CHAPTER 4

AN INVENTORY MODEL WITH FINITE REPLENISHMENT, TRADE CREDIT AND PROBABILISTIC DETERIORATION

4.1 Introduction

In framing the traditional inventory model, it was assumed that the payment must be made to the supplier for the items immediately after receiving the consignment. However, a supplier permits the retailer a period of time (say 30 days), to settle the total amount owed to him/her. Usually, interest is not charged for the outstanding amount if it is paid within the permissible delay period. This credit term in financial management is denoted as “net 30”. But if the payment is not paid within the permissible delay period, then interest is charged on the outstanding amount under the previously agreed terms and conditions. Hence, the retailer earns the interest on the accumulated revenue received. The retailer can delay the payment up to the last moment of the permissible period allowed by the supplier. Thus the retailer is very much benefitted by making possible use of the permissible delay period. This reduces the amount of capital invested in stock for the duration of the permissible period. Allowing credit period is a marketing strategy for the supplier to attract new customers who consider it to be a type of cost (or price) reduction.
In the present chapter, a deterministic inventory model for deteriorating items with finite replenishment and permissible delay in payments is proposed. Also in this model, we have considered that the deterioration function follows probability distributions such as (a) uniform distribution, (b) triangular distribution and (c) beta distribution. Most of the inventory models are developed without considering the factors such as finite replenishment, probabilistic deterioration and permissible delay in payments. So, in this chapter, we want to investigate these issues together and derive a comprehensive model to determine the optimal total cost. To the authors' best of knowledge, this type of model has not yet been considered by any of the researchers in inventory literature.

The rest of the chapter is organized as follows: The problem description follows this section. In section 4.3, the notations and assumptions, which are used throughout this chapter, are described. In Section 4.4, the mathematical model to minimize the total annual inventory cost is established. Section 4.5 presents solution procedure to find the optimal cycle length and optimal order quantity. Numerical examples are provided in Section 4.6 to illustrate the theory and the solution procedure. This is followed by sensitivity analysis and conclusion.

4.2 Problem description

This chapter develops an inventory model for deteriorating items with finite replenishment rate under a progressive payment scheme within the cycle time. In this model, the deterioration function follows probability distributions such as (a) uniform distribution, (b) triangular distribution and
(c) beta distribution. Here the retailer is allowed a trade-credit offered by the supplier to buy more items. During the credit period, the retailer can earn more by selling their products. The interest on purchasing cost is charged for the delay of payment by the retailer. The major objective is to determine the optimal cycle length, the optimal time of replenishment and the optimal order quantity simultaneously such that the total inventory cost of the retailer is minimized. A theorem has been framed to characterize the optimal solutions. The necessary and sufficient conditions of the existence and uniqueness of the optimal solutions are also provided. The optimal solution of the inventory model is illustrated with the help of numerical examples and numerical comparisons among the three models are also given. Finally, sensitivity analysis and graphical representations are given to demonstrate the model.

4.3 Notations and assumptions

4.3.1 Notations

The mathematical model in this chapter is developed on the basis of the following some additional notations.

\[ t_1 \] Duration of the replenishment rate.

\[ T \] The length of the inventory cycle.

\[ q_1(t) \] The inventory level at time \( t, \ 0 \leq t \leq t_1 \).

\[ q_2(t) \] The inventory level at time \( t, \ t_1 \leq t \leq T \).

\[ C_1 \] The holding cost (excluding interest charges) per unit per unit time.

\[ TC \] The total cost of the system.
4.3.2 Assumptions

To develop the mathematical model, the following assumptions are made.

1. A single item is considered over an infinite planning horizon.
2. The replenishment takes place at a finite rate.
3. Deterioration follows continuous probability distribution functions as (a) uniform distribution, (b) triangular distribution, (c) beta distribution.
4. There is no replacement or repair of deteriorated items in a given cycle.
5. The lead time is zero and shortages are not allowed.
6. The permissible delay in payment is offered by the supplier to the retailer.
7. The imperfect (deteriorating) items are considered.

4.4 Formulation of the model

The inventory system is developed as follows: the inventory cycle starts at \( t = 0 \) with zero inventory and increases up to time \( t_1 \) at a rate \( K \), and also simultaneously decreases due to demand and deterioration. In the interval \([t_1, T]\), the inventory level is decreasing only due to demand rate and deterioration. Finally, the inventory reaches the zero level at time \( T \). The figure of the model is as follows (Figure 4.1).
Based on the above description, during the time interval \([0, t_1]\), the differential equation representing the inventory status is given by

\[
\frac{dq_1(t)}{dt} + \theta q_1(t) = K - D, \quad 0 \leq t \leq t_1
\]  

(26)

With the condition \(q_1(0) = 0\), the solution of Equation (26) is

\[
q_1(t) = \frac{K - D}{\theta} \left[1 - e^{-\theta t}\right] \quad 0 \leq t \leq t_1
\]  

(27)

In the second interval \([t_1, T]\), the differential equation below represents the inventory status:

\[
\frac{dq_2(t)}{dt} + \theta q_2(t) = -D, \quad t_1 \leq t \leq T
\]  

(28)

With the condition \(q_2(T) = 0\), we get the solution of Equation (28) as

\[
q_2(t) = \frac{D}{\theta} \left[e^{\theta(T-t)} - 1\right] \quad t_1 \leq t \leq T
\]  

(29)

Putting \(t = t_1\) in Equations (27) and (29) we find the value of \(t_1\) as
\[ t_1 = \frac{1}{\theta} \ln \left( 1 + \frac{D}{K} \left[ e^{\theta T} - 1 \right] \right) \]  

(30)

Since the replenishment occurs in the continuous time-span \([0, t_1]\), then the ordering lot size in the problem is, \(Q = Kt_1\).  

