 \) gives the following nonlinear equation in \( t_0 \).

\[
\frac{DC^r_{hr}}{R} e^{\beta(t_r - \mu_2)} (1 - e^{-R\mu_2}) + \frac{D}{\beta + R} \left( C_2 \beta + C_{hr} \right) \left( e^{\beta t_r} e^{-(\beta + R)\mu_2} - e^{-Rt_r} \right) + C_4 D e^{-Rt_0} (T - t_0) - (pI_e + C_{ho} + C_2 \alpha) \frac{D}{(\alpha + R)} \\
\times \left[ e^{-Rt_0} + \frac{1}{\alpha} \left( Re^{Rt_r} e^{\frac{\alpha(\mu_1 + \frac{1}{\alpha} \ln D)}{D}} \right) \right] + \frac{DpI_c}{\beta + R} \left( e^{\beta t_r} e^{-(\beta + R)M} - e^{-Rt_r} \right) \\
+ \frac{C_3 D}{R} \left[ \left( e^{-Rt_0} - e^{-Rf} \right) (\delta T - \delta t_0 - 1) \right] \\
+ W_1 e^{-(\alpha + R)t_r} \left[ e^{\alpha \mu_t} (C_{ho} + C_2 \alpha) + pI_e e^{\alpha M} \right] = 0
\]  \hspace{1cm} (162)

and provided \( t_0 \) satisfies the sufficient condition \( \frac{d^2TC_3(t_0)}{dt_0^2} > 0 \).

By solving Equation (162) the optimal value of \( t_0 = t_0^* \) can be obtained and then from Equations (127), (135), (140) and (157), the optimal value of \( W_2 = W_2^* \), \( t_r = t_r^* \), \( Q = Q^* \) and \( TC = TC_3^* \) can be found out respectively.

164
7.5.4 Case 4: \( t_r < M \leq t_0 \)

The necessary and sufficient conditions to minimize \( TC_4(t_0) \) are respectively \( \frac{dT C_4(t_0)}{dt_0} = 0 \) and \( \frac{d^2TC_4(t_0)}{dt_0^2} > 0 \).

Now, \( \frac{dTC_4(t_0)}{dt_0} = 0 \) gives the following nonlinear equation in \( t_0 \).

\[
\frac{DC_{hr}}{R} e^{\beta_r(t_r-\mu_2)} \left( 1 - e^{-R\mu_2} \right) + \frac{D}{\beta + R} \left( C_2 \beta + C_{hr} \right) \left( e^{\beta_r} e^{-(\beta + R)\mu_2} - e^{-Rt_r} \right) + C_4 D \delta e^{-Rt_0} (T - t_0) + \left( C_{ho} + C_2 \alpha \right) \left( W_1 e^{\alpha \mu_1} e^{-(\alpha + R)\mu_1} \right) - \frac{P}{(\alpha + R)}
\]

\[
e^{Rt_0} + \frac{1}{\alpha} \left( \Re^{Rt_r} e^{\alpha \left( \mu_1 + \frac{1}{\alpha} \ln \frac{W_{1\alpha}}{R} \right)} \right) + \frac{C_3 D}{R} \left( e^{-Rt_0} - e^{-RT} \right) \left( \delta T - \delta t_0 - 1 \right)
\]

\[
+ \frac{pI_c}{\alpha + R} \left[ \alpha e^{\alpha \mu_1} e^{-(\alpha + R)M} - D e^{-Rt_0} \right] = 0 \tag{163}
\]

and provided \( t_0 \) satisfies the sufficient condition \( \frac{d^2TC_4(t_0)}{dt_0^2} > 0 \).

By solving Equation (163) the optimal value of \( t_0 - t_0^* \) can be obtained and then from Equations (127), (135), (140) and (158), the optimal value of \( W_2 = W_2^*, t_r = t_r^*, Q = Q^* \) and \( TC = TC_4^* \) can be found out respectively.

