Chapter 1

Preliminary Concepts

This chapter sheds light on introduction and basic notions of the Approximation Theory which are used in ensuing chapters. We attempt to present a fair detail of the subject.

1.1 Introduction

The Weierstrass approximation theorem, given by the German mathematician K. Weierstrass [122] in the year of 1885, is regarded as the inception of the Approximation Theory which provides a criterion for the uniform convergence of a sequence of polynomials towards a continuous function. The theorem is stated as follows: Let $f$ be any continuous function defined on the finite interval $[a, b]$ ($f \in C[a, b]$) and $\varepsilon > 0$. Then there exists a sequence of algebraic polynomials $p_n(x)$ with real coefficients which converges uniformly to $f$ on $[a, b]$, i.e., $|f(x) - p_n(x)| < \varepsilon$, $\forall x \in [a, b]$, $\forall n \in \mathbb{N}$. There are many proofs of this celebrated theorem but one of the most elegant proofs was given by S.N. Bernstein [14] in 1912. He constructed the sequence of polynomials by using probabilistic approach. E. Borel’s contribution [16] in 1905 gave another remarkable impetus to the field.

The polynomials constructed by Bernstein to prove the Weierstrass approximation theorem are as follows:

$$B_n(f; x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} f \left( \frac{k}{n} \right), \text{ for any } f \in C[0, 1], x \in [0, 1].$$

These polynomials are known as Bernstein classical operators. These are the widely and intensively studied polynomials in approximation theory. Many researchers have constructed and studied various kinds of positive linear operators inspired by these operators.

in 1953, individually discovered a simple and succinct criterion for the uniform convergence of a sequence of positive linear operators towards the identity operator. The criterion lays down an easy condition for the uniform convergence of the sequence of positive linear operators. It states that the necessary and sufficient condition for the uniform convergence of a sequence \((P_n)\) of positive linear operators to a continuous function \(f\) on the compact interval \([a, b]\), is the uniform convergence of the sequence \(P_n t^i \to t^i\) for \(i = 0, 1, 2\). For this criterion, it is natural to know what happens if the domain of definition of \(f\) is unbounded (for example \([0, \infty)\)). In that case this result remains valid only for those continuous functions that have a finite limit at infinity. In the process of approximation by positive linear operators on \(C[a, b]\), the functions \(1, t, t^2\) are vital. They bear special name: the Korovkin test-functions. A comprehensive detail of the Korovkin-type theory can be found in [7].

For the approximating investigations and analysis by a positive linear operator \(P\), the following quantities are essential. The calculation of the test functions \(P(t^i; x)\), for \(i \in \mathbb{N} \cup \{0\}\) and the moments \(P((t - x)^i; x)\) of order \(i\). For more details of the moments of operators the reader is referred to [43]. The convergence criterion discussed above is popularly known as Bohman-Korovkin theorem, because T. Popoviciu’s contribution [107] remained obscure for long.

For a quantitative approximation we use certain tools: the Ditzian-Totik modulus of smoothness, the modulus of smoothness of first and second order and the \(K\)-functionals. They will be given in Section 1.4.

### 1.2 Positive linear operators

Let \(U\) be a nonempty set and \(\mathbb{R}\) be the set of real numbers. We denote by \(\mathcal{L}\), the collection of all functions \(f : U \to \mathbb{R}\). This forms a linear space over \(\mathbb{R}\) with respect to the usual addition and scalar multiplication of functions. In the following we denote by \(X\), a linear subspace of \(\mathcal{L}(U, \mathbb{R})\) and \(Y\) will represent a linear subspace of \(\mathcal{L}(V, \mathbb{R})\), where \(V\) is a nonvoid set.
1.2.1 Definition of positive linear operators