(31)

The maximum inventory level \(S\) in the problem is given by,

\[ S = q(t_1) = \frac{K - D}{\theta} \left[ 1 - e^{-\theta t_1} \right] \]  

(32)

Now we want to find the different inventory costs as follows:

Ordering cost is \(OC = A\).

The holding cost \(HC\) is given by

\[ HC = C_1 \left[ \int_0^{t_1} q_1(t) \, dt + \int_{t_1}^T q_2(t) \, dt \right] \]

\[ = \frac{C_1}{\theta^2} \left\{ K \left[ \theta t_1 + e^{-\theta t_1} - 1 \right] + D \left[ e^{\theta(T-t_1)} - \theta T - e^{-\theta t_1} \right] \right\} \]  

(33)

Since \(q_1(t_1) = q_2(t_1)\), this implies that Equation (33) can be rearranged as follows:

\[ HC = \frac{C_1}{\theta} \left[ Kt_1 - DT \right] \]  

(34)

The deteriorating cost \(DC\) is given by

\[ DC = \rho \left[ Kt_1 - DT \right] \]  

(35)

The following cases arise due to different types of delay periods.

1) Before the settlement of an account by a retailer, the sales revenue is used to earn interest with an annual rate \(I_c\). When \(T \geq M\), the account is settled at \(T = M\), and the retailer starts paying the interest charges on the
items in stock with an annual rate $I_c$. When $T \leq M$, the account is settled
at $T = M$, and the retailer does not pay interest charges.

2) The retailer can accumulate revenue and earn interest during the period 0
to $M$ with rate $I_c$ under the condition of a permissible delay in payments.

Therefore, the total cost of the system per unit time is given by

$$\begin{align*}
TC &= \begin{cases} 
TC_1, & T \leq M \\
TC_2, & t_1 \leq M \leq T \\
TC_3, & M \leq t_1 \leq T
\end{cases} \\
&= \begin{cases} 
TC_1, & T \leq M \\
TC_2, & t_1 \leq M \leq T \\
TC_3, & M \leq t_1 \leq T
\end{cases} 
\tag{36}
\end{align*}$$

where $TC_i(T), i = {1, 2, 3}$ are discussed as follows.

\subsection*{4.4.1 Case 1: $T \leq M$}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4_2.png}
\caption{Inventory level as a function of time for case 1 ($T \leq M$).}
\end{figure}

In this case, the period of delay in payment ($M$) is more than the cycle
length, and the retailer has received all the payment of sales goods from the
customers at time $T$ and pays all the amounts for purchased goods to the
supplier at $M$ (Figure 4.2). Therefore, the retailer uses the sales revenue to
earn interest at a rate of $I_c$, and no interest is payable.
Hence, the retailer’s interest earned is

\[ IE_1 = sI_e \left[ \int_0^T D t \, dt + (M - T)DT \right] = sI_e DT \left[ M - \frac{T}{2} \right] \]  \hspace{1cm} (37)

Total cost per cycle = ordering cost + inventory holding cost + deterioration cost – interest earned.

So, the total cost per unit time is

\[ TC_1 = \frac{1}{T} \left( OC + HC + DC - IE_1 \right) \]

\[ = \frac{A}{T} + \frac{C_1 + \theta p}{\theta T} (Kt_1 - DT) - sI_e D \left[ M - \frac{T}{2} \right] \]  \hspace{1cm} (38)

4.4.2 Case 2: \( t_1 \leq M \leq T \)

![Inventory level](image)

**Figure 4.3: Inventory level as a function of time for case 2 \((t_1 \leq M \leq T)\).**

In this case, the retailer pays the supplier at the end of the credit period, \( M \), which is before the inventory is depleted completely. Hence, the retailer still has some stock on hand during the time interval \([M, T]\) (Figure 4.3). Therefore, the interest charged by the supplier to the retailer per unit time is
\[ IC_2 = pI_c \int_M^T q_2(t) \, dt \]

\[ = \frac{pI_c D}{\theta^2} \left[ e^{\theta(T-M)} - 1 - \theta(T-M) \right] \tag{39} \]

The interest earned by the retailer is

\[ IE_2 = sI_c \int_M^T D t \, dt = \frac{sI_c DM^2}{2} \tag{40} \]

Total cost per cycle = ordering cost + inventory holding cost + deterioration cost + interest payable − interest earned.

So, the total cost per unit time is

\[ TC_2 = OC + HC + DC + IC_2 - IE_2 \]

\[ = \frac{A}{T} + \frac{C_1}{\theta T} (Kt_1 - DT) + \frac{pI_c D}{\theta^2 T} \left[ e^{\theta(T-M)} - 1 - \theta(T-M) \right] \]

\[ - \frac{sI_c DM^2}{2T} \tag{41} \]

4.4.3 Case 3: \( M \leq t_1 \leq T \)

![Inventory level as a function of time for case 3 (M ≤ t₁ ≤ T).](image)

Figure 4.4: Inventory level as a function of time for case 3 (\( M \leq t_1 \leq T \)).
In this case (Figure 4.4), the interest charged by the supplier to the retailer is

\[ IC_3 = pI_c \left[ \int_{t_1}^{t} q_1(t) \, dt + \int_{t_1}^{T} q_2(t) \, dt \right] \]

\[ = \frac{pI_c}{\theta^2} \left[ (K - D)(1 - \theta M - e^{-\theta M}) + \theta (Kt_1 - DT) \right] \] (42)

Furthermore, regarding the items that have already sold but have not yet been paid during the time interval \([0, M]\), the retailer earns an interest per unit time as follows:

\[ IE_3 = sI_c \int_{0}^{M} D \, dt \, dt = \frac{sI_c DM^2}{2} \] (43)

Total cost per cycle = ordering cost + inventory holding cost + deterioration cost + interest payable − interest earned.