7.5.5 Case 5: \( M > t_0 \)

The necessary and sufficient conditions to minimize \( TC_5(t_0) \) are respectively \( \frac{dT C_5(t_0)}{dt_0} = 0 \) and \( \frac{d^2TC_5(t_0)}{dt_0^2} > 0 \).
Now \( \frac{dTC_5(t_0)}{dt_0} = 0 \) gives the following nonlinear equation in \( t_0 \).

\[
\frac{DC_{hr}}{R} e^{\beta (t_r - \mu_2)} \left( 1 - e^{-R \mu_2} \right) + \frac{D}{\beta + R} \left( C_2 \beta + C_{hr} \right) \left( e^{\beta t_r} e^{-\left( \beta + R \right) \mu_3} - e^{-R t_r} \right) \\
+ C_4 D \delta e^{-R t_0} (T - t_0) + \left( C_{h0} + C_2 \alpha \right) \\
\times \left( W_1 e^{\alpha \mu_1} e^{-(\alpha + R) t_r} - \frac{D}{\alpha + R} \left( e^{-R t_0} + \frac{1}{\alpha} \left( \text{Re}^{-R t_r} e^{\frac{\alpha}{\alpha + 1} \ln \left( \frac{W_0(\alpha)}{D} \right)} \right) \right) \right) \\
+ \frac{C_3 D}{R} \left( e^{-R t_0} - e^{-RT} \right) (\delta T - \delta t_0) - 1 \\
+ \frac{s I e D}{R} \left( e^{-R t_0} \left( 1 + RM \right) - 1 \right) = 0
\] (164)

and provided \( t_0 \) satisfies the sufficient condition \( \frac{d^2 TC_5(t_0)}{dt_0^2} > 0 \).

By solving Equation (164) the optimal value of \( t_0 = t_0^* \) can be obtained and then from Equations (127), (135), (140) and (159), the optimal value of \( W_s = W_s^* \), \( t_r = t_r^* \), \( Q = Q^* \) and \( TC = TC_s^* \) can be found out respectively.

Since \( TC_i(t_0) \) for \( i = \{1, 2, 3, 4, 5\} \) are very complicated functions due to high-power expression of the exponential function, it is unlikely to show analytically the validity of the above sufficient conditions. However, it can be assessed numerically in the following illustrative examples. By examining the second order sufficient condition for above five cases, it can be verified that the total cost \( TC_i \) for \( i = \{1, 2, 3, 4, 5\} \) are convex functions with
respect to $t_0$. The convexity of the total cost function is graphically illustrated through the following numerical examples.

### 7.6 Numerical Examples

The numerical examples given below cover all the five cases that arise in the model.

**Example 7.1** (case 1)

Consider an inventory system with the following data: $T = 1$ year; $W_1 = 500$ units; $p = \$20$; $s = \$25$; $A = \$1000$; $C_{hr} = \$6$; $C_{ho} = \$4$; $C_2 = \$2$; $C_3 = \$15$; $C_4 = \$25$; $\alpha = 0.9$; $\beta = 0.2$; $\delta = 0.01$; $M = 0.5/12$ year; $\mu_1 = 1/12$ year; $\mu_2 = 2/12$ year; $m = 0.5$; $n = 0.9$; $\alpha = 15$; $b = 5$; $A_c = \$150$; $R = 0.1$; $I_c = 0.15$; $I_c = 0.10$.

Then, we get the optimal values as $t_r^* = 0.2680$ year, $t_0^* = 0.3332$ year, $W_2^* = 123.1045$ units, $Q^* = 927.0594$ units, $TC_1^* = \$3369.1685$. Figure 7.3 shows that the total cost decreases with $t_r$ and $t_0$ and $TC_1$ attains the minimum value $3369.1685$ at $t_r = 0.2680$ and $t_0 = 0.3332$. If $t_r$ and $t_0$ crosses $0.2680$ and $0.3332$ respectively, the total cost then increases. The graph (Figure 7.3) shows that the function $TC_1$ is convex with respect to $t_r$ and $t_0$.

