CHAPTER 5

AN INVENTORY MODEL WITH VARIABLE PRODUCTION COST, TIME DEPENDENT HOLDING COST, PARTIAL BACKLOGGING AND INFLATION

5.1 Introduction

Generally, high selling price makes a negative impact on a major part of the customers to buy the products. That is, the market demand is inversely related to selling price. Thus, change in price affects the demand, which in turn affects the decisions on production and inventory policies. The advertising by the sales team is one of the most important factors to increase the retailer’s profit in modern marketing system. The purpose of the advertisement is to enhance potential customer’s responses to a business organization. In general, this strategy is only to sell more items in a short time. More demand implies more selling of the items. The advertising intensity increases not only the probability of successful marketing targets but also the demand of the customers. Therefore, the more investment in advertising gives more profits for the company. In this direction, this chapter encourages the retailers to consider the demand as increasing function of advertising parameter with decreasing value of selling price.

In inventory management, holding cost is defined as the money spent to keep and maintain a stock of goods in storage. The most obvious holding costs include rent for the required space; equipment, materials, and labor to
operate the space; insurance; security; interest on money invested in the inventory and space, and other direct expenses. This holding cost is not always constant for the entire inventory cycle. This is particularly true in the storage of deteriorating and perishable items such as foodstuffs, milk, fruit, vegetables and meat, whose quality drops with each passing day and, as a result, increasing holding costs are necessary to maintain the freshness of the items and to prevent spoilage. That is, if these products are kept longer and longer they need more sophisticated storage facilities and maintenance. Therefore, the holding cost would be higher. Moreover, due to inflation, bank interest, hiring charge, etc., increase with time. Thus some factors contributing to the holding cost changes with time and other remain constant. Hence the holding cost per unit time changes with time and can be assumed to be time dependent.

In general, almost all the researchers assume that the production rate and the unit production cost are assumed to be constant. In reality, it has been observed that the unit production cost vary with changes in the production rate. In this chapter, the production cost is assumed to be a function of the production rate.

The problem of inventory systems under inflationary conditions has received attention in recent years. Due to high inflation and consequent sharp decline in the purchasing power of money, especially in the developing countries, the financial situation has been changed and so it is impossible to ignore the effect of inflation and time value of money.
For the case of perishable product, the retailer/manufacturer may need to backlog demand to avoid costs due to deterioration. When the shortage occurs, some customers are willing to wait for back order and others would turn to buy from other sellers. This situation is modelled by the consideration of partial backordering in the formulation of the mathematical models.

In the present chapter, a deterministic EPQ model for perishable items with advertisement dependent demand, variable production rate, unit production cost is a function of some factors such as raw materials, labour charges, advertisement cost, produced units, etc., and selling price is a mark-up over the unit production cost, are proposed. Also, inflation and time value of money, the holding cost is expressed as linearly increasing functions of time, are considered. And shortages are allowed and partially backlogged in this model. To the authors' best of knowledge, this type of model has not yet been considered by any of the researchers in inventory literature.

This chapter is organized as follows: The problem description follows this section. The various notations and assumptions used throughout the chapter are given in section 5.3. The mathematical modeling of the problem and the total cost calculations are given in section 5.4. In section 5.5, the solution procedure to find the optimal values are given. In section 5.6, the model formulation is given for non perishable product. This section is followed by numerical examples. In section 5.8, sensitivity analysis is carried out and the results are discussed in detail. Section 5.9 concludes the chapter.
5.2 Problem description

This chapter considers an EPQ model with price and advertisement dependent demand under the effect of inflation and time value of money. Here, the rate of replenishment is considered to be a variable and the generalized unit production cost function is formulated by incorporating the several factors such as raw material, labour cost, replenishment rate, advertisement cost and so on of the manufacturing system. The selling price of a unit is determined by a mark-up over the production cost. In this model, the holding cost per unit of the item per unit time is assumed to be an increasing linear function of time spent in storage. Also, shortages are allowed and we consider that shortage occurs before the starting of inventory. This type of inventory is called SFI (shortage followed by inventory) policy. In this model, the customers are viewed to be impatient and a fraction of the demand is exponentially backlogged. This fraction is a function of the waiting time of the customers. Here, the purpose is to minimize the total inventory cost of the manufacturer by finding the optimal cycle length and the optimal production quantity. The model is extended to the case of non-perishable product also. The optimal solution of the model is illustrated with the help of a numerical example.

5.3 Notations and assumptions

5.3.1 Notations

The following notations are additionally considered to develop the mathematical model of the chapter.

\[ q(t) \]  The inventory level at any time \( t \).
\( \beta, \psi \)  The interim time spans.

\( T \)  The duration of an inventory cycle when stockout condition exists.