So, the total cost per unit time is

\[ TC_3 = OC + HC + DC + IC_3 - IE_3 \]

\[ = \frac{A}{T} + \frac{C_1 + (\theta + I_c) p}{\theta T} (Kt_1 - DT) + \frac{pI_c}{T\theta^2} \left[ (K - D)(1 - \theta M - e^{-\theta M}) \right] \]

\[ - \frac{sI_c DM^2}{2T} \] (44)

Now, we consider the deterioration rate \(\theta\) which follows three different types of probability distribution functions as \(\theta = E[f(x)]\), where \(f(x)\) follows (a) uniform distribution, (b) triangular distribution and (c) beta distribution.
4.4.4 \( \theta \) follows uniform distribution

We consider that \( \theta \) follows uniform distribution as \( \theta = E[f(x)] = \frac{a+b}{2} \), \( a > 0, b > 0, a < b \), i.e., \( \theta \) follows uniform a distribution on the interval \([a, b], 0 < a < b\).

Now from Equations (38), (41) and (44), we have

\[
TC_{1,1} = \frac{A}{T} + \frac{2C_1 + (a+b)p}{(a+b)T} (Kt_1 - DT) - sI_c D \left[ M - \frac{T}{2} \right]
\]  

\[
TC_{2,1} = \frac{A}{T} + \frac{2C_1 + (a+b)p}{(a+b)T} (Kt_1 - DT) - \frac{sI_c D M^2}{2T}
\]

\[
+ \frac{4pI_c D}{(a+b)^2 T} \left[ e^{(a+b)(T-M)/2} - 1 - \frac{(a+b)(T-M)}{2} \right]
\]  

\[
TC_{3,1} = \frac{A}{T} + \frac{2C_1 + (a+b+2I_c)p}{(a+b)T} (Kt_1 - DT) - \frac{sI_c D M^2}{2T}
\]

\[
+ \frac{4pI_c}{T(a+b)^2} \left[ (K - D) \left( 1 - \frac{(a+b)M}{2} - e^{-(a+b)M/2} \right) \right]
\]

4.4.5 \( \theta \) follows triangular distribution

Now, we assume that \( \theta \) follows triangular distribution as \( \theta = E[f(x)] = \frac{a+b+c}{3} \), where \( f(x) \) is the probability density function of a triangular distribution with lower limit \( a \), upper limit \( b \) and mode \( c \) as well as \( a < b \), and \( a \leq c \leq b \).

Therefore, from Equations (38), (41) and (44), the equation of \( TC \) can be written as
\[ TC_{1,3} = \frac{A}{T} + \frac{3C_1 + (a + b + c)p}{(a + b + c)T} (K_{t_1} - DT) - sI_cD \left[ M - \frac{T}{2} \right] \] (48)

\[ TC_{2,2} = \frac{A}{T} + \frac{3C_1 + (a + b + c)p}{(a + b + c)T} (K_{t_1} - DT) - \frac{sI_cDM^2}{2T} \]

\[ + \frac{9pI_cD}{(a + b + c)^2T} e^{(a+b+c)(T-M)/3} - 1 - \frac{(a + b + c)(T - M)}{3} \] (49)

\[ TC_{3,2} = \frac{A}{T} + \frac{3C_1 + (a + b + c + 3I_c)p}{(a + b + c)T} (K_{t_1} - DT) - \frac{sI_cDM^2}{2T} \]

\[ + \frac{9pI_c}{T(a + b + c)^2} \left[ (K \quad D) \left( \begin{array}{c} 1 \\ (a + b + c)M/3 \\ e^{-(a+b+c)M/3} \end{array} \right) \right] \] (50)

4.4.6 \( \theta \) follows beta distribution

Now, we consider that \( \theta \) follows beta distribution as

\[ \theta = E[f(x)] = \frac{\alpha}{\alpha + \beta} \]

where \( f(x) \) follows beta distribution which is a continuous probability distribution defined on the interval \((0, 1)\) parameterized by two positive parameters, denoted by \( \alpha \) and \( \beta \).

From Equations (38), (41) and (44), we have

\[ TC_{1,3} = \frac{A}{T} + \frac{(\alpha + \beta)C_1 + \alpha p}{\alpha T} (K_{t_1} - DT) - sI_cD \left[ M - \frac{T}{2} \right] \] (51)

\[ TC_{2,3} = \frac{A}{T} + \frac{(\alpha + \beta)C_1 + \alpha p}{\alpha T} (K_{t_1} - DT) - \frac{sI_cDM^2}{2T} \]

\[ + \frac{pI_cD(\alpha + \beta)^2}{\alpha^2 T} \left[ e^{\alpha(T-M)/(\alpha + \beta)} - 1 - \frac{\alpha(T - M)}{(\alpha + \beta)} \right] \] (52)

\[ TC_{3,3} = \frac{A}{T} + \frac{(\alpha + \beta)C_1 + p(\alpha + (\alpha + \beta)I_c)}{\alpha T} (K_{t_1} - DT) - \frac{sI_cDM^2}{2T} \]
\[ + \frac{p_1}{T \alpha^2} (\alpha + \beta)^2 \left[ (K - D) \left( 1 - \frac{\alpha M}{(\alpha + \beta)} - e^{-\alpha M / (\alpha + \beta)} \right) \right] \] (53)

4.5 Solution procedure

To find the optimal solution, the following procedures are considered.

4.5.1 Determination of the optimal cycle length \( T \)

Here, the objective is to minimize the expected total inventory cost by finding the optimal cycle length.