Definition 1.1. Let $X$ and $Y$ be two normed linear spaces and $L : X \rightarrow Y$, where $X \subseteq Y$ be an operator. Then the norm of $L$, denoted by $\|L\|$, is defined by

$$\|L\| := \sup_{\|f\| = 1} \|Lf\| = \sup_{0 < \|f\| \leq 1} \|Lf\|.$$ 

Definition 1.2. Let $X$, $Y$ be two linear spaces of real functions. The mapping $L : X \rightarrow Y$ is called a linear operator if

$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g), \text{ for every } f, g \in X, \alpha, \beta \in \mathbb{R},$$

and positive if

$$L(f) \geq 0, \text{ for every } f \in X \text{ with the property } f \geq 0.$$

Remark 1.3. We shall use the notation $L(f; x)$ or $(Lf)(x)$ to denote the argument of the function $Lf \in Y$.

1.2.2 Properties of positive linear operators

Proposition 1.4. For the positive linear operators the followings hold:

(i) A positive linear operator is monotone.

(ii) If $L$ is a positive linear operator, then for every $f \in X$ we have $|Lf| \leq L(|f|)$.

Proposition 1.5. (Hölder inequality for positive linear operators). Let $L : X \rightarrow Y$ be a positive linear operator and let $p, q > 1$ be real numbers such that $1/p + 1/q = 1$. Then

$$L(|f \cdot g|) \leq (L(|f|^{p}))^{1/p} \cdot (L(|g|^{q}))^{1/q}, \text{ for every } f, g \in X.$$

Remark 1.6. On putting $p = q = 2$ in the Hölder’s inequality the following important inequality, known as the Cauchy-Schwarz inequality for positive linear operators is obtained

$$|L(f \cdot g; x)| \leq \sqrt{L(f^{2}; x)} \cdot \sqrt{L(g^{2}; x)}.$$

1.3 Different types of positive linear operators

Example 1.7. (Bernstein operators): Let $n \geq 1$ and $f \in C[0, 1]$. The Bernstein operators $B_n : C[0, 1] \rightarrow C[0, 1]$ are defined by

$$B_n(f, x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f \left( \frac{k}{n} \right), \quad x \in [0, 1].$$

Each $B_n(f)$ is a polynomial of degree not greater than $n$. 

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Example 1.8. (Bernstein-Chlodowsky operators): let $D \subset C(I)$ be a linear space of continuous real functions defined on $I = [0, \infty)$. Then operators $C_n : D \to C(I)$ defined by

$$C_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} \left( \frac{x}{\beta_n} \right)^k \left( 1 - \frac{x}{\beta_n} \right)^{n-k} f \left( \frac{k}{n} \beta_n \right),$$

for $0 \leq x \leq \beta_n$ and $C_n(f; x) = f(x)$, for $x > \beta_n$, where $(\beta_n)_{n \in \mathbb{N}}$ is a sequence of positive real numbers having the properties

$$\lim_{n \to \infty} \beta_n = \infty \text{ and } \lim_{n \to \infty} \frac{\beta_n}{n} = 0,$$

are called Bernstein-Chlodowsky operators, introduced by I. Chlodowsky [27] in 1937.

Example 1.9. (Bernstein-Kantorovich operators): The Bernstein polynomials are not suitable to approximate Lebesgue integrable functions, or in other words, general discontinuous functions. Since the space $C[0, 1]$ is dense in $L^p[0, 1]$ with respect to the natural norm $\| f \|_p := \left( \int_0^1 |f(t)|^p \, dt \right)^{1/p}$ ($f \in L^p[0, 1]$). So, by replacing $f(k/n)$ in the definition of Bernstein polynomial by an integral mean of $f(x)$ over a small interval around $k/n$, we may obtain better results. To approximate Lebesgue integrable functions on the interval $[0, 1]$, Kantorovich [59] introduced modified Bernstein polynomials as

$$K_n(f; x) = (n + 1) \sum_{k=0}^{n} p_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) \, dt,$$

where

$$p_{n,k}(x) = \binom{n}{k} x^n (1 - x)^{n-k}, \quad 0 \leq x \leq 1.$$

Each $K_n(f; x)$ is a polynomial of degree not greater than $n$ and $K_n$ is a positive linear operator from $L^p[0, 1]$ (and, in particular from $C[0, 1]$) into $C[0, 1]$.