\( \lambda \)  The duration of an inventory cycle when there is positive inventory.

\( TC \)  The total cost of the system with perishable product.

\( TC_N \)  The total cost of the system with non-perishable product.

### 5.3.2 Assumptions
To develop the mathematical model, the following assumptions are being made.

1. A single item is considered over an infinite planning horizon.

2. The demand rate \( D \) is a deterministic function of selling price, \( S \) and advertisement cost, \( A_c \) per unit item. i.e.) \( D(A_c, S) = A_c^\gamma (x - yS), \)
   \( x, y, \gamma \geq 0. \)

3. The production rate \( P \) is variable, which is more than the demand rate.

4. The unit production cost \( v(P) = C_{rw} + A_c + L/P^c + KP^d \) where \( C_{rw}, L \) and \( K \) are non-negative real numbers to be provide the best fit for the estimated unit cost function. \( C_{rw}, L \) and \( K \) represent the raw material cost, labour charges and a positive constant. Also \( c, d \) are chosen to provide the feasible solution to the model. We shall use \( v \) and \( v(P) \) interchangeable in the rest of the chapter.
5. The selling price is determined by a mark\-up over the unit production cost, i.e. \( s = n v, n \geq 1 \), where \( n \) is a mark\-up.

6. The items deteriorate continuously at a rate \( \sigma \).

7. There is no replacement or repair of deteriorated items takes place in a given cycle.

8. The inflation rate is constant.

9. The lead time is zero.

10. \( C_1(t) = a + bt, a \geq 0, b \geq 0 \) is the holding cost, which is linear function of time.

11. Shortages are allowed and during stock out period, only a fraction \( B(\eta) \) of the demand is backlogged where \( \eta \) is the amount of time for which the customer waits before receiving goods and remaining fraction \( 1 - B(\eta) \) is lost. \( B(\eta) \) is given by \( B(\eta) = k_0 e^{-k_1 \eta} \), \( k_0 < 1, k_1 \geq 0 \).

5.4 Formulation of the model

The inventory system is developed as follows: The inventory starts with zero stock at zero time. So, shortage begins to build up at the early stage in inventory cycle. The production starts at time \( \beta \) to meet the current and backlogged demand. \( T \) is the time when the shortage level reaches to zero; afterwards a positive level of inventory begins to accumulate. At time \( T + \psi \), the production process stops and the inventory level then starts declining. Finally, the cycle ends with zero stock at time \( T + \lambda \). The behaviour of the inventory model is demonstrated in Figure 5.1.
Based on the above description, during the time interval \([0, \beta]\), the differential equation representing the inventory status is given by

\[
\frac{dq(t)}{dt} = -DB(\beta - t), \quad 0 \leq t \leq \beta 
\]  

(75)

With the condition \(q(0) = 0\), the solution of Equation (75) is

\[
q(t) = \frac{Dk_0}{k_1} e^{-k_1 \beta} \left[ 1 - e^{k_1 t} \right] \quad 0 \leq t \leq \beta
\]  

(76)

In the second interval \([\beta, T]\), the differential equation below represents the inventory status:

\[
\frac{dq(t)}{dt} = P - D, \quad \beta \leq t \leq T
\]  

(77)

With the condition \(q(T) = 0\), we get the solution of Equation (77) is

\[
q(t) = (P - D)(t - T) \quad \beta \leq t \leq T
\]  

(78)

Put \(t = \beta\) in Equations (76) and (78) we find the value of \(T\) as
\[ T = \beta - \frac{Dk_0}{k_1(P-D)}[e^{-k_1\beta} - 1] \] (79)

During the third interval \([T, T + \psi]\), the inventory level at time \(t\) is governed by the following differential equation:

\[ \frac{dq(t)}{dt} + \sigma q(t) = P - D, \quad T \leq t \leq T + \psi \] (80)

With the condition \(q(T) = 0\), the solution of Equation (80) is

\[ q(t) = \frac{P - D}{\sigma} \left[1 - e^{\sigma(T-t)}\right] \quad T \leq t \leq T + \psi \] (81)

During the time interval \([T + \psi, T + \lambda]\), the inventory status is given by

\[ \frac{dq(t)}{dt} + \sigma q(t) = -D, \quad T + \psi \leq t \leq T + \lambda \] (82)

With the condition \(q(T + \lambda) = 0\), the solution of Equation (82) is

\[ q(t) = \frac{D}{\sigma} \left[e^{\sigma(T+\lambda-t)} - 1\right] \quad T + \psi \leq t \leq T + \lambda \] (83)

Equating the expressions (81) and (83) at \(t = T + \psi\), we have the value of \(\psi\) as

\[ \psi = \psi(\lambda) = \frac{1}{\sigma} \log \left[1 + \frac{D}{P}[e^{\sigma\lambda} - 1]\right]. \] (84)

We shall use \(\psi\) and \(\psi(\lambda)\) interchangeable in the rest of the chapter.