4.5.1.1 When 0 follows uniform distribution:

We have \( \frac{dTC_{i,1}(T)}{dT} = \frac{f_{i,1}(T)}{T^2} \) for \( i = \{1,2,3\} \)

where:

\[ f_{1,1}(T) = -A + \frac{K(2C_1 + (a + b) p)}{(a + b)} \left\{ \frac{T dt_1}{dT} - t_1 \right\} + \frac{s_1 D T^2}{2} \] (54)

\[ f_{2,1}(T) = -A + \frac{K(2C_1 + (a + b) p)}{(a + b)} \left\{ \frac{T dt_1}{dT} - t_1 \right\} + \frac{s_1 D M^2}{2} \]

\[ + \frac{4p_1 D}{(a + b)^2} \left\{ e^{a+b/M} - \frac{(a + b)T}{2} - 1 + \frac{(a + b)M}{2} \right\} \] (55)

\[ f_{3,1}(T) = -A + \frac{K(2C_1 + (a + b + 2I_c) p)}{(a + b)} \left\{ \frac{T dt_1}{dT} - t_1 \right\} + \frac{s_1 D M^2}{2} \]

\[ - \frac{4p_1 (K - D)}{(a + b)^2} \left\{ 1 - \frac{(a + b)M}{2} - e^{(a+b)/M/2} \right\} \] (56)
Then both $f_{i,1}(T)$’s and $\frac{dTC_{i,1}(T)}{dT}$’s for $i = \{1,2,3\}$ have the same sign.

The optimal values of $T$, say $T_{i,1}^*$, are obtained by solving the equations $f_{i,1}(T) = 0$ for $i = \{1,2,3\}$.

We also have:

$$\frac{df_{1,1}(T)}{dT} = \frac{K(2C_1 + (a + b) p)}{(a + b)} T \frac{d^2t_1}{dT^2} + sI_e DT > 0 \text{ if } T > 0. \quad (57)$$

$$\frac{df_{2,1}(T)}{dT} = \frac{K(2C_1 + (a + b) p)}{(a + b)} T \frac{d^2t_1}{dT^2} + pI_e DTe^{(a+b)(T-M)/2} > 0 \text{ if } T > 0. \quad (58)$$

$$\frac{df_{3,1}(T)}{dT} = \frac{K(2C_1 + (a + b + 2I_c) p)}{(a + b)} T \frac{d^2t_1}{dT^2} > 0 \text{ if } T > 0. \quad (59)$$

where $\frac{d^2t_1}{dT^2} = \frac{D\theta e^{\theta T} (K - D)}{[K + D(e^{\theta T} - 1)]} > 0$, since $K > D$ and $e^{\theta T} - 1 > 0$.

Hence $f_{i,1}(T)$ are increasing on $(0, \infty)$, and so $\frac{dTC_{i,1}(T)}{dT}$ are increasing on $(0, \infty)$ for $i = \{1,2,3\}$.

From Lemma 1.1, $TC_{i,1}(T)$ are convex functions on $(0, \infty)$ for $i = \{1,2,3\}$.

Also $f_{i,1}(0) < 0$, since:

$f_{1,1}(0) = -A$

$f_{2,1}(0) = -A + \frac{sl_cDM^2}{2} - \frac{2pI_eDM}{(a + b)}$

$f_{3,1}(0) = -A + \frac{sl_cDM^2}{2} - \frac{4pI_e(K-D)}{(a + b)^2} \left\{ \frac{(a + b)M}{2} - e^{(a+b)M/2} \right\}$

and $\lim_{T \to -\infty} f_{i,1}(T) = \infty > 0$
implies that
\[
\frac{dTC_{i,1}(T)}{dT} = \begin{cases} 
< 0, & \text{if } T \in (0,T_{i,1}^*) \\
= 0, & \text{if } T = T_{i,1}^* \\
> 0, & \text{if } T \in (T_{i,1}^*, \infty) 
\end{cases} \quad \text{for } i = \{1,2,3\} \tag{60}
\]

Based upon the above arguments, the intermediate value theorem (Theorem 1.1) shows that the optimal solutions, \( T_{i,1}^* \), exist and unique.

4.5.1.2 When \( \theta \) follows triangular distribution:

We have
\[
\frac{dTC_{i,2}(T)}{dT} = \frac{f_{i,2}(T)}{T^2} \quad \text{for } i = \{1,2,3\}
\]
where:

\[
f_{1,2}(T) = -A + \frac{K(3C_1 + (a + b + c) \rho P)}{(a + b + c)} \left\{ \frac{Td_t}{dT} - t_1 \right\} + \frac{sI_c DT^2}{2} \tag{61}
\]

\[
f_{2,2}(T) = -A + \frac{K(3C_1 + (a + b + c) \rho P)}{(a + b + c)} \left\{ \frac{Td_t}{dT} - t_1 \right\} + \frac{sI_c DM^2}{2}
+ \frac{9 pI_c D}{(a + b + c)^2} \left\{ e^{(a+b+c)MT} \left[ \frac{(a+b+c)T}{3} - 1 \right] - \frac{(a+b+c)M}{3} + 1 \right\} \tag{62}
\]

\[
f_{3,1}(T) = -A + \frac{K(3C_1 + (a + b + c + 3I_c) \rho P)}{(a + b + c)} \left\{ \frac{Td_t}{dT} - t_1 \right\} + \frac{sI_c DM^2}{2}
- \frac{9 pI_c (K - D)}{(a + b + c)^2} \left\{ 1 - \frac{(a + b + c)M}{3} - e^{(a+b+c)M/3} \right\} \tag{63}
\]

Then both \( f_{i,2}(T) \)'s and \( \frac{dTC_{i,2}(T)}{dT} \)'s for \( i = \{1,2,3\} \) have the same sign.

