Chapter 2

On Stancu type Jakimovski-Leviatan-Durrmeyer and generalized Dunkle analogue of Szász operators

2.1 Introduction

The approximation process given by Korovkin gave a new shoot up to the approximation theory. It arises naturally in many situations connected with measure theory, functional analysis, partial differential equations, harmonic analysis, probability theory, etc. One of the most useful operators of this type are the Favard-Szász operators defined in (1.3.3).

Several authors have investigated many interesting properties of these operators in [12, 25, 41, 44, 45, 66, 79, 116, 120]. Later Jakimovski and Leviatan generalized these operators in [56] using Appell polynomials defined as follows: Let \( g(z) = \sum_{k=0}^{\infty} a_k z^k \) \( (a_0 \neq 0) \) be an analytic function defined in the disk \(|z| < R, \ (R > 0)\) with \( g(1) \neq 0 \). The Appell polynomials are generated by the functions of the type

\[
g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k
\]

with the condition that \( p_k(x) \geq 0 \) for every \( x \in [0, \infty) \). Jakimovski and Leviatan introduced the following positive linear operators:

\[
P_n(f; x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f \left( \frac{k}{n} \right).
\]

They investigated approximation properties of these opeaors in [56]. Ismail [52] generalized these operators using Sheffer polynomials and studied some approximation properties. Büyükyazıcı et al. [20] obtained Chlodowsky type generalization of these operators and investigated many poroperties by using Appell polynomials. Mursaleen et al. obtained another Chlodowsky type generalization and examined several approximation properties of

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these operators in [88]. Karaisa gave Durrmeyer type generalization of these operators and investigated the approximation properties in [61]. The Durrmeyer type generalization of these operators is defined as follows:

\[ S_n(f; x) = e^{-nx} \frac{\sum_{k=0}^{\infty} p_k(nx)}{g(1)} \frac{B(n + 1, k)}{B(n + 1, k)} \int_0^\infty \frac{t^{k-1}}{(1 + t)^{n+k+1}} f(t) dt, \quad x \geq 0, \]  

(2.1.2)

where \( B(n + 1, k) \) is the beta function defined by

\[ B(x, y) = \int_0^\infty \frac{t^{x-1}}{(1 + t)^{x+y}} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x, y > 0. \]

2.2 Stancu type Jakimovski-Leviatan-Durrmeyer operators

In this section we introduce and study the Stancu type generalization of the Jakimovski-Leviatan-Durrmeyer operators and examine their approximation properties. We investigate the convergence of these operators with the help of Korovkin’s approximation theorem. Also, we study local approximation properties and some direct theorems for these operators.

2.2.1 Construction of Operators and approximating results

We introduce the Stancu type generalization of the Jakimovski-Leviatan-Durrmeyer operators as follows:

\[ L_n(f; x) = e^{-nx} \frac{\sum_{k=0}^{\infty} p_k(nx)}{g(1)} \frac{B(n + 1, k)}{B(n + 1, k)} \int_0^\infty \frac{t^{k-1}}{(1 + t)^{n+k+1}} f\left(\frac{nt + \alpha}{n + \beta}\right) dt, \quad x \geq 0, \]  

(2.2.1)

where \( B(x, y) \) is the beta function defined above and \( \alpha, \beta \) are such that \( 0 \leq \alpha \leq \beta \). Taking \( \alpha = 0, \beta = 0 \) in (2.2.1), we get the Jakimovski-Leviatan-Durrmeyer operators (2.1.2). To examine the approximation results for the newly constructed operators, we need the following lemmas.

**Lemma 2.1.** By the Appell polynomials (2.1.1), we easily get

\[ (1) \sum_{k=0}^{\infty} p_k(nx) = e^{nx}g(1), \]
\[
(2) \sum_{k=0}^{\infty} kp_k(nx) = e^{nx}[nxg(1) + g'(1)],
\]
\[
(3) \sum_{k=0}^{\infty} k^2p_k(nx) = e^{nx}[n^2x^2g(1) + nx(2g'(1) + g(1)) + g''(1) + g'(1)],
\]
\[
(4) \sum_{k=0}^{\infty} k^3p_k(nx) = e^{nx}[n^3x^3g(1) + n^2x^2(3g'(1) + 4g(1)) + nx(3g''(1) + 8g'(1)) + g(1)) + g''(1) + 4g''(1) + g'(1)],
\]
\[
(5) \sum_{k=0}^{\infty} k^4p_k(nx) = e^{nx}[n^4x^4g(1) + n^3x^3(4g'(1) + 10g(1)) + n^2x^2(6g''(1) + 30g'(1) + 14g(1)) + nx(4g''(1) + 30g''(1) + 28g'(1)) + g(1)) + g''(1) + 10g''(1) + 14g''(1) + g'(1)].
\]

Lemma 2.2. For \(n > 3\), we have the following:

(1) \(L_n(e_0; x) = 1\).

(2) \(L_n(e_1; x) = \frac{1}{n+\beta}(nx + (A_0 + \alpha))\).

(3) \(L_n(e_2; x) = \frac{n^2}{n(n+1)(n+\beta)}[n^2x^2 + nx(2A_0+1) + (A_0 + B_0)] + \frac{2n\alpha}{(n+1)(n+\beta)^2}nx + (A_0 + 1)]
+ \frac{\alpha^2}{(n+\beta)^2}.

Proof. (1) \(L_n(e_0; x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} \frac{p_k(nx)}{B(n+1,k)} \int_{0}^{\infty} \frac{t^{k-1}}{(1+t)^{n+k+1}} dt = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} \frac{p_k(nx)}{B(n+1,k)} B(k, n+1) = 1\).