Since the production occurs in the continuous time-span \([\beta, T]\) and \([T, T + \psi]\). Thus the order size in the problem during total time interval \([\beta, T]\) and \([T, T + \psi]\) is,

\[ Q = P(T - \beta) + P\psi. \] (85)
The maximum inventory level $S$ is given by

$$S = q(T + \psi) = \frac{\mu - D}{\sigma} \left[1 - e^{-\sigma \psi}\right]$$  \hspace{1cm} (86)

Now we want to find the different inventory costs with the effect of inflation as:

Setup cost $= A$.  \hspace{1cm} (87)

The production cost $PC = \int_{\beta}^{T} vP e^{-R \tau} d\tau + \int_{T}^{T+\psi} vP e^{-R \tau} d\tau$

$$PC = \frac{vP}{R} e^{-RT} \left[ e^{R(T-\beta)} - e^{-R\psi} \right]$$  \hspace{1cm} (88)

The deteriorating cost $DC$ during the period $[T, T + \psi]$ and $[T + \psi, T + \lambda]$ is

$$DC = \int_{T}^{T+\psi} vP e^{-R \tau} d\tau - \int_{T}^{T+\lambda} vP e^{-R \tau} d\tau$$

$$DC = \frac{v}{R} e^{-RT} \left[ D(e^{-R\lambda} - 1) - P(e^{-R\psi} - 1) \right]$$  \hspace{1cm} (89)

The holding cost $HC$ is given by

$$HC = \int_{T}^{T+\psi} C_1(t) e^{-R \tau} q(t) dt + \int_{T+\psi}^{T+\lambda} C_1(t) e^{-R \tau} q(t) dt$$

$$HC = \frac{P}{R} e^{-RT} \left\{ \frac{1 - e^{-R\psi}}{R} \left[ a + bT + \frac{b}{R} \right] + \frac{e^{-\psi(R+\sigma)} - 1}{R + \sigma} \left[ a + bT + \frac{b}{R + \sigma} \right] \right\}$$

$$+ b \psi e^{-R\psi} \left( \frac{e^{-\sigma \psi}}{R + \sigma} - \frac{1}{R} \right) + \frac{D}{R} e^{-RT} \left\{ \frac{e^{-R\lambda} - 1}{R} \left[ a + bT + \frac{b}{R} \right] \right\}$$
\[\begin{align*}
+ \left( \frac{1-e^{-R\lambda}}{R+\sigma} \right) \left( a + bT + \frac{b}{R+\sigma} \right) + \left( \frac{e^{\sigma\lambda} - 1}{R+\sigma} \right) e^{-\psi(R+\sigma)} \\
\times \left( a + bT + b\psi + \frac{b}{R+\sigma} \right) + b\lambda e^{-R\lambda} \left( \frac{2R+\sigma}{R(R+\sigma)} \right) \right]
\end{align*}\]  
(90)

Total shortage cost $SC$ during the period $[0, T]$ is given by

\[SC = C_3 \left[ \int_0^\beta - q(t)e^{-Rt} dt + \int_\beta^T - q(t)e^{-Rt} dt \right] = \frac{Dk_0}{k_1} e^{-k_1\beta} \left[ \frac{1}{R} \left( 1 - e^{-R\beta} \right) + \frac{1}{k_1 - R} \left( 1 - e^{(k_1-R)\beta} \right) \right] \]

\[+ (P - D) \left[ \frac{e^{-R\beta}}{R} (\beta - T) + \frac{1}{R} \left( e^{-R\beta} - e^{-RT} \right) \right] \]  
(91)

The lost sales cost $LC$ during the period $[0, T]$ is

\[LC = C_4 \int_0^\beta \left( 1 - k_0 e^{-k_1(\beta-t)} \right) D e^{-RT} dt = C_4 D \left[ \frac{k_0}{k_1 - R} \left( e^{-k_1\beta} - e^{-R\beta} \right) + \frac{1}{R} \left( 1 - e^{-R\beta} \right) \right] \]  
(92)

Total average cost per cycle = setup cost + production cost + deterioration cost + inventory holding cost + shortage cost + lost sales cost.