The optimal values of \( T \), say \( T_{i,2}^* \), are obtained by solving the equations \( f_{i,2}(T) = 0 \) for \( i = \{1,2,3\} \).
We also have:

\[
\frac{df_{1,2}(T)}{dT} = \frac{K(3C_1 + (a + b + c) p)}{(a + b + c)} T \frac{d^2 t_1}{dT^2} + s\ell_c DT > 0 \text{ if } T > 0. \tag{64}
\]

\[
\frac{df_{2,2}(T)}{dT} = \frac{K(3C_1 + (a + b + c) p)}{(a + b + c)} T \frac{d^2 t_1}{dT^2} + p\ell_c DT e^{(a+b+c)(T-M)/3} > 0
\]

if \( T > 0. \tag{65} \)

\[
\frac{df_{3,2}(T)}{dT} = \frac{K(3C_1 + (a + b + c + 3\ell_c) p)}{(a + b + c)} T \frac{d^2 t_1}{dT^2} > 0 \text{ if } T > 0. \tag{66}
\]

Hence \( f_{i,2}(T) \) are increasing on \((0, \infty)\), and so \( \frac{dTC_{i,2}(T)}{dT} \) are increasing on \((0, \infty)\) for \( i = \{1,2,3\} \).

From Lemma 1.1, \( TC_{i,2}(T) \) are convex functions on \((0, \infty)\) for \( i = \{1,2,3\} \).

Also \( f_{i,2}(0) = 0 \), since:

\[
f_{1,2}(0) = -A
\]

\[
f_{2,2}(0) = -A + \frac{s\ell_c DM^2}{2} - \frac{3 p\ell_c DM}{(a + b + c)}
\]

\[
f_{3,2}(0) = -A + \frac{s\ell_c DM^2}{2} - \frac{9 p\ell_c (K - D)}{(a + b + c)^2} \left\{ 1 - \frac{(a + b + c)M}{3} - e^{(a+b+c)M/3} \right\}
\]

and \( \lim_{T \to \infty} f_{i,2}(T) = \infty > 0 \)

implies that \( \frac{dTC_{i,2}(T)}{dT} = \begin{cases} <0, & \text{if } T \in (0,T^*_{i,2}) \\ =0, & \text{if } T = T^*_{i,2} \\ >0, & \text{if } T \in (T^*_{i,2}, \infty) \end{cases} \tag{67} \)

for \( i = \{1,2,3\} \).
Based upon the above arguments, the intermediate value theorem (Theorem 1.1) shows that the optimal solutions, $T_{i,2}^*$, exist and unique.

### 4.5.1.3 When $\theta$ follows beta distribution:

We have $\frac{dTC_{i,3}(T)}{dT} = \frac{f_{i,3}(T)}{T^2}$ for $i = \{1,2,3\}$

where:

\[
f_{1,3}(T) = -A + \frac{K((\alpha + \beta)C_1 + \alpha p)}{\alpha} \left\{ \frac{td_{t_1}}{dT} - t_1 \right\} + \frac{sl_e DT^2}{2}
\]

\[
f_{2,3}(T) = -A + \frac{K((\alpha + \beta)C_1 + \alpha p)}{\alpha} \left\{ \frac{td_{t_1}}{dT} - t_1 \right\} + \frac{sl_e DM^2}{2} + \frac{(\alpha + \beta)^2}{\alpha^2} pI_c D e^{\alpha(T-M)/(\alpha + \beta)} \left\{ \frac{\alpha T}{(\alpha + \beta)} - 1 \right\} - \frac{\alpha M}{(\alpha + \beta)} + 1
\]

\[
f_{3,3}(T) = -A + \frac{K((\alpha + \beta)C_1 + (\alpha + (\alpha + \beta)I_c) p)}{\alpha} \left\{ \frac{td_{t_1}}{dT} - t_1 \right\} + \frac{sl_e DM^2}{2} - \frac{(\alpha + \beta)^2}{\alpha^2} pI_c (K - D) \left\{ 1 - \frac{\alpha M}{(\alpha + \beta)} - e^{\alpha M/(\alpha + \beta)} \right\}
\]

Then both $f_{i,3}(T)$’s and $\frac{dTC_{i,3}(T)}{dT}$’s for $i = \{1,2,3\}$ have the same sign.

The optimal values of $T$, say $T_{i,3}^*$, are obtained by solving the equations $f_{i,3}(T) = 0$ for $i = \{1,2,3\}$.

We also have:

\[
\frac{df_{i,3}(T)}{dT} = \frac{K((\alpha + \beta)C_1 + \alpha p)}{\alpha} T \frac{d^2 t_1}{dT^2} + sl_e DT > 0 \text{ if } T > 0.
\]
\[
\frac{df_{2,3}(T)}{dT} = \frac{K((\alpha + \beta)C_1 + \alpha p) T}{\alpha} \frac{d^2 t_1}{dT^2} + pI_c DTe^{(T-M)/(\alpha+\beta)} > 0
\]
if \( T > 0 \).  
(72)

\[
\frac{df_{3,3}(T)}{dT} = \frac{K((\alpha + \beta)C_1 + (\alpha + (\alpha + \beta)I_c) p) T}{\alpha} \frac{d^2 t_1}{dT^2} > 0 \text{ if } T > 0 .
\]
(73)

Hence \( f_{i,3}(T) \) are increasing on \((0, \infty)\), and so \( \frac{dTC_{i,3}(T)}{dT} \) are increasing on \((0, \infty)\) for \( i = \{1,2,3\} \).

From Lemma 1.1, \( TC_{i,3}(T) \) are convex functions on \((0, \infty)\) for \( i = \{1,2,3\} \).