(2) \(L_n(e_1; x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} \frac{p_k(nx)}{B(n+1,k)} \int_{0}^{\infty} \frac{t^{k-1}}{(1+t)^{n+k+1}} \left(\frac{nt+\alpha}{n+\beta}\right) dt
\]
\[= \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} \frac{p_k(nx)}{B(n+1,k)} \frac{1}{n+\beta} \left[nB(k+1, n) + \alpha B(k, n+1)\right]
= \frac{1}{n+\beta}(nx + (A_0 + \alpha)).\]

(3) \(L_n(e_2; x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} \frac{p_k(nx)}{B(n+1,k)} \int_{0}^{\infty} \frac{t^{k-1}}{(1+t)^{n+k+1}} \left(\frac{nt+\alpha}{n+\beta}\right)^2 dt
\]
\[= \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} \frac{p_k(nx)}{B(n+1,k)} \left[\frac{n^2}{(n+\beta)^2} B(k+2, n-1) + \frac{2n\alpha}{(n+\beta)^2} B(k+1, n) + \frac{\alpha^2}{(n+\beta)^2 n(n+1)}\right]
= \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} \frac{p_k(nx)}{B(n+1,k)} \left[\frac{n^2}{(n+\beta)^2} \frac{(k+2)(k+1)}{n(n+1)} + \frac{2n\alpha}{(n+\beta)^2} \frac{k+1}{n(n+1)} + \frac{\alpha^2}{(n+\beta)^2}\right]
= \frac{n^2}{n(n+1)(n+\beta)^2} \left[n^2x^2 + nx(2A_0+1) + (A_0 + B_0)\right] + \frac{2n\alpha}{n(n+1)(n+\beta)^2} nx + (A_0 + 1)\)
Hence the lemma is proved. \qed

We prove the following theorem.

**Theorem 2.3.** For \( f \in C[0, \infty), \) the operators \( L_n(\cdot; x) \) converge uniformly to \( f \) on the compact domain \([0, a], a > 0\) as \( n \to \infty. \)

**Proof.** By the Lemma 2.2, we have

\[
\lim_{n \to \infty} L_n(e_0; x) = 1,
\]

\[
\lim_{n \to \infty} L_n(e_1; x) = x,
\]

\[
\lim_{n \to \infty} L_n(e_2; x) = x^2.
\]

Thus the operators \( L_n(\cdot; x) \) converge uniformly to \( f \), where \( f \) is one of the test functions 1, \( t \), \( t^2 \) on the compact interval \([0, a]\). So by the Korovkin approximation theorem [65, 84], the result is true for every continuous function \( f \) defined on the compact interval \([0, a]\). Hence the theorem is proved. \qed

### 2.2.2 Convergence in weighted space

We investigate some approximation properties of the operators \( L_n(\cdot; x) \) in the weighted space of continuous functions \( Q^k_{p, \rho}(\mathbb{R}^{+}) \) defined in Section 1.6.

**Lemma 2.4.** Let \( L_n(\cdot; x) \) be the operators defined by (2.2.1). Then for the weight function \( \rho(x) = 1 + x^2 \), we obtain

\[
\| L_n(\rho; x) \|_{x^2} \leq K,
\]

where \( K \) is a positive constant greater than unity.

**Proof.** For \( n > 1 \), using Lemma 2.2, we get

\[
L_n(\rho; x) = 1 + \frac{n^4}{n(n+1)(n+\beta)^2} x^2 + \left( \frac{n^3(2A_0 + 1)}{n(n+1)(n+\beta)^2} + \frac{2\alpha n^2}{(n+1)(n+\beta)^2} \right) x
\]

\[
+ \frac{n^2((1+2\alpha)A_0 + B_0 + 2\alpha)}{n(n+1)(n+\beta)^2} + \frac{\alpha^2}{(n+\beta)^2}.
\]
Therefore,
\[
\|L_n(\rho; x)\|_{x^2} = \sup_{x \geq 0} \left\{ \frac{1}{1 + x^2} + \frac{n^4}{n(n + 1)(n + \beta)^2} \frac{x^2}{1 + x^2} + \frac{n^3(2A_0 + 1)}{n(n + 1)(n + \beta)^2} \frac{x}{1 + x^2} \right. \\
\left. + \frac{2\alpha n^2}{(n + 1)(n + \beta)^2} \frac{x}{1 + x^2} \right\}
\]
\[
\leq 1 + \frac{n^4}{n(n + 1)(n + \beta)^2} + \frac{n^3(2A_0 + 1)}{n(n + 1)(n + \beta)^2} + \frac{2\alpha n^2}{(n + 1)(n + \beta)^2}
\]
\[
+ \frac{n^2((1 + 2\alpha)A_0 + B_0 + 2\alpha)}{n(n + 1)(n + \beta)^2} + \frac{\alpha^2}{(n + \beta)^2}.
\]

Now since \(\lim_{n \to \infty} \frac{n^3}{n(n + 1)(n + \beta)^2} = 1, \lim_{n \to \infty} \frac{n^2}{n(n + 1)(n+\beta)^2} = 0, \lim_{n \to \infty} \frac{n^2}{n(n+1)(n+\beta)^2} = 0, \lim_{n \to \infty} \frac{n^3}{(n+1)(n+\beta)^2} = 0, \lim_{n \to \infty} \frac{n^2}{n(n+1)(n+\beta)^2} = 0\), and \(\lim_{n \to \infty} \frac{1}{(n+\beta)^2} = 0\), there exists a positive constant \(K > 1\) such that
\[
\|L_n(\rho; x)\|_{x^2} \leq K.
\]

This completes the proof. \(\Box\)

By the Lemma 2.4, it is easily seen that the operators \(L_n(\cdot; x)\) act from the space \(C_{x^2}[0, \infty)\) to the space \(B_{x^2}[0, \infty)\).

**Theorem 2.5.** Let \(L_n(\cdot; x)\) be the sequence of positive linear operators defined by (2.2.1) and \(\rho(x) = 1 + x^2\) be the weight function. Then, for each \(f \in C_{x^2}[0, \infty)\),
\[
\lim_{n \to \infty} \|L_n(f; x) - f(x)\|_{x^2} = 0.
\]

**Proof.** In view of the Korovkin Theorem it is sufficient to prove that
\[
\lim_{n \to \infty} \|L_n(t^i; x) - x^i\|_{x^2} = 0, \quad i = 0, 1, 2.
\]

By Lemma 2.2 (1), it is obvious that
\[
\lim_{n \to \infty} \|L_n(1; x) - 1\|_{x^2} = 0.
\]

Making use of Lemma 2.2 (2), we have
\[
\|L_n(e_1; x) - e_1(x)\|_{x^2} = \sup_{x \geq 0} \frac{n}{n + \beta} \frac{x}{1 + x^2} + A_0 + \frac{\alpha}{n + \beta} \frac{1}{1 + x^2} - \frac{x}{1 + x^2}
\]

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\[ \begin{align*}
= & \sup_{x \geq 0} \left| \left( \frac{n}{n+\beta} - 1 \right) \frac{x}{1+x^2} + \frac{A_0 + \alpha}{n+\beta} \frac{1}{1+x^2} \right| \\
= & \frac{\beta}{n+\beta} + \frac{A_0 + \alpha}{n+\beta}.
\end{align*} \]