So, the total variable cost per unit time is

\[TC = \frac{1}{T + \lambda} \left[ A + PC + DC + HC + SC + LC \right] = \left\{ A + \frac{vP}{R} e^{-RT} \left[ e^{R(T-\beta)} - e^{-R\psi} \right] + \frac{v}{R} e^{-RT} \left[ D(e^{-R\lambda} - 1) - P(e^{-R\psi} - 1) \right] \right\} \]
\[ + \frac{P}{\sigma} e^{-RT} \left\{ \left( \frac{1-e^{-R\psi}}{R} \right)(a+bT + \frac{b}{R}) + \left( \frac{e^{-\psi(R+\sigma)} - 1}{R + \sigma} \right)(a+bT + \frac{b}{R}) \right\} \\
+ b\psi e^{-R\psi} \left( e^{-\sigma\psi} - \frac{1}{R} \right) + \frac{D}{\sigma} e^{-2RT} \left( \frac{e^{-R\lambda} - 1}{R} \right)(a+bT + \frac{b}{R}) \\\
+ \left( \frac{1-e^{-R\lambda}}{R + \sigma} \right)(a+bT + \frac{b}{R+\sigma}) + \left( \frac{e^{\sigma\lambda} - 1}{R + \sigma} \right)e^{-\psi(R+\sigma)} \times \left( a+bT + b\psi + \frac{b}{R+\sigma} \right) + b\lambda e^{-R\lambda} \left( \frac{2R+\sigma}{R(R+\sigma)} \right) \right\} \\
- C_3 \left[ \frac{Dk_0}{k_1} e^{-k_1\beta} \left( \frac{1}{R} (1-e^{-R\beta}) + \frac{1}{k_1-R} (1-e^{(k_1-R)\beta}) \right) \right] \\
+ (P-D) \left[ \frac{e^{-R\beta}}{R} (\beta-T) + \frac{1}{R^2} (e^{-R\beta} - e^{-RT}) \right] \right\} \right\} / (T+\lambda) \tag{93} \]

### 5.5 Solution procedure

Now the production rate that minimizes the unit production cost is given by 
\[ \nu'(P) = 0. \]

Therefore \( \nu'(P) = 0 \) implies 
\[ - \frac{cL}{P^{c+1}} + K d^{d-1} = 0 \]

\[ \Rightarrow P = \left( \frac{L c}{K d} \right)^{\frac{1}{c+d}} \tag{94} \]
The necessary and sufficient conditions to minimize $TC(\lambda)$ are respectively \( \frac{dTC(\lambda)}{d\lambda} = 0 \) and \( \frac{d^2TC(\lambda)}{d\lambda^2} > 0 \).

Now \( \frac{dTC(\lambda)}{d\lambda} = 0 \) gives the following equation in $\lambda$.

\[
\left\{ vP + vP \left[ P + \left[ e^{-(R+\sigma)\lambda} (D - P) - De^{-\lambda} \right] \frac{P + De^\lambda - D}{P} \right] \right\}^{(R/\sigma)}
+ \frac{PD}{\sigma} \left( \frac{e^{\sigma\lambda} - 1}{P + De^\sigma\lambda - D} \right) \left[ a + b \left( T + \frac{1}{\sigma} \log \left( \frac{P + De^\sigma\lambda - D}{D} \right) \right) \right]
+ \frac{Pe^{-(R+\sigma)\lambda}}{\sigma} \left[ \frac{P + De^\sigma\lambda - D}{P} \right] \frac{R}{P(R + \sigma)} \left\{ - (a + bT + b/R) + \frac{(a + bT + b/(R + \sigma))}{R + \sigma} \right\}
- \frac{b(2R + \sigma)(R\lambda - 1)}{R(R + \sigma)} \left[ \frac{P + De^\sigma\lambda - D}{P(R + \sigma)^2} \right] \left[ R \log \left( \frac{P + De^\sigma\lambda - D}{P} \right) \right]
\times \left[ b \left( \frac{P\sigma}{R} + D \right) \right] - bDR \left( e^{\sigma\lambda} + 1 \right)
\]

\[- RD \left( e^{\sigma\lambda} - 1 \right) \left\{ a \left( 1 + \frac{R}{\sigma} \right) + bT \left( 1 + \frac{R}{\sigma} - \frac{b}{R} \right) + P(a + bT)(1 - R) + bP \right\} \right\} / (T + \lambda)
\]

\[- \left\{ A + \frac{vP}{R} e^{-RT} \left[ e^{(T - \beta)} - e^{-R\psi} \right] + \frac{v}{R} e^{-RT} \left[ De^{-R\lambda} - 1 - P(e^{-R\psi} - 1) \right] \right\}
\]

\[+ \frac{P}{\sigma} e^{-RT} \left( \left( \frac{1 - e^{-R\psi}}{R} \right) \left( a + bT + b/R \right) + \left( e^{-R\psi(R + \sigma)} - 1 \right) \left( a + bT + b/R + \sigma \right) \right)
+ b\psi e^{-R\psi} \left( \frac{e^{-\sigma\psi}}{R + \sigma} - \frac{1}{R} \right) + \frac{D}{\sigma} e^{-RT} \left( \left\{ \frac{e^{-R\lambda} - 1}{R} \right\} \left( a + bT + b/R \right) \right) \]