**Example 7.2** (case 2)

Taking all values of the parameters from Example 7.1 except $M$ and put $M = 1.5/12$ year, we get the optimal solutions as $t_r^* = 0.2745$ year, $t_0^* = 0.3396$ year, $W_2^* = 126.1023$ units, $Q^* = 927.1410$ units.
$TC_2^* = \$3266.1331$. From Figure 7.4, it is observed that the total cost $TC_2$ decreases with respect to $t_r$ and $t_o$ and attains the minimum value 3266.1331 at 0.2745 and 0.3396 respectively. After that the total cost will be increased. The graph (Figure 7.4) shows that the function $TC_2$ is convex with respect to $t_r$ and $t_o$.

**Example 7.3 (case 3)**

Consider an inventory system with all data from Example 7.1 except $M$ and put $M = 2.5/12$ year, we get the optimal values as $t_r^* = 0.2759$ year, $t_o^* = 0.3411$ year, $W_2^* = 126.7804$ units, $Q^* = 927.1599$ units, $TC_3^* = \$3160.1099$. The graph (Figure 7.5) shows that the convexity of the total cost function $TC_3$ with respect to $t_r$ and $t_o$.

**Example 7.4 (case 4)**

For this case, consider all data from Example 7.1 except $M$ and put $M = 5/12$ year, we get the optimal solutions as $t_r^* = 0.4049$ year, $t_o^* = 0.4701$ year, $W_2^* = 187.8739$ units, $Q^* = 929.5993$ units, $TC_4^* = \$2918.8401$. Figure 7.6 shows that the total cost decreases with $t_r$ and $t_o$ and $TC_4$ attains the minimum value 2918.8401 at $t_r = 0.4049$ and $t_o = 0.4701$. If $t_r$ and $t_o$ crosses 0.4049 and 0.4701 respectively, the total cost then increases. The graph (Figure 7.6) shows that the function $TC_4$ is convex with respect to $t_r$ and $t_o$. 

168
Example 7.5 (case 5)

For this case, taking all values from Example 7.1 except $M$, and put $M = 7.5/12$ year, we get the optimal values as $t_r^* = 0.4360$ year, $t_0^* = 0.5012$ year, $W_2^* = 202.8587$ units, $Q^* = 930.4144$ units, $TC_5^* = \$2752.2599$. From Figure 7.7, it is observed that the total cost $TC_5$ decreases with respect to $t_r$ and $t_0$ and attains the minimum value 2752.2599 at 0.4360 and 0.5012 respectively. After that the total cost will be increased. The graph (Figure 7.7) shows that the function $TC_5$ is convex with respect to $t_r$ and $t_0$.

![Figure 7.3: Case 1: The total cost with respect to $t_r$ and $t_0$.](image1)

![Figure 7.4: Case 2: The total cost with respect to $t_r$ and $t_0$.](image2)
From the above five cases, the optimal minimum total cost of the retailer occurs in case 5 ($TC^* = $2752.2599). That is, if the retailer gets the permissible delay after $t_0$, then the total cost of the retailer will be minimized. Moreover, if $\mu_1 = 0$ and $\mu_2 = 0$ this model becomes the instantaneous deteriorating item case, and the optimal minimum total cost
can be found as $TC^* = $3201.0125. It can be seen that there is a decrease in total cost from the non-instantaneous deteriorating item model. This implies that if the retailer can convert the instantaneously deteriorating items to non-instantaneous deteriorating items by improving stock control, then the total cost per unit time will decrease. Also, when the supplier does not provide a trade credit period, the optimal retailer total cost can be found as $TC^* = $3062.2366. It can be seen that optimal total cost increases. So, retailers should try to get credit periods for their payments if they wish to decrease their total cost.

7.7 Sensitivity analysis

We now study the effects of changes in the values of the system parameters $s$, $p$, $A$, $C_{hr}$, $C_{ho}$, $C_2$, $C_3$, $C_4$, $\beta$, $\delta$, $m$, $a$, $A_e$, $I_e$, $I_c$ and $R$ on the optimal replenishment policy of the Example 7.1. We change one parameter at a time keeping the other parameters unchanged. The results are summarized in Table 7.1.