Therefore,

\[ \lim_{n \to \infty} \| L_n(e_1; x) - e_1(x) \|_{x^2} = 0. \]

In the spirit of Lemma 2.2 (3), it implies that

\[ \begin{align*}
\| L_n(e_2; x) - e_2(x) \|_{x^2} &= \sup_{x \geq 0} \left| \left( \frac{n^4}{(n(n+1)(n+\beta)^2)} - 1 \right) \frac{x^2}{(1+x^2)} + \left( \frac{n^3(2A_0 + 1)}{n(n+1)(n+\beta)^2} \right) \frac{x}{1+x^2} \\
&+ \left( \frac{2\alpha n^2}{(n+1)(n+\beta)^2} \right) \frac{x}{1+x^2} + \left( \frac{n^2(A_0 + B_0)}{n(n+1)(n+\beta)^2} \right) \frac{1}{1+x^2} \\
&+ \left( \frac{2\alpha n^2(A_0 + 1)}{n(n+1)(n+\beta)^2} + \frac{\alpha^2}{(n+\beta)^2} \right) \frac{1}{1+x^2} \right| \\
&\leq \left\{ \left( \frac{n^4}{(n(n+1)(n+\beta)^2)} - 1 \right) + \left( \frac{n^3(2A_0 + 1)}{n(n+1)(n+\beta)^2} \right) + \left( \frac{2\alpha n^2}{n(n+1)(n+\beta)^2} \right) \right\}.
\end{align*} \]

Hence we get,

\[ \lim_{n \to \infty} \| L_n(e_2; x) - e_2(x) \|_{x^2} = 0. \]

This proves the theorem. \( \square \)

### 2.2.3 On rates of approximation

In this subsection, we compute the rate of convergence of operators \( L_n(\cdot; x) \) in terms of the modulus of continuity defined in (1.4.3). We will use (1.4.7).

We denote by \( C_E[0, \infty) \), the set of all continuous functions \( f \) on \( [0, \infty) \) with the property that \( |f(x)| \leq be^{ax} \) for all \( x \geq 0 \) and for some positive finite constants \( a, b \).

**Theorem 2.6.** Let \( f \in C_E[0, \infty) \) and \( L_n(\cdot; x) \) be the operators defined by (2.2.1). Then for all \( x \geq 0 \) and \( n > 1 \), we obtain

\[ |L_n(f; x) - f(x)| \leq 2\omega_f(\delta_{n,x}), \]

where

\[ \delta_{n,x}^2 = \omega_{n,1}x^2 + \omega_{n,2}x + \omega_{n,3} + \frac{\alpha^2}{(n+\beta)^2}; \]
with
\[ \omega_{n,1} = \frac{n((n+1)\beta^2) - 2(n+\beta) - 2n\beta - 3n^2}{(n+1)^2(n+\beta)^2}, \]
\[ \omega_{n,2} = \frac{n^5 - n^4(2\alpha(1+\beta) + 2A_0(2+\beta)) - 2n^3((1+2\beta)(A_0 + \alpha) - 2n^2\beta(1+A_0))}{n^2(n+1)^2(n+\beta)^2}, \]
\[ \omega_{n,3} = \frac{n^4(A_0 + B_0)}{n^2(n+1)^2(n+\beta)^2} + \frac{n^3(A_0 + 1)}{n(n+1)(n+\beta)^2}. \]

**Proof.** By linearity and positivity of the operators \( L_n(\cdot; x) \), we get
\[
|L_n(f; x) - f(x)| \leq \omega(f, \delta) \left\{ 1 + \frac{1}{\delta}(L_n((t-x)^2; x))^\frac{1}{2} \right\}.
\]

Making use of Lemma 2.2 and linearity of operators \( L_n(\cdot; x) \), for \( n > 1 \), we have
\[
L_n((t-x)^2; x) = \frac{1}{n^2(n+1)^2(n+\beta)^2} \left( \left( n^3(n+1)\beta^2 - 2(n+\beta) - 2n\beta - 3n^2 \right) x^2 
+ \left( n^5 - n^4(2\alpha(1+\beta) + 2A_0(2+\beta) - 2n^3(1+2\beta)(A_0 + \alpha) 
- 2n^3(1+2\beta)(A_0 + \alpha) - 2n^2\beta(A_0 + 1)) \right) x 
+ n^4((A_0 + B_0) + (n+1)(A_0 + 1)) + n^2\alpha^2(n+1)^2 \right). 
\]

Using this and taking \( \delta_{n,x} = \delta \), we arrive at the required inequality. Hence the theorem is proved. \( \Box \)

### 2.2.4 Direct theorems

By \( C_B[0, \infty) \) we denote the space of all bounded and continuous functions \( f \) on \( [0, \infty) \) with the norm
\[
\| f \| = \sup_{x \in [0, \infty)} |f(x)|.
\]

From (1.4.11) for the domain \([0, \infty)\) and by [30, Theorem 2.4, p. 177], there exists a constant \( C > 0 \) such that
\[
K_2(f, \delta) \leq C \omega_2(f, \delta), \tag{2.2.2}
\]
where \( \omega_2(f, \delta) \), the second order modulus of continuity is defined in (1.4.4).

For \( f \in C_B[0, \infty) \), we define the following associated operators:
\[
\tilde{L}_n(f; x) = L_n(f; x) - \frac{1}{n+\beta} \left( (n+\beta)x + (A_0 + \alpha) \right). \tag{2.2.3}
\]
where \( x \geq 0 \).

We prove the following lemma.

**Lemma 2.7.** Let \( g \in C^2_B[0, \infty) \) and \( \mathcal{L}_n(\cdot; x) \) be the operators defined by (2.2.3). Then for all \( x \geq 0 \) and \( n > 1 \), we obtain

\[
|\mathcal{L}_n(g; x) - g(x)| \leq \phi_n^\alpha,\beta(x)\|g''\|,
\]

where

\[
\phi_n^\alpha,\beta(x) = \frac{1}{n^2(n+1)^2(n+\beta)^2} \left( a_{n,x^2} x^2 + b_{n,x} x + c_n + \left( \frac{A_0 + \alpha}{n + \beta} \right)^2 \right),
\]

\[
a_{n,x^2} = n^3 \left( (n+1)\beta^2 - 2(n+\beta) - 1n\beta - 3n^2 \right),
\]

\[
b_{n,x} = n^5 - n^4 \left( 2\alpha(1+\beta) + 2A_0(2+\beta) - 2n^3(1+2\beta)(A_0 + \alpha) - 2n^2\beta(A_0 + 1) \right)
\]

and

\[
c_n = n^2 \left( n^2(A_0 + B_0) + (n+1)((A_0 + 1) + \alpha(n+1)) \right).
\]