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\[ + \left( \frac{1-e^{-R\lambda}}{R+\sigma} \right) \left( a+bT + \frac{b}{R+\sigma} \right) + \left( \frac{e^{\sigma\lambda}-1}{R+\sigma} \right) e^{-\psi(R+\sigma)} \left( a+bT + b\psi + \frac{b}{R+\sigma} \right) \]

\[ + b\lambda e^{-R\lambda} \left( \frac{2R+\sigma}{R(R+\sigma)} \right) - C_3 \left\{ \frac{Dk_0}{k_1} e^{-k_1\beta} \left[ \frac{1}{R} \left( 1-e^{-R\beta} \right) + \frac{1}{k_1-R} \left( 1-e^{(k_1-R)\beta} \right) \right] \right\} \]

\[ + (P-D) \left[ \frac{e^{-R\beta}}{R} (\beta-T) + \frac{1}{R^2} \left( e^{-R\beta} - e^{-RT} \right) \right] \]

\[ + C_4 D \left[ \frac{k_0}{k_1-R} \left( e^{-k_1\beta} - e^{-R\beta} \right) + \frac{1}{R} \left( 1-e^{-R\beta} \right) \right] \] / \((T+\lambda)^2 = 0 \quad (95)\]

By examining the second order sufficient condition, it can be verified that the total cost \(TC\) is a convex function of \(\lambda\), because the second order differential equation of \(TC\) with respect to \(\lambda\) is

\[ \frac{d^2 TC(\lambda)}{d\lambda^2} = \left\{ vRD e^{-R\lambda} \left[ \frac{P + De^{\sigma\lambda}-D}{P} \right]^{(R/\sigma)} \right\} \times \left\{ \left( \frac{\sigma e^{\sigma\lambda}}{P + De^{\sigma\lambda} - D} \right) e^{-\sigma\lambda} (D-P) - D \right\} + 1 \left( 1 - \frac{P+De^{\sigma\lambda}-D}{P} \right) + \left[ \frac{PD e^{\sigma\lambda}}{(P+De^{\sigma\lambda} - D)^2} \right] \]

\[ \times \left\{ bD(e^{\sigma\lambda} - 1) + P \left( a + b \left( T + \frac{1}{\sigma} \log \left( \frac{P + De^{\sigma\lambda} - D}{P} \right) \right) \right) \right\} \]

\[ + \left( \frac{(a+bT+b/R+\sigma)}{R+\sigma} \right) - \left( a+bT+b/R+\sigma \right) \left( \frac{b(2R+\sigma)(R\lambda-1)}{R(R+\sigma)} \right) \]

\[ \times \left[ \frac{P + De^{\sigma\lambda} - D}{P} \right]^{(R/\sigma)} \left[ \frac{P \left( \frac{1}{\sigma} - 1 \right) + D(1-e^{\sigma\lambda})}{\sigma} \right] \]

\[ + \left( \frac{D\sigma e^{\sigma\lambda}}{P(R+\sigma)^2} \right) \left[ \frac{R}{\sigma} \log \left( \frac{P + De^{\sigma\lambda} - D}{P} \right) \left[ b \left( \frac{P\sigma}{R} + D \right) - b\sigma (e^{\sigma\lambda} + 1) \right] \right] \]
\[ + RD(1 - e^{-\sigma \lambda}) \left[ a \left( 1 + \frac{R}{\sigma} \right) + bT \left( 1 + \frac{R}{\sigma} \right) - \frac{b}{R} \right] + P(a + bT)(1 - R) + bP \] 

\[ + \left[ \frac{P + De^{-\sigma \lambda} - D}{P(R + \sigma)^2} \right] \left[ \left( \frac{P + D}{R} + D \right) - \frac{bDR}{\sigma} (e^{-\sigma \lambda} + 1) \right] \left[ \frac{RD e^{-\sigma \lambda}}{(P + D e^{-\sigma \lambda} - D)} \right] \] 

\[ - \log \left( \frac{P + De^{-\sigma \lambda} - D}{P} \right) \left[ \frac{PDR^2 e^{-\sigma \lambda}}{\sigma} \right] - RD \sigma e^{-\sigma \lambda} \] 

\[ \times \left[ a \left( 1 + \frac{R}{\sigma} \right) + bT \left( 1 + \frac{R}{\sigma} \right) - \frac{b}{R} \right] \left[ \frac{P + De^{-\sigma \lambda} - D}{P} \right]^{(R + \sigma) + 1} \] 