Also \( f_{1,3}(0) < 0 \), since:

\[
f_{1,3}(0) = -A
\]

\[
f_{2,3}(0) = -A + \frac{sI_c DM^2}{2} - \frac{(\alpha + \beta)pI_c DM}{\alpha}
\]

\[
f_{3,3}(0) = -A + \frac{sI_c DM^2}{2} - \frac{(\alpha + \beta)^2 pI_c (K - D)}{\alpha^2} \left\{ 1 - \frac{\alpha M}{(\alpha + \beta)} - e^{\alpha M/(\alpha + \beta)} \right\}
\]

and \( \lim_{T \to \infty} f_{i,3}(T) = \infty > 0 \)

implies that

\[
\frac{dTC_{i,3}(T)}{dT} = \begin{cases} 
< 0, & \text{if } T \in (0, T_{i,3}^*) \\
= 0, & \text{if } T = T_{i,3}^* \quad \text{for } i \in \{1,2,3\} \\
> 0, & \text{if } T \in (T_{i,3}^*, \infty)
\end{cases}
\]
(14)

Based upon the above arguments, the intermediate value theorem (Theorem 1.1) shows that the optimal solutions, \( T_{i,3}^* \), exist and unique.
**Theorem 4.1:**

a) $TC_{i,1}(T)$ has the unique optimal solution $T_{i,1}^*$ on the non-negative interval $(0, \infty)$ for $i = \{1,2,3\}$.

b) $TC_{i,2}(T)$ has the unique optimal solution $T_{i,2}^*$ on the non-negative interval $(0, \infty)$ for $i = \{1,2,3\}$.

c) $TC_{i,3}(T)$ has the unique optimal solution $T_{i,3}^*$ on the non-negative interval $(0, \infty)$ for $i = \{1,2,3\}$.

**Proof:** The above arguments imply that Theorem 4.1 holds.

**4.6 Numerical examples**

**4.6.1 When $\theta$ follows uniform distribution:**

**Example 4.1:** (for $T \leq M$)

Consider an inventory system with the following data in appropriate units: $A = $1000; $C_1 = $15; $p = $30; $s = $55; $K = 4000$ units; $D = 2500$ units; $I_c = $0.10; $I_e = $0.12; $M = 0.3$; $a = 0.15$; $b = 0.25$.

Then we get the optimal values as $t_1 = 0.1479$ years, $T = 0.2346$ years, $Q = 592$ units, and $TC = $3548.2.

Figure 4.5 shows that, if $T \leq M$ and if the deterioration follows uniform distribution, then the total cost $TC$ decreases with $T$ and it attains the minimum value $3548.2$ at $T = 0.2346$ years. If $T$ crosses 0.2346, the total cost then increases. The graph (Figure 4.5) shows that the function $TC$ is convex with respect to $T$ when $\theta$ follows uniform distribution and when $T \leq M$. 

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Example 4.2: (for \( t_1 \leq M \leq T \))

Consider an inventory system with the following data in appropriate units: \( A = $1000 \); \( C_1 = $15 \); \( p = $30 \); \( s = $55 \); \( K = 4000 \) units; \( D = 2500 \) units; \( I_c = $0.10 \); \( I_e = $0.12 \); \( M = 0.2 \); \( a = 0.15 \); \( b = 0.25 \).

Then we get the optimal values as \( t_1 = 0.1554 \) years, \( T = 0.2463 \) years, \( Q = 621 \) units, and \( TC = $5167.4 \).

Figure 4.6 shows that, if \( t_1 \leq M \leq T \) and if the deterioration follows uniform distribution, then the total cost \( TC \) decreases with \( T \) and it attains the minimum value $5167.4 at \( T = 0.2463 \) years. If \( T \) crosses 0.2463, the total cost then increases. The graph (Figure 4.6) shows that the function \( TC \) is convex with respect to \( T \) when \( \theta \) follows uniform distribution and when \( t_1 \leq M \leq T \).
Figure 4.6: The total cost function with respect to $T$
when $\theta$ follows uniform distribution and $t_1 \leq M \leq T$.

**Example 4.3**: (for $M \leq t_1 \leq T$)

Consider an inventory system with the following data in appropriate units: $A = \$1000$; $C_1 = \$15$; $p = \$30$; $s = \$55$; $K = 4000$ units; $D = 2500$ units; $I_c = \$0.10$; $I_e = \$0.12$; $M = 0.1$; $a = 0.15$; $b = 0.25$.

Then we get the optimal values as $t_1 = 0.1773$ years, $T = 0.2808$ years, $Q = 709$ units, and $TC = \$6331.7$.

Figure 4.7 shows that, if $M \leq t_1 \leq T$ and if the deterioration follows uniform distribution, then the total cost $TC$ decreases with $T$ and it attains the minimum value $\$6331.7$ at $T = 0.2808$ years. If $T$ crosses 0.2808, the total cost then increases. The graph (Figure 4.7) shows that the function $TC$ is convex with respect to $T$ when $\theta$ follows uniform distribution and when $M \leq t_1 \leq T$. 

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4.6.2 When $\theta$ follows triangular distribution:

**Example 4.4:** (for $T \leq M$)

Consider an inventory system with the following data in appropriate units: $A = $1000; $C_1 = $15; $p = $30; $s = $55; $K = 4000$ units; $D = 2500$ units; $I_c = $0.10; $I_e = $0.12; $M = 0.3$ years; $a = 0.15$; $b = 0.35$; $c = 0.25$.

Then we get the optimal values as $t_1 = 0.1453$ years, $T = 0.2301$ years, $Q = 581$ units and $TC = $3709.2.

From Figure 4.8, it is observed that the total cost $TC$ decreases with $T$ and it attains the minimum value $3709.20$ at $T = 0.2301$ years, when the deterioration follows triangular distribution and when $T \leq M$. If $T$ crosses 0.2301, the total cost then increases. The graph (Figure 4.8) shows that the
function $TC$ is convex with respect to $T$ when $θ$ follows triangular distribution and when $T ≤ M$.

Figure 4.8: The total cost function with respect to $T$ when $θ$ follows triangular distribution and $T ≤ M$.