**Proof.** It is obvious that \( \mathcal{L}_n(c_1(x) - x; x) = 0 \). Let \( g \in C^2_B[0, \infty) \). Then by Taylor’s expansion of \( g \), we have

\[
g(t) - g(x) = (t - x)g'(x) + \int_x^t (t - u)g''(u)du,
\]

where \( t \in [0, \infty) \). Operating by \( \mathcal{L}_n(\cdot; x) \) on both sides of the above equality, we get

\[
\mathcal{L}_n(g; x) - g(x) = g'(x)\mathcal{L}_n(t - x; x) + \mathcal{L}_n \left( \int_x^t (t - u)g''(u)du; x \right)
\]

\[
= \mathcal{L}_n \left( \int_x^t (t - u)g''(u)du; x \right)
\]

\[
= L_n \left( \int_x^t (t - u)g''(u)du; x \right)
\]

\[
- \int_x^{(n+\beta)x + (A_0 + \alpha)} \left( \frac{(n + \beta)x + A_0}{n + \beta} - u \right) g''(u)du.
\]

Therefore

\[
|L_n(g; x) - g(x)| \leq L_n \left( \int_x^t (t - u)g''(u)du; x \right) + \int_x^{(n+\beta)x + (A_0 + \alpha)} \left( \frac{(n + \beta)x + A_0}{n + \beta} - u \right) g''(u)du \tag{2.2.4}
\]
Since \(| \int_x^1 (t - u) g''(u) du | \leq (t - x)^2 \| g'' \| ,\) we get

\[
\left| \int_x^1 \left( \frac{(n + \beta)x + A_0}{n + \beta} - u \right) g''(u) du \right| \leq \left( \frac{A_0 + \alpha}{n + \beta} \right)^2 \| g'' \| .
\]

In view of (2.2.3), we have

\[
| \tilde{L}_n(g; x) - g(x) | \leq \left\{ L_n ((t - x)^2; x) + \left( \frac{A_0 + \alpha}{n + \beta} \right)^2 \right\} \| g'' \| .
\]

Now using the expression for \( L_n ((t - x)^2; x) \) from the Theorem 2.6, we get

\[
| \tilde{L}_n(g; x) - g(x) | \leq \frac{1}{n^2(n + 1)^2(n + \beta)^2} \left\{ n^3((n + 1)\beta^2 - 2(n + \beta) - 1n\beta - 3n^2)x^2 + (n^5 - n^4(2\alpha(1 + \beta) + 2A_0(2 + \beta)) - 2n^3(1 + 2\beta)(A_0 + \alpha) - 2n^2\beta(A_0 + 1)x + (n^2(A_0 + B_0) + (n + 1)((A_0 + 1) + \alpha(n + 1))) + \left( \frac{A_0 + \alpha}{n + \beta} \right)^2 \right\} \| g'' \|
\]

where \( a_{n,x^2}, b_{n,x} \) are respectively the coefficients of \( x^2, x \) in the above expression and \( c_n = n^2(A_0 + B_0) + (n + 1)((A_0 + 1) + \alpha(n + 1)). \) More succinctly, it can be written as

\[
| \tilde{L}_n(g; x) - g(x) | \leq \phi_n^{\alpha,\beta}(x) \| g'' \| ,
\]

where

\[
\phi_n^{\alpha,\beta}(x) = \frac{1}{n^2(n + 1)^2(n + \beta)^2} \left( a_{n,x^2}x^2 + b_{n,x}x + c_n + \left( \frac{A_0 + \alpha}{n + \beta} \right)^2 \right)
\]

and hence the lemma is proved. \( \square \)

**Theorem 2.8.** Let \( f \in C_B[0, \infty) \) and \( L_n(.; x) \) be given by (2.2.1). Then for every \( x \geq 0, \) there exists a constant \( K > 0 \) such that

\[
| L_n(f; x) - f(x) | \leq K \omega_2 \left( f, \sqrt{\phi_n^{\alpha,\beta}(x)} \right) + \omega \left( f, A_0 + \alpha \right),
\]

where \( \phi_n^{\alpha,\beta}(x) \) is as in Lemma 2.7 and \( \omega(f, \cdot), \omega_2(f, \cdot) \) are respectively the first order modulus of continuity and the second order modulus of continuity of \( f \) given by (1.4.5), (1.4.6) respectively.
\textbf{Proof.} For \( f \in C_B^1[0, \infty), \ g \in C_B^2[0, \infty), \) by the definition of the operators \( \bar{L}_n(.; x), \) we have
\[
\left| L_n(f; x) - f(x) \right| \leq \left| \bar{L}_n(f - g; x) \right| + |(f - g)(x)| \\
+ \left| \bar{L}_n(g; x) - g(x) \right| + \left| f \left( \frac{(n + \beta)x + (A_0 + \alpha)}{n + \beta} \right) - f(x) \right|.
\]
But
\[
\left| \bar{L}_n(f; x) \right| \leq \| f \| L_n(1; x) + 2\| f \| = 3\| f \|,
\]
so we have
\[
\left| L_n(f; x) - f(x) \right| \leq 4\| f - g \| + \left| \bar{L}_n(g; x) - g(x) \right| + \omega \left( f, \frac{A_0 + \alpha}{n + \beta} \right).
\]
Using Lemma 2.7, we easily get
\[
\left| L_n(f; x) - f(x) \right| \leq 4 \left( \| f - g \| + \phi_n^{a,b}(x)g'' \right) + \omega \left( f, \frac{A_0 + \alpha}{n + \beta} \right).
\]
On taking the infimum over all \( g \in C_B^2[0, \infty) \) on the right hand side of above inequality and using the relation (2.2.2) we arrive at the desired result. \( \square \)

\subsection*{2.2.5 Conclusions}

In the previous section we obtained Stancu type generalization of the well known Jakimovski-Leviatan-Durrmeyer operators. In the light of the Korovkin’s approximation theorem, we studied approximating behaviour of these operators. Error bounds were obtained and some local approximation properties investigated. Also, some theorems were studied for the functions of class \( C^1[0, \infty). \)

\section*{2.3 Generalized Dunkle analogue of Szász operators}

In this section we present a generalization of Szász operators using the Dunkl generalization of the exponential function. We investigate approximating properties for these operators using the Korovkin approximation theorem and the weighted Korovkin-type theorem. We obtain quantitative estimates by using the modulus of continuity and the rate of convergence for functions belonging to the Lipschitz class. Furthermore, we obtain the rate
of convergence in terms of the classical, the second order, and the weighted modulus of continuity.