\[ \times \left[ bP(2R + \sigma) e^{-\sigma (R + \sigma) \lambda} \right]/(T + \lambda) \] 

\[ - 2 \left\{ vP + v \left( P + \left[ e^{-\sigma (R + \sigma) \lambda} (D - P) - De^{-R \lambda} \right] \left[ \frac{P + De^{-\sigma \lambda} - D}{P} \right] \right) \right\} \] 

\[ + \frac{PD}{\sigma} \left( \frac{e^{\sigma \lambda} - 1}{P + De^{-\sigma \lambda} - D} \right) \left\{ a + b \left( T + \frac{1}{\sigma} \log \left[ \frac{P + De^{-\sigma \lambda} - D}{P} \right] \right) \right\} + \frac{Pe^{-(R + \sigma) \lambda}}{\sigma} \] 

\[ \times \left[ \frac{P + De^{-\sigma \lambda} - D}{P} \right]^{R + 1} \left\{ \frac{(a + bT + b/(R + \sigma))}{R + \sigma} - (a + bT + b/R) \right\} \] 

\[ - \frac{b(2R + \sigma)(R\lambda - 1)}{R(R + \sigma)} + \left[ \frac{P + De^{-\sigma \lambda} - D}{P(R + \sigma)^2} \right] \frac{R}{\sigma} \log \left( \frac{P + De^{-\sigma \lambda} - D}{P} \right) \] 

\[ \times \left[ b \left( \frac{P\sigma}{R} + D \right) - \frac{bDR}{\sigma} (e^{-\sigma \lambda} + 1) \right] \] 

\[ + RD(1 - e^{-\sigma \lambda}) \left[ a \left( 1 + \frac{R}{\sigma} \right) + bT \left( 1 + \frac{R}{\sigma} \right) - \frac{b}{R} \right] + P(a + bT)(1 - R) + bP \] 

\[ \right\} / (T + \lambda)^2 \] 

\[ + 2 \left\{ \frac{R}{P} e^{-\sigma T - \beta} e^{-\sigma \psi} + \psi e^{-\sigma T} D(e^{-\sigma \lambda} - 1) - P(e^{-\sigma \psi} - 1) \right\} \] 

\[ + \frac{P}{\sigma} e^{-\sigma T} \left\{ \left( \frac{1 - e^{-\sigma \psi}}{R} \right) (a + bT + \frac{b}{R}) + \left( \frac{e^{-\psi (R + \sigma)} - 1}{R + \sigma} \right) (a + bT + \frac{b}{R + \sigma}) \right\} \] 

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\[ + b \psi e^{-R\psi} \left( \frac{e^{-\sigma\psi}}{R + \sigma} - \frac{1}{R} \right) + \frac{D}{\sigma} e^{-RT} \left\{ \left( \frac{e^{-R\lambda}}{R} - 1 \right) \left( a + bT + \frac{b}{R} \right) \right. \\
\left. + \left( \frac{1 - e^{-R\lambda}}{R + \sigma} \right) \left( a + bT + \frac{b}{R + \sigma} \right) \right\} \] \\
\left. + e^{-R(\lambda - \psi)} \left( \frac{2R + \sigma}{R(R + \sigma)} \right) \right\} - C_3 \left\{ \frac{Dk_0}{k_1} e^{-k_1\beta} \left[ \frac{1}{R} \left( 1 - e^{-R\beta} \right) + \frac{1}{k_1 - R} \left( 1 - e^{(k_1 - R)\beta} \right) \right] \right\} \\
+ (P - D) \left\{ \frac{e^{-R\beta}}{R} (\beta - T) + \frac{1}{R^2} (e^{-R\beta} - e^{-RT}) \right\} \\
+ C_4 D \left\{ \frac{k_0}{k_1 - R} \left( e^{-k_1\beta} - e^{-R\beta} \right) + \frac{1}{R} \left( 1 - e^{-R\beta} \right) \right\} / (T + \lambda)^3 > 0 \]

since \( P > D, k_0 < 1, a, b, k_1 \geq 0 \) and \( 0 < \sigma < 1 \).

By solving Equation (95) the optimal value of \( \lambda = \lambda^* \) can be obtained and then from Equations (79), (84), (85), (86), (94) and (93), the optimal value of \( T = T^*, \psi = \psi^*, Q = Q^*, S = S^*, P = P^* \) and \( TC = TC^* \) can be found out respectively.