Example 4.5: (for $t_1 ≤ M ≤ T$)

Consider an inventory system with the following data in appropriate units: $A = $1000; $C_1 = $15; $p = $30; $s = $55; $K = 4000$ units; $D = 2500$ units; $I_c = $0.10; $I_e = $0.12; $M = 0.2$ years; $a = 0.15$; $b = 0.35$; $c = 0.25$.

Then we get the optimal values as $t_1 = 0.1519$ years, $T = 0.2404$ years, $Q = 608$ units and $TC =$5335.0.

From Figure 4.9, it is observed that the total cost $TC$ decreases with $T$ and it attains the minimum value $5335$ at $T = 0.2404$ years, when the deterioration follows triangular distribution and when $t_1 ≤ M ≤ T$. If $T$ crosses 0.2404, the total cost then increases. The graph (Figure 4.9) shows
that the function $TC$ is convex with respect to $T$ when $\theta$ follows triangular distribution and when $t_1 \leq M \leq T$.

![Figure 4.9: The total cost function with respect to $T$ when $\theta$ follows triangular distribution and $t_1 \leq M \leq T$.](image)

**Example 4.6:** (for $M \leq t_1 \leq T$)

Consider an inventory system with the following data in appropriate units: $A = $1000; $C_1 = $15; $p = $30; $s = $55; $K = 4000$ units; $D = 2500$ units; $I_c = $0.10; $I_e = $0.12; $M = 0.1$ years; $a = 0.15; b = 0.35; c = 0.25$.

Then we get the optimal values as $t_1 = 0.1723$ years, $T = 0.2722$ years, $Q = 689$ units and $TC = $6523.4.

From Figure 4.10, it is observed that the total cost $TC$ decreases with $T$ and it attains the minimum value $6523.40$ at $T = 0.2722$ years, when the deterioration follows triangular distribution and when $M \leq t_1 \leq T$. If $T$ crosses $0.2722$, the total cost then increases. The graph (Figure 4.10) shows
that the function $TC$ is convex with respect to $T$ when $\theta$ follows triangular distribution and when $M \leq t_1 \leq T$.

![Graph of the total cost function with respect to $T$ when $\theta$ follows triangular distribution and $M \leq t_1 \leq T$.](image)

**Figure 4.10:** The total cost function with respect to $T$ when $\theta$ follows triangular distribution and $M \leq t_1 \leq T$.

### 4.6.3 When $\theta$ follows beta distribution:

**Example 4.7:** (for $T \leq M$)

Consider an inventory system with the following data in appropriate units: $A = $1000; $C_1 = $15; $p = $30; $s = $55; $K = 4000$ units; $D = 2500$ units; $I_c = $0.10; $I_e = $0.12; $M = 0.3$; $\alpha = 0.15$; $\beta = 0.35$.

Then we get the optimal values as $t_1 = 0.1429$ years, $T = 0.2258$ years, $Q = 572$ units and $TC = $3867.1.

Figure 4.11 illustrates that, supposing deterioration follows beta distribution and if $T \leq M$, the total cost $TC$ decreases with $T$ and it attains...
the minimum value $3867.10$ at $T = 0.2258$ years. The total cost then increases, if $T$ crosses $0.2258$. The graph (Figure 4.11) shows that the function $TC$ is convex with respect to $T$ when $\theta$ follows beta distribution and when $T \leq M$.

![Figure 4.11: The total cost function with respect to $T$ when $\theta$ follows beta distribution and $T \leq M$.]

**Example 4.8:** (for $t_1 \leq M \leq T$)

Consider an inventory system with the following data in appropriate units: $A = $1000; $C_1 = $15; $p = $30; $s = $55; $K = 4000$ units; $D = 2500$ units; $I_c = $0.10; $I_e = $0.12; $M = 0.2$; $\alpha = 0.15$; $\beta = 0.35$.

Then we get the optimal values as $t_1 = 0.1487$ years, $T = 0.2348$ years, $Q = 595$ units and $TC = $5498.5.
Figure 4.12: The total cost function with respect to $T$
when $\theta$ follows beta distribution and $t_1 \leq M \leq T$.

Figure 4.12 illustrates that, supposing deterioration follows beta
distribution and if $t_1 \leq M \leq T$, the total cost $TC$ decreases with $T$ and it
attains the minimum value $5498.50$ at $T = 0.2348$ years. The total cost
then increases, if $T$ crosses $0.2348$. The graph (Figure 4.12) shows that the
function $TC$ is convex with respect to $T$ when $\theta$ follows beta distribution
and when $t_1 \leq M \leq T$.

**Example 4.9:** (for $M \leq t_1 \leq T$)

Consider an inventory system with the following data in appropriate
units: $A = \$1000$; $C_1 = \$15$; $p = \$30$; $s = \$55$; $K = 4000$ units; $D = 2500$
units; $I_c = \$0.10$; $I_e = \$0.12$; $M = 0.1$; $\alpha = 0.15$; $\beta = 0.35$.

Then we get the optimal values as $t_1 = 0.1676$ years,
$T = 0.2643$ years, $Q = 671$ units and $TC = \$6709.4$.  

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Figure 4.13 illustrates that, supposing deterioration follows beta distribution and if $M \leq t_1 \leq T$, the total cost $TC$ decreases with $T$ and it attains the minimum value $6709.40$ at $T = 0.2643$ years. The total cost then increases, if $T$ crosses 0.2643. The graph (Figure 4.13) shows that the function $TC$ is convex with respect to $T$ when $\theta$ follows beta distribution and when $M \leq t_1 \leq T$.

![Figure 4.13: The total cost function with respect to $T$.](image)

Suppose $M = t_1$:

Then we get the optimal values as follows:

For uniform distribution: $t_1 = 0.2317$ years, $T = 0.3657$ years, $Q = 927$ units, and $TC = 5286.8$.