In 1912, Bernstein [14] constructed the following sequence of operators \( B_n : C[0, 1] \to C[0, 1] \)

\[
B_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} f \left( \frac{k}{n} \right), \quad x \in [0, 1],
\]

for \( n \in \mathbb{N} \) and \( f \in C[0, 1] \).

Later, in 1950, Szász [120] introduced the following sequence of linear positive operators

\[
S_n(f; x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f \left( \frac{k}{n} \right), \quad f \in C[0, \infty),
\]

where \( x \geq 0 \) and \( f \) is a continuous and nondecreasing function on \([0, \infty)\). He obtained approximation results and rate of convergence for these operators.

### 2.3.1 Construction of \( L_{n}^{*}(\cdot, \cdot) \) and auxiliary results

For any \( x \in [0, \infty), \quad n \in \mathbb{N}, \mu \geq 0 \) and \( f \in C[0, \infty) \), Sucu [118] defined the following sequence of positive linear operators

\[
L_n(f; x) = \frac{1}{e_{\mu(n\lambda)}} \sum_{k=0}^{\infty} \frac{(nx)^k}{\gamma_{\mu}(k)} f \left( \frac{k + 2\mu \theta_k}{n} \right), \quad (2.3.1)
\]

where the generalized exponential function is defined by

\[
e_{\mu}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\gamma_{\mu}(k)}.
\]

and the coefficients \( \gamma_{\mu} \) are defined as follows for \( k \in \mathbb{N}_0 \) and \( \mu > -\frac{1}{2} \)

\[
\gamma_{\mu}(2k) = \frac{2^{2k} k! \Gamma \left( k + \mu + \frac{1}{2} \right)}{\Gamma \left( \mu + \frac{1}{2} \right)}
\]

and

\[
\gamma_{\mu}(2k + 1) = \frac{2^{2k+1} k! \Gamma \left( k + \mu + \frac{3}{2} \right)}{\Gamma \left( \mu + \frac{3}{2} \right)}.
\]

A recursion formula for \( \gamma_{\mu} \) is given by

\[
\gamma_{\mu}(k + 1) = (k + 1 + 2\mu \theta_{k+1}) \gamma_{\mu}(k), \quad k = 0, 1, 2, \ldots,
\]
where
\[
\theta_k = \begin{cases} 
0 & \text{if } k \in 2\mathbb{N} \\
1 & \text{if } k \in 2\mathbb{N} + 1.
\end{cases}
\]

The operators in (2.3.1) have been called as Dunkl analogue of the Szász operators. Sucu investigated their approximating properties and some direct results. Motivated by the work of Sucu [118] and Duman [32], we introduce the Dunkl generalization of the Szász operators as follows: we replace \( x \) by \( r_n(x) = x - \frac{1}{2n}, \ n \in \mathbb{N}, \) in (2.3.1) to define the following sequence of operators
\[
L_n^*(f; x) = \frac{1}{e_{\mu}(nr_n(x))} \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_{\mu}(k)} \frac{f\left(\frac{k + 2\mu \theta_k}{n}\right)}{n},
\]
and call them as generalized Dunkl analogue of Szász operators. It is easily seen that the operators \( L_n^*(\cdot; x) \) are positive and linear.

For the operators 2.3.2, the following results are derived.

**Lemma 2.9.** Let \( L_n^*(\cdot; x) \) be the operators given by (2.3.2). Then we have the following identities and estimates:

1. \( L_n^*(1; x) = 1, \)

2. \( L_n^*(t; x) = r_n(x) = x - \frac{1}{2n}, \)

3. \( L_n^*(t^2; x) = x^2 + 2\mu \frac{e_{\mu}(nr_n(x))}{e_{\mu}(nr_n(x))} \frac{x}{n} - \frac{1}{n^2} \left( \frac{1}{4} + \mu \frac{e_{\mu}(nr_n(x))}{e_{\mu}(nr_n(x))} \right), \)

4. \( L_n^*(t^3; x) \leq x^3 + \left( 3 - 2\mu \frac{e_{\mu}(nr_n(x))}{e_{\mu}(nr_n(x))} \right) \frac{x^2}{n} + \left( 3 + 4\mu^2 + (4 + 2\mu) \frac{e_{\mu}(nr_n(x))}{e_{\mu}(nr_n(x))} \right) \frac{x}{n^2} + \frac{1}{8n^3}, \)

5. \( L_n^*(t^4; x) \leq x^4 + \left( 6 + 4\mu \frac{e_{\mu}(nr_n(x))}{e_{\mu}(nr_n(x))} \right) \frac{x^3}{n} + \left( 3 + 12\mu^2 - 12\mu \frac{e_{\mu}(nr_n(x))}{e_{\mu}(nr_n(x))} \right) \frac{x^2}{n^2} + \left( 17 + 6\mu^2 (2 + n^2) + (42\mu + 8\mu^3) \frac{e_{\mu}(nr_n(x))}{e_{\mu}(nr_n(x))} \right) \frac{x}{n^3} + \left( 3 + 2\mu^2 (4 + 2\mu) \frac{e_{\mu}(nr_n(x))}{e_{\mu}(nr_n(x))} \right) \frac{x^2}{n} \)

**Proof.** Using the definition of the generalized exponential function, (1) is obvious.

\[
L_n^*(t; x) = \frac{1}{e_{\mu}(nr_n(x))} \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_{\mu}(k)} \frac{f\left(\frac{k + 2\mu \theta_k}{n}\right)}{n} = \frac{1}{ne_{\mu}(nr_n(x))} \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_{\mu}(k)} (k + 2\mu \theta_k)
\]
\[
= \frac{1}{ne_\mu(nr_n(x))} \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{(k + 2\mu\theta_k)\gamma_\mu(k - 1)(k + 2\mu\theta_k)}
= \frac{1}{ne_\mu(nr_n(x))} \sum_{k=0}^{\infty} \frac{nr_n(x)^{k+1}}{\gamma_\mu(k)}
= r_n(x)
= x - \frac{1}{2},
\]

which proves (2).