### 5.6 Non-perishable product

Suppose the produced items are non-perishable, then we have \( \sigma = 0 \). Therefore taking limit as \( \sigma \to 0 \) or by substituting \( \sigma = 0 \) in the Equations (80) and (82), we get

\[ q(t) = (P - D)(t - T), \quad T \leq t \leq T + \psi \quad (96) \]

\[ q(t) = -D(t - T - \lambda), \quad T + \psi \leq t \leq T + \lambda \quad (97) \]

\[ \psi = \psi(\lambda) = \frac{D\lambda}{P} \quad (98) \]
Therefore, the holding cost $HC_N$ with the effect of inflation is given by

$$
HC_N = \int_T^{T+\psi} (a + bt)e^{-Rt} (P - D)(t - T)dt + \int_T^{T+\lambda} -(a + bt)e^{-Rt}D(t - T - \lambda)dt
$$

$$
= P \left\{ \frac{-e^{-RT}}{R^2} \left[ b \left( T + 2\psi + \frac{2}{R} + R\psi(\psi + T) \right) + a(1 + R\psi) \right] \\
+ \frac{e^{-RT}}{R^2} \left[ a + b \left( T + \frac{2}{R} \right) \right] - D \left\{ \frac{e^{-RT}}{R^2} \left[ a + b \left( T + \frac{2}{R} \right) \right] \\
- \frac{e^{-R(T+\lambda)}}{R^2} \left[ a + b \left( T + \lambda + \frac{2}{R} \right) \right] - \frac{\lambda e^{-R(T+\psi)}}{R} \left[ a + b \left( T + \psi + \frac{1}{R} \right) \right] \right\}
\right\} (99)
$$

So, the total variable cost for non-perishable product per unit time is

$$
TC_N = \frac{1}{T + \lambda} \left[ A + PC + HC_N + SC + LC \right] = \left\{ A + \frac{\psi P}{R} e^{-RT} \left[ e^{R(T-\beta)} - e^{-R\psi} \right] \right\}
$$

$$
= P \left\{ \frac{-e^{-RT}}{R^2} \left[ b \left( T + 2\psi + \frac{2}{R} + R\psi(\psi + T) \right) + a(1 + R\psi) \right] \\
+ \frac{e^{-RT}}{R^2} \left[ a + b \left( T + \frac{2}{R} \right) \right] - D \left\{ \frac{e^{-RT}}{R^2} \left[ a + b \left( T + \frac{2}{R} \right) \right] \\
- \frac{e^{-R(T+\lambda)}}{R^2} \left[ a + b \left( T + \lambda + \frac{2}{R} \right) \right] - \frac{\lambda e^{-R(T+\psi)}}{R} \left[ a + b \left( T + \psi + \frac{1}{R} \right) \right] \right\} \\
- C_3 \left\{ \frac{Dk_0}{k_1} e^{-k_1\beta} \left[ \frac{1}{R} \left( 1 - e^{-R\beta} \right) + \frac{1}{k_1 - R} \left( 1 - e^{(k_1 - R)\beta} \right) \right] \\
+ (P - D) \left[ \frac{e^{-R\beta}}{R} \left( \beta - T \right) + \frac{1}{R^2} \left( e^{-R\beta} - e^{-RT} \right) \right] \right\} \\
+ C_4 D \left[ \frac{k_0}{k_1 - R} \left( e^{-k_1\beta} - e^{-R\beta} \right) + \frac{1}{R} \left( 1 - e^{-R\beta} \right) \right] \right\} / (T + \lambda) (100)
$$
5.7 Numerical example

Consider an inventory system with the following data: \( b = 10; \) 
\( A_c = 50; \ y = 0.01; \ x = 200; \ y = 0.6; \ L = 1500; \ c = 0.76; \ d = 1.5; \ C_{rw} = 45; \)
\( K = 0.01; \ n = 1.18; \ A = 100; \ a = 3; \ b = 0.8; \ \sigma = 0.2; \ k_0 = 0.9; \ k_1 = 0.2; \)
\( R = 0.1; \ C_3 = 4; \ C_4 = 10 \) in appropriate units.

Then we get the optimal values as \( T^* = 18.7678, \ \psi^* = 1.4946, \)
\( \lambda^* = 2.0374, \ P^* = 144.4282, \ Q^* = 1482.1822, \ S^* = 57.3526 \) and
\( TC^* = 2887.2717 \) in appropriate units.

For non-perishable product, the optimal values of \( Q^* \) and \( TC^* \) are
\( Q^* = 337.9287 \) and \( TC^* = 3147.3193 \) in appropriate units.

The result shows that the total cost of the system with perishable product is less than the total cost of the system without perishable product.

Figure 5.2 shows that the total cost \( TC \) decreases with \( \lambda \) and it attains the minimum value 2887.2717 at \( \lambda = 2.0374 \). If \( \lambda \) crosses 2.0374, the total cost then increases. The graph (Figure 5.2) shows that the function \( TC \) is convex with respect \( \lambda \).