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<th>$t_o$</th>
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<th>$Q$</th>
<th>$TC$</th>
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171
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Based on the numerical results, we obtain the following managerial phenomena:

1. When the selling price $s$ is increasing, the total optimal cost ($TC$), order quantity ($Q$), maximum inventory level in RW ($W_2$), the time at which the inventory level become zero in OW ($t_o$) and the time at which the inventory level become zero in RW ($t_r$) are decreasing. That is, the increasing of selling price will decrease the demand as well as decrease the order quantity. So, the total cost of the retailer will decrease.

2. When the purchasing price $p$ is increasing, the total optimal cost ($TC$) is highly increasing. But the order quantity ($Q$), maximum inventory level
in RW ($W_2$), the time at which the inventory level become zero in OW ($t_o$) and the time at which the inventory level become zero in RW ($t_r$) are decreasing. That is, the increasing of purchasing price will increase the total cost of the retailer.

3. When the setup cost $A$ is increasing, the total cost is also increasing. That is, minimum setup cost will minimize the total cost of the retailer.

4. When the holding cost $C_{hr}$ and $C_{ho}$ for RW and OW respectively are increasing, the total optimal cost ($TC$) highly increasing. But the order quantity ($Q$), maximum inventory level in RW ($W_2$), the time at which the inventory level become zero in OW ($t_o$) and the time at which the inventory level become zero in RW ($t_r$) are decreasing. That is, the minimum cost for holding the items will minimize the total cost of the retailer.

5. If the deterioration cost ($C_2$) and the shortage cost ($C_3$) increase, then the total cost of the retailer also increases. But the increase in the lost sales cost ($C_4$) will decrease the total cost of the retailer.

6. When the deterioration rate $\beta$ is increasing, then the total cost of the retailer and the order quantity are also increasing. That is, if the retailer minimizes the deterioration rate of the item, then the total cost will be reduced.

7. If the backlogging parameter $\delta$ increases, then the ordering quantity will decrease. Therefore the total cost of the retailer also decreases. That is, in order to minimize the cost, the retailers should increase the backlogging parameter.

8. When the parameter $m$ is increasing, the total optimal cost ($TC$), order quantity ($Q$), maximum inventory level in RW ($W_2$), the time at which
the inventory level become zero in OW \((t_o)\) and the time at which the inventory level become zero in RW \((t_r)\) are decreasing. But the increase in the parameter \(a\) will increase all the above.

9. When the net discount rate of inflation \(R\) is increasing, the optimal cost is decreasing and the order quantity is also decreasing. That is, for higher values of the net discount rate of inflation \((R)\), the total cost of the retailer will be minimum.

10. When the advertisement cost \(A_c\) is increasing, the total cost and the order quantity are highly increasing. That is, the minimum advertisement cost will minimize the total cost of the retailer but more advertisement cost implies more demand as well as more ordering quantity.

11. If the rate of interest earned \((I_c)\) by the retailer increases, then the total optimal cost will be decreased. But, when the interest charged \((I_c)\) by the supplier increases, the total cost of the retailer also increases.

### 7.8 Conclusion

For the capacity of any warehouse is limited, it has to rent warehouse for storing the excess units over the fixed capacity \(W_i\) of the own warehouse in practice. The RW is assumed to offer better preserving facilities than the OW resulting in a lower rate of deterioration and is assumed to charge higher holding cost than the OW. In the business transactions, the supplier usually offers a permissible delay in payment to his retailer to attract more sales. Based on the above phenomena, in this chapter, two-warehouse inventory model for non-instantaneous deteriorating items with price and advertisement dependent demand is developed where delay in payment is allowed. In the present situation,
inflation and time value of money are also main factors. In keeping with this reality, these factors are incorporated in the present model. For the case of perishable product, the retailer may need to backlog demand to avoid costs due to deterioration. Therefore, shortage is allowed and can be partially backlogged, where the backlogging rate is dependent on the time of waiting for the next replenishment.

The results show that there is decrease in total cost from the non-instantaneously deteriorating items compared with instantaneously deteriorating items. Also, when a delay in payments is allowed, the total cost for the retailer also decreases. Moreover, when the net discount rate of inflation and the backlogging rate increase, the optimal total cost will decrease. Finally, sensitivity analysis is carried out with respect to the key parameters and useful managerial insights are obtained.