\[
L_n^*(t^2; x) = \frac{1}{e_\mu(nr_n(x))} \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_\mu(k)} \left( \frac{k + 2\mu\theta_k}{n} \right)^2
= \frac{1}{n^2e_\mu(nr_n(x))} \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_\mu(k)}(k + 2\mu\theta_k)^2
= \frac{1}{n^2e_\mu(nr_n(x))} \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_\mu(k)(k - 1)}(k + 2\mu\theta_k)
= \frac{1}{n^2e_\mu(nr_n(x))} \sum_{k=0}^{\infty} \frac{(nr_n(x))^{k+1}}{\gamma_\mu(k)}(k + 1 + 2\mu\theta_{k+1}),
\]

using the relation
\[
\theta_{k+1} = \theta_k + (-1)^k,
\]

the following is obtained

\[
L_n^*(t^2; x) = \frac{r_n(x)}{ne_\mu(nr_n(x))} \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_\mu(k)}(k + 2\mu\theta_k) + \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_\mu(k)}
+ \frac{r_n(x)}{ne_\mu(nr_n(x))} 2\mu \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_\mu(k)}
= (x - \frac{1}{2n})^2 + \frac{1}{n}(x - \frac{1}{2n}) + \frac{2\mu}{n}(x - \frac{1}{2n}) \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}
= x^2 + 2\mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))} \frac{x}{n} - \frac{1}{n^2} \left( \frac{1}{4} + \mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))} \right),
\]

and this proves (3).

\[
L_n^*(t^3; x) = \frac{1}{e_\mu(nr_n(x))} \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_\mu(k)} \left( \frac{k + 2\mu\theta_k}{n} \right)^3
\]
\[
L_n^*(t^4; x) = \frac{1}{e_\mu(nr_n(x))} \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_\mu(k)} \left( k + 2\mu \theta_k \right)^4
\]

and after some simple computations we arrive at the desired inequality.

Some intermediary calculations yield the required inequality. □

**Lemma 2.10.** Let the operators \( L_n^*(.; x) \) be given by (2.3.2). Then

1. \( L_n^*(t - x; x) = -\frac{1}{2n} \),

2. \( L_n^*((t - x)^2; x) = \left( 1 + 2\mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))} \right) \frac{x}{n} - \frac{1}{n^2} \left( \frac{1}{4} + \mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))} \right) \),

3. \( L_n^*((t - x)^4; x) \leq \left( 6 + 24\mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))} \right) \frac{x^2}{n} + \left( 3 - 4\mu^2 - 20\mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))} \right) \frac{x^2}{n^2} + \left( 17 + 6\mu^2(2 + \mu^2) + (42\mu + 8\mu^3) \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))} \right) \frac{x}{n^3} \).

**Proof.**

\[
L_n^*((t - x)^2; x) = L_n^*(t^2; x) - 2xL_n^*(t; x) + x^2L_n^*(1; x) = x^2 + 2\mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))} \frac{x}{n} - \frac{1}{n^2} \left( \frac{1}{4} + \mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))} \right) - 2x(x - \frac{1}{2}) + x^2 = 2\mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))} \frac{x}{n} - \frac{1}{n^2} \left( \frac{1}{4} + \mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))} \right) + \frac{x}{n}.
\]
\[
(1 + 2\mu \frac{\epsilon_\mu(-n\tau_n(x))}{\epsilon_\mu(n\tau_n(x))}) \frac{x}{n} - \frac{1}{n^2} \left( \frac{1}{4} + \mu \frac{\epsilon_\mu(-n\tau_n(x))}{\epsilon_\mu(n\tau_n(x))} \right),
\]

this ends the proof of (2).

Noting that

\[
L_n^*((t^4; x) = L_n^*(t^4; x) - 4xL_n^*(t^3; x) + 6x^2L_n^*(t^2; x) - 4x^3L_n^*(t; x) + x^4L_n^*(1; x)
\]

and using the Lemma 2.9, the desired inequality is obtained. \(\square\)

### 2.3.2 Korovkin-type theorem for the operators \(L_n^*(.;.;.)\)

We prove the following results by using Korovkin’s theorem.

**Theorem 2.11.** Let \(L_n^*(.; x)\) be the operators given by (2.3.2) and let

\[
H := \left\{ f : x \in [0, \infty), \frac{f(x)}{1 + x^2} \text{ is convergent as } x \to \infty \right\},
\]

then

\[
\lim_{n \to \infty} L_n^*(f; x) = f
\]

uniformly on each compact subset of \([0, \infty)\).

**Proof.** The proof is based on the well known Korovkin’s theorem [7] regarding the convergence of a sequence of linear positive operators, so it is enough to prove the conditions

\[
\lim_{n \to \infty} L_n^*(t^j; x) = x^j, \quad j = 0, 1, 2, \quad \{\text{as } n \to \infty}\}
\]

uniformly on each compact subset of \([0, 1]\). Using Lemma 2.9 it is easily seen that the above assertions are supplied. Hence the theorem is proved. \(\square\)

Recall the weighted space of functions \(Q_k^\beta(\mathbb{R}^+)\) defined in Section 1.6.

**Theorem 2.12.** Let \(L_n^*(.; ;.)\) be the operators given by (2.3.2). Then for any function \(f \in Q_k^\beta(\mathbb{R}^+)\), we have

\[
\lim_{n \to \infty} \| L_n^*(f; x) - f \| = 0.
\]
Proof. In view of (1) and (2) of Lemma 2.9 implies that
\[ \| L_n^*(1; x) - 1 \|_\rho = 0 \]
and
\[ \| L_n^*(t; x) - x \|_\rho = 0 \]
respectively. Using (3) of Lemma 2.9 and the definition of norm, the following is obtained
\[ \sup_{x \in [0, \infty)} \frac{| L_n^*(t^2; x) - x^2 |}{1 + x^2} \leq \frac{((8n + 1)\mu + 1)}{4n^2} \sup_{x \in [0, \infty)} \frac{x}{1 + x^2}. \]
From where we easily find that
\[ \lim_{n \to \infty} \| L_n^*(t^2; x) - x^2 \|_\rho = 0. \]
This completes the proof by using the weighted Korovkin-type theorems proved by Gadzhiev [39]. \( \square \)

2.3.3 Rates of convergence

In what follows we calculate the rate of convergence of the operators (2.3.2) by means of modulus of continuity defined by (1.4.5), (1.4.6) and Lipschitz type functions given in (1.4.2).