From Figure 5.3, it is observed that the total cost \( TC \) decreases with respect to \( \lambda \) and \( \psi \) and \( TC \) attains the minimum value 2887.2717 at \( \lambda = 2.0374 \) and \( \psi = 1.4946 \) respectively. After that the total cost will be increased. The graph (Figure 5.3) shows that the convexity of the total cost function with respect to \( \lambda \) and \( \psi \).
5.8 Sensitivity analysis

We now study the effects of changes in the values of the system parameters $\beta$, $A$, $\sigma$, $a$, $b$, $k_0$, $k_1$, $C_3$, $C_4$, $L$, $C_{tw}$ and $n$ on the optimal replenishment policy of the above numerical example. We change one parameter at a time keeping the other parameters unchanged. The results are summarized in Table 5.1.

Table 5.1: Sensitivity analysis for various parameters.

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Based on the numerical results, we obtain the following managerial phenomena:

1. When the labour charge $L$ is increasing, the production rate ($P$), unit production cost ($v$), unit selling price ($s$), duration of an inventory cycle when there is positive inventory ($\lambda$), optimal order quantity ($Q$) and the total optimal cost ($TC$) are also increasing. That is, the increasing of labour charge will increase the total cost of the manufacturer. In order to minimize the cost, the manufacturers should decrease the labour charge.

2. When the raw material cost $C_{rw}$ is increasing, there is no change in the production rate ($P$). But the increasing of raw material cost $C_{rw}$ will increase the unit production cost ($v$), unit selling price ($s$), duration of an inventory cycle when there is positive inventory ($\lambda$), optimal order quantity ($Q$) and the total optimal cost ($TC$). That is, if the manufacturer minimize the raw material cost, then the total cost will be reduced.

3. When the mark-up value $n$ is increasing, there are no changes in the production rate ($P$) and the unit production cost ($v$). But the unit selling price ($s$), duration of an inventory cycle when there is positive inventory ($\lambda$), optimal order quantity ($Q$) and the total optimal cost ($TC$) will increase as $n$ increase. That is, the increasing of selling price will increase the order quantity. So, the total cost of the manufacturer will increase.
4. If the interim time span $\beta$ increases, then the duration of an inventory cycle when there is positive inventory ($\lambda$) and optimal order quantity ($Q$) are increases. But the total optimal cost ($TC$) will decrease. That is, to minimize the cost, the manufacturer should increase the shortage period.

5. When the setup cost $A$ is increasing, the order quantity ($Q$) and total optimal cost ($TC$) are also increasing. That is, minimum setup cost will minimize the total cost of the manufacturer.

6. When the parameters $a$ and $b$ are increasing, the total optimal cost ($TC$) is also increasing. That is, the minimum cost of holding the items will minimize the total cost of the manufacturer.

7. When the deterioration rate $\sigma$ is increasing, there are no changes in the production rate ($P$), unit production cost ($v$) and the unit selling price ($s$). But the duration of an inventory cycle when there is positive inventory ($\lambda$), optimal order quantity ($Q$) and the total optimal cost ($TC$) will decrease. That is, the increase in $\sigma$ will decrease the order quantity, therefore the total cost of the manufacturer will be minimized.

8. When the parameters $k_0$ and $k_1$ are increasing, the duration of an inventory cycle when there is positive inventory ($\lambda$), optimal order quantity ($Q$) and the total optimal cost ($TC$) are also increasing. That is, the minimum backlogging rate of the items will minimize the total cost of the manufacturer.

9. If the shortage cost ($C_3$) and the lost sales cost ($C_4$) are increasing, then the optimal order quantity ($Q$) and the total optimal cost ($TC$) of the manufacturer also increasing. But there are no changes in the production rate ($P$), unit production cost ($v$) and the unit selling price ($s$).
5.9 Conclusion

In this chapter, the economic production lot size model for determining the optimal production length and the optimal order quantity for perishable items are developed. Holding cost is expressed as linearly increasing functions of time in this model. This model is very practical for the industries in which the holding cost is depending upon the time. In the present situation, inflation and time value of money are also main factors. In keeping with this reality, these factors are incorporated in the present model. We also developed the model incorporating both marketing decision and variable unit price depending on the rate of production. When a product is perishable, the manufacturer may need to backlog demand in order to avoid high cost due to deterioration. So, shortage is allowed and can be partially backlogged, where the backlogging rate is dependent on the waiting time of the customer. Moreover, we considered that the shortage occurs before the starting of inventory, called SFI policy. The model is extended to the case of non-perishable product also. The result shows that the total cost of the model will minimum when the items are defective. Furthermore, numerical examples are provided to illustrate the model and the solution procedure. Finally, sensitivity analysis is carried out with respect to the key parameters and useful managerial insights are obtained.