For triangular distribution: $t_1 = 0.2211$ years, $T = 0.3481$ years, $Q = 884$ units, and $TC = 5533.8$. 

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For beta distribution \( t_1 = 0.2119 \) years, \( T = 0.3329 \) years, 
\( Q = 848 \) units, and \( TC = \$5770.5 \).

Suppose \( M = T \):
Then we get the optimal values as follows:

For uniform distribution: \( t_1 = 0.5720 \) years, \( T = 0.8862 \) years, 
\( Q = 2288 \) units, and \( TC = \$2403.0 \).

For triangular distribution: \( t_1 = 0.4031 \) years, \( T = 0.6269 \) years, 
\( Q = 1613 \) units, and \( TC = \$2943.4 \).

For beta distribution \( t_1 = 0.3558 \) years, \( T = 0.5523 \) years, 
\( Q = 1423 \) units, and \( TC = \$3376.1 \).

### 4.6.4 Comparison among the models by the graphical representations

The comparison among the three probabilistic deteriorated models is done with the help of graphical representations. The following plots are due to the change of the three probabilistic deterioration functions for different delay periods. Consider the three figures (Figure 4.14, Figure 4.15 and Figure 4.16); each figure contains a combination of three plots in which the top plot is for beta distribution, the middle-plot is for triangular distribution and the bottom plot is for uniform distribution. From figures 4.14, 4.15 and 4.16, it can be observed that the total cost will be minimized when the deterioration follows uniform distribution. From figures 4.17, 4.18 and 4.19, we can see that the optimal minimum total cost of the retailer occurs in case \( T \leq M \). That is, if the retailer gets the permissible delay after \( T \), then the total cost of the retailer will be minimized.
Figure 4.14: Total cost with respect to cycle length for different distributions of $\theta$ when $T \leq M$.

Figure 4.15: Total cost with respect to cycle length for different distributions of $\theta$ when $t_1 \leq M \leq T$.

Figure 4.16: Total cost with respect to cycle length for different distributions of $\theta$ when $M \leq t_1 \leq T$.

Figure 4.17: Total cost with respect to cycle length for different delay periods when $\theta$ follows uniform distribution.
4.7 Sensitivity analysis

As mentioned earlier, the change in the values of the parameters may happen due to uncertainties in any decision making situation. In order to examine the implications of these changes, a sensitivity analysis will be of great help in decision-making. We now study the effects of changes in the values of the system parameters $\Lambda$, $C_1$, $p$, $s$, $K$, $D$ and $M$ on the optimal replenishment policy based on the first case ($T \leq M$). We change one parameter at a time, keeping the other parameters unchanged. The results of the sensitivity analysis are summarized in Table 4.1.
Table 4.1: Sensitivity analysis with respect to some major parameters.

<table>
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<tr>
<th>Parameter</th>
<th>Parameter value</th>
<th>$TC$ (uniform)</th>
<th>$TC$ (triangular)</th>
<th>$TC$ (beta)</th>
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<td>3867.1</td>
</tr>
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<td>4693.1</td>
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<tr>
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Based on the numerical results, we obtain the following managerial phenomena:

1. When the ordering cost $A$ and the holding cost $C_1$ are increasing, total cost $TC$ is also increasing. This is not unexpected because in practice when the retailer controls his ordering cost, then the inventory total cost automatically decreases. To avoid deterioration or to reduce the deterioration rate, the retailer must provide good facilities to hold the items. Due to increasing holding cost, the total cost will increase. That is, the minimum ordering cost and minimum cost for holding the items will minimize the total cost of the retailer.

2. When the purchasing price $p$ is increasing, the total cost $TC$ is increasing. That is, an increase of the purchasing price will increase the total cost of the retailer. In order to minimize the cost, the retailer should decrease the purchasing cost.

3. When the selling price $s$ is increasing, the total optimal cost $TC$ is decreasing. But increasing the selling price will decrease the demand as well as decrease the order quantity. Since the order quantity is reduced, automatically holding the cost of the items will also be reduced. Therefore, the total cost of the retailer will be minimized.

4. When the replenishment rate $K$ is increasing, the total optimal cost $TC$ is also increasing. That is, the minimum replenishment rate will minimize the total cost of the retailer. This fact occurs in real life because we can observe that if the retailer increases his/her replenishment lot-size then the cost for holding the items or the cost of deterioration will be increased; therefore, the total cost increases.
5. When the demand rate \( D \) is increasing, the total cost \( TC \) is decreasing. This fact may occur in real-life business, because if the buyer’s demand increases, this will give more profit to the retailer, so the total cost is automatically decreased.

6. When the permissible delay period \( M \) is increasing, the total cost \( TC \) is decreasing. This is expected because when the trade credit period goes up, the customer orders a large quantity. This large quantity will increase the profit of the retailer. Therefore, increasing the trade credit will decrease the total cost of the retailer.

4.8 Conclusion

In this chapter, an EOQ model for determining the optimal cycle length, optimal order quantity and optimal total cost for deteriorating items is developed. Here the deterioration function follows probability distributions such as (a) uniform distribution, (b) triangular distribution and (c) beta distribution. In each case, we find the minimum total cost associated with the system. Moreover, the retailer is allowed a trade-credit offered by the supplier to buy more items. A theorem has been framed to characterize the optimal solutions. The necessary and sufficient conditions of the existence and uniqueness of the optimal solutions are also provided. The proposed solution procedure in this model is simple and does not require tedious computation effort. Furthermore, numerical examples along with graphical representations are provided to illustrate the model and the solution procedure. Also, a numerical comparison among the models is shown graphically. Finally, a sensitivity analysis is carried out with respect to the key parameters.