Theorem 2.13. Let \( L_n^*(\cdot; \cdot) \) be the operators defined by (2.3.2). Then for each \( f \in \text{Lip}_M(\nu) \) satisfying (1.4.2), we have
\[ | L_n^*(f; x) - f(x) | \leq M (\lambda_n(x))^{\frac{\nu}{2}}, \]
where
\[ \lambda_n(x) = L_n^* \left( (t - x)^2; x \right). \]

Proof. By using (1.4.2) and the linearity property, the following is obtained
\[ | L_n^*(f; x) - f(x) | = | L_n^* (f(t) - f(x); x) | \]
\[ \leq L_n^* (| f(t) - f(x) |; x) \]
\[ \leq M L_n^* (| t - x |^\nu; x). \]
Therefore, by Lemma 2.9 and the Hölder’s inequality, we obtain
\[
| L_n^*(f; x) - f(x) | \\
\leq M \frac{1}{e_{\mu}(nr_n(x))} \sum_{k=0}^{\infty} \left( \frac{(nr_n(x))^k}{\gamma_{\mu}(k)} \right)^{\frac{2n}{2}} \left( \frac{(nr_n(x))^k}{\gamma_{\mu}(k)} \right)^{\frac{n}{2}} | k + 2\mu\theta_k - x |^{\nu} \\
\leq M \frac{1}{e_{\mu}(nr_n(x))} \left( \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_{\mu}(k)} \right)^{\frac{2n}{2}} \left( \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_{\mu}(k)} \right)^{\frac{n}{2}} | k + 2\mu\theta_k - x |^{2} \frac{1}{n} \\
= M \left( L_n^*(t - x)^2; x \right)^{\frac{1}{2}},
\]
which completes the proof. □

**Theorem 2.14.** Let \( \tilde{C}[0, \infty) \) denote the space of uniformly continuous functions defined on \([0, \infty)\) and \( II \) be the function space defined in Theorem 2.11. Then for \( f \in \tilde{C}[0, \infty) \cap II \), the following holds
\[
| L_n^*(f; x) - f(x) | \leq \left\{ 1 + \left( x \left( 1 + 2\mu \frac{e_{\mu}(-nr_n(x))}{e_{\mu}(nr_n(x))} \right) \right)^{\frac{1}{2}} \right\} \omega(f, \frac{1}{\sqrt{n}}).
\]

**Proof.** Making use of Lemma 4.2, definition of modulus of continuity and the Cauchy-Schwarz inequality, we have
\[
| L_n^*(f; x) - f(x) | \leq \frac{1}{e_{\mu}(nr_n(x))} \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_{\mu}(k)} \left| f \left( \frac{k + 2\mu\theta_k}{n} \right) - f(x) \right| \\
\leq \left\{ 1 + \frac{1}{\delta} \frac{1}{e_{\mu}(nr_n(x))} \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_{\mu}(k)} \left| \frac{k + 2\mu\theta_k}{n} - x \right| \right\} \omega(f, \delta) \\
\leq \left\{ 1 + \frac{1}{\delta} \left( \frac{1}{e_{\mu}(nr_n(x))} \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_{\mu}(k)} \left( \frac{k + 2\mu\theta_k}{n} - x \right)^2 \right)^{\frac{1}{2}} \right\} \omega(f, \delta) \\
\leq \left\{ 1 + \frac{1}{\delta} \left( \frac{x}{n} \left( 1 + 2\mu \frac{e_{\mu}(-nr_n(x))}{e_{\mu}(nr_n(x))} \right) \right)^{\frac{1}{2}} \right\} \omega(f, \delta) \\
- \frac{1}{n^2} \left( \frac{1}{4\mu} \frac{e_{\mu}(-nr_n(x))}{e_{\mu}(nr_n(x))} \right)^{\frac{1}{2}} \omega(f, \delta) \\
\leq \left\{ 1 + \frac{1}{\delta} \left( \frac{x}{n} \left( 1 + 2\mu \frac{e_{\mu}(-nr_n(x))}{e_{\mu}(nr_n(x))} \right) \right)^{\frac{1}{2}} \right\} \omega(f, \delta).
\]
We rearrange the resulting terms and arrive at the desired inequality. □
Lemma 2.15. Let \( L^*_n(\cdot;\cdot) \) be the operator defined by (2.3.2). Then for any \( g \in C^2_B(\mathbb{R}^+) \), we have

\[
| L^*_n(g; x) - g(x) | \leq \left(1 + \frac{\lambda_n(x)}{2}\right) \| g \|_{C^2_B(\mathbb{R}^+)},
\]

where

\[
\lambda_n(x) = |L^*_n((t - x)^2; x)|
\]

and \( \| g \|_{C^2_B(\mathbb{R}^+)} \) is given by (1.5.3).

Proof. Let \( g \in C^2_B(\mathbb{R}^+) \). Then by using the generalized Mean Value Theorem in the Taylor series expansion we have

\[
g(t) = g(x) + g'(x)(t - x) + g''(\psi)\frac{(t - x)^2}{2}, \quad \psi \in (x, t).
\]

Operating by \( L^*_n(\cdot; x) \) on both sides, we have

\[
L^*_n(g, x) - g(x) = g'(x)L^*_n((t - x); x) + \frac{g''(\psi)}{2}L^*_n((t - x)^2; x).
\]

Combining this with the Lemma 2.10, the following is obtained

\[
| L^*_n(g, x) - g(x) | \leq \frac{1}{2n} \| g' \|_{C^1_B(\mathbb{R}^+)} + \frac{\| g'' \|_{C^2_B(\mathbb{R}^+)}}{2}
\times \left| \left(1 + 2\mu \frac{e_{\mu}(-nr_n(x))}{e_{\mu}(nr_n(x))}\right) \frac{x}{n} - \frac{1}{n^2} \left(1 + \mu \frac{e_{\mu}(-nr_n(x))}{e_{\mu}(nr_n(x))}\right) \right|
\leq \| g \|_{C^2_B(\mathbb{R}^+)} + \frac{\| g \|_{C^2_B(\mathbb{R}^+)}}{2}
\times \left| \left(1 + 2\mu \frac{e_{\mu}(-nr_n(x))}{e_{\mu}(nr_n(x))}\right) \frac{x}{n} - \frac{1}{n^2} \left(1 + \mu \frac{e_{\mu}(-nr_n(x))}{e_{\mu}(nr_n(x))}\right) \right|
= \| g \|_{C^2_B(\mathbb{R}^+)} + \frac{\| g \|_{C^2_B(\mathbb{R}^+)}}{2} \lambda_n(x)
= \| g \|_{C^2_B(\mathbb{R}^+)} \left(1 + \frac{\lambda_n(x)}{2}\right),
\]

where

\[
\lambda_n(x) = |L^*_n((t - x)^2; x)|.
\]

This completes the proof of the lemma. \( \square \)

Theorem 2.16. Let \( L^*_n(\cdot; x) \) be the operators defined by (2.3.2). Then for every \( f \in C_B[0, \infty) \) and \( x \in [0, \infty) \), there holds the following

\[
| L^*_n(g; x) - g(x) | \leq 2\tilde{M} \left\{ \omega_2 \left(f; \frac{\lambda_n(x)}{4}\right)^{\frac{1}{2}} + \min \left(1, \frac{\lambda_n(x)}{4}\right) \| f \|_{C_B(\mathbb{R}^+)} \right\},
\]

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where $\tilde{M}$ is a positive constant independent of $n$ and $\omega_2$ is the second modulus of continuity given by (1.4.6) and $\| f \|_{C^2_B(\mathbb{R}^+)}$ is defined by (1.5.1).

**Proof.** Let $g \in C^2_B([0, \infty))$. Then using triangle inequality and the Lemma 2.15, we obtain the following

$$| L^*_n(f; x) - f(x) | \leq | L^*_n(f - g; x) | + | L^*_n(g; x) - g(x) | + | f(x) - g(x) |$$

$$\leq 2\| f - g \|_{C^2_B([0, \infty))} + \frac{\lambda_n(x)}{2} \| g \|_{C^2_B([0, \infty))}.$$

Using the definition (1.4.11) of Peetre’s K-functional, we get

$$| L^*_n(f; x) - f(x) | \leq 2K \left( f, \frac{\lambda_n(x)}{4} \right).$$

Using the relation (2.2.2) between the second modulus of continuity and the Peetre’s K-functional, we arrive at the desired estimate. $\square$

### 2.3.4 Order of approximation

We shall determine the order of convergence for the functions in the space $Q^k_\rho(\mathbb{R}^+)$ as described in Section 1.6. For $f \in Q^k_\rho(\mathbb{R}^+)$, the weighted modulus of continuity, introduced by Ispir and Atakut [13], is defined by

$$\Omega(f; \delta) = \sup_{0 \leq x < \infty, |h| \leq \delta} \frac{|f(x + h) - f(x)|}{(1 + h^2)(1 + x^2)}.$$

The importance of this type of modulus of continuity is due to its following properties. For $f \in Q^k_\rho(\mathbb{R}^+)$,

$$\lim_{\delta \to 0} \Omega(f; \delta) = 0,$$

$$|f(t) - f(x)| \leq 2 \left( 1 + \frac{1}{\delta} |t - x| \right) (1 + \delta^2)(1 + x^2)(1 + (t - x)^2) \Omega(f; \delta); \ 0 \leq x, t < \infty.$$

(2.3.3)

Details of this modulus of continuity can be found in [13].

We prove the following theorem.

**Theorem 2.17.** Let $L^*_n(\cdot; x)$ be the operators in (2.3.2) and $f \in Q^k_\rho(\mathbb{R}^+)$. Then

$$\sup_{0 \leq x < \infty} \frac{|L^*_n(f; x) - f(x)|}{(1 + x^2)^3} \leq M_\mu \left( f, \frac{1}{n} \right) \omega \left( f; \frac{1}{n^{\frac{1}{3}}} \right),$$

where $M_\mu$ is constant independent of $n$.  

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\textbf{Proof.} Making use of Lemma 2.9 and the expression (2.3.3), the following is obtained

\[ | L_n^*(f; x) - f(x) | \leq \frac{1}{e_\mu(nr_n(x))} \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_\mu(k)} \left| f \left( \frac{k + 2 \mu \theta_k}{n} \right) - f(x) \right| \]

\[ \leq 2(1 + \delta^2)(1 + x^2) \omega(f; \delta) \frac{1}{e_\mu(nr_n(x))} \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_\mu(k)} \]

\[ \times \left( 1 + \left| \frac{k + 2 \mu \theta_k}{n} - x \right| \right) \left( 1 + \left( \frac{k + 2 \mu \theta_k}{n} - x \right)^2 \right) \]

\[ = 2(1 + \delta^2)(1 + x^2) \omega(f; \delta) \frac{1}{e_\mu(nr_n(x))} \left\{ \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_\mu(k)} \left| \frac{k + 2 \mu \theta_k}{n} - x \right| \right. \]

\[ + \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_\mu(k)} \left( \frac{k + 2 \mu \theta_k}{n} - x \right)^2 + \frac{1}{\delta} \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_\mu(k)} \left| \frac{k + 2 \mu \theta_k}{n} - x \right| \left( \frac{k + 2 \mu \theta_k}{n} - x \right)^2 \left\} \right. \]

Applying Cauchy-Schwarz inequality in the above, we obtain

\[ | L_n^*(f; x) - f(x) | \leq (1 + \delta^2)(1 + x^2) \omega(f; \delta) \left\{ 1 + L_n^*((t - x)^2; x) + \frac{1}{\delta} (L_n^*((t - x)^2; x))^{\frac{1}{2}} \right. \]

\[ + \frac{1}{\delta} (L_n^*((t - x)^2; x)) \times (L_n^*((t - x)^4; x)))^{\frac{1}{2}} \right\}. \] (2.3.4)

In the light of Lemma 2.10, we have the following estimates

\[ L_n^*((t - x)^2; x) \leq \frac{1 + 2 \mu}{n} x, \quad L_n^*((t - x)^4; x) \leq \frac{14 + 68 \mu + 16 \mu^2 + 8 \mu^3 + 6 \mu^4}{n} (x + x^2 + x^3). \]

Combining these with (2.3.4), we obtain the following

\[ | L_n^*(f; x) - f(x) | \leq (1 + \delta^2)(1 + x^2) \omega(f; \delta) \left\{ 1 + (1 + 2 \mu)x + \frac{1}{\delta} \frac{1}{\sqrt{n}} \sqrt{(1 + 2 \mu)x} \right. \]

\[ + \frac{1}{\delta} \frac{1}{\sqrt{n}} \sqrt{(x^2 + x^3 + x^4)(14 + 68 \mu + 16 \mu^2 + 8 \mu^3 + 6 \mu^4)} \right\}. \]

On choosing \( \delta = \frac{1}{n^2} \), the theorem follows. \( \square \)

\subsection*{2.3.5 Conclusions}

In the last section of this chapter a Dunkl generalization of Szász operators was studied. The approximation properties were investigated in the spirit of the Korovkin theorem. Approximation behaviour was also investigated in the weighted spaces of functions. Errors
were obtained in terms of Lipschitz class of functions. The convergence properties were studied by means of modulii of continuity.