Example 1.10. (Bernstein-Schurer operators): For $f : S_{m,\ell} : C[0, \ell + 1] \to C[0, 1]$ the operators $S_{m,\ell}(f; x)$ defined as

$$S_{m,\ell}(f; x) = \sum_{k=0}^{m+\ell} \binom{m+\ell}{k} x^k (1 - x)^{m+\ell-k} f \left( \frac{k}{m} \right), \quad x \in [0, 1],$$

where $\ell$ be fixed in $\mathbb{N}$ and $m \in \mathbb{N}$.

Example 1.11. (Bernstein-Stancu operators): Let $0 \leq \alpha \leq \beta$ be real numbers. For $n \geq 1$, the relation

$$p_n^{(\alpha, \beta)}(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} f \left( \frac{k + \alpha}{n + \beta} \right),$$

defines the Bernstein-Stancu operators $P_n^{(\alpha, \beta)} : C[0, 1] \to C[0, 1]$, introduced by D.D. Stancu [115] in 1969.
Example 1.12. (Bleimann-Butzer and Hahn operators): For \( x \geq 0, n \in \mathbb{N} \), the operators \( B_n(f)(x) \) defined by

\[
B_n(f; x) = \frac{1}{(1+x)^n} \sum_{k=0}^{n} \binom{n}{k} f \left( \frac{k}{n - k + 1} \right) x^k.
\]

Example 1.13. (Durrmeyer operators): To approximate Lebesgue integrable functions on the interval \([0, 1]\) Durrmeyer introduced the integral modification of the well known Bernstein polynomials. In 1981 M. M. Derriennic [29] first studied these operators in details. The Durrmeyer operators \( D_n \) are defined as

\[
D_n(f; x) = (n + 1) \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t)dt, \quad x \in [0, 1],
\]

where the Bernstein basis function is defined by \( p_{n,k} = \binom{n}{k} x^k (1-x)^{n-k} \).

Example 1.14. (Favard-Szász operators):[121] Let \( \mathbb{N} \) denote the set of natural numbers and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Let \( f \) be real-valued function defined on the closed interval \([0, 1]\). Then for each \( x \in [0, \infty), n \in \mathbb{N} \) and \( m \in \mathbb{N}_0 \), we have

\[
S_n(f; x) = \sum_{m=0}^{\infty} \frac{e^{-nx}(nx)^m}{(m)!} f \left( \frac{m}{n} \right). \quad (1.3.3)
\]

Example 1.15. (Szász-Mirakjan operators): For \( I = [0, \infty) \) the operators \( S_n : D \to C(I) \) defined by

\[
S_n(f; x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f \left( \frac{k}{n} \right), \quad x \in [0, \infty), \quad n \geq 1,
\]

are called Szász-Mirakjan operators (where \( D \subset C(I) \) be a linear space of continuous real functions defined on \( I \)). They were introduced by G. Mirakjan [81] in 1941 (some authors spell this name as Mirakyan) and later studied by J. Favard [36] in 1944 and by O. Szász [120] in 1950. The domain of definition of \( S_n \) is the set of all functions \( f(x) = \mathcal{O}(e^{\alpha x \ln x}), \quad \alpha > 0 \), proved by T. Hermann in [47].

Example 1.16. (Szász-Mirakjan-Bernstein operators):[121] Let \( \mathbb{N} \) denote the set of natural numbers and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Let \( f \) be real-valued function defined on the closed interval \([0, 1]\). Then for each \( x \in [0, 1], n \in \mathbb{N} \) and \( m \in \mathbb{N}_0 \), we have

\[
E_n(f; x) = \sum_{m=1}^{\infty} \frac{e^{-nx}(nx)^{m-1} m^{\alpha_n}}{(m-1)!} \sum_{k=0}^{m^{\alpha_n}} \binom{m^{\alpha_n}}{k} x^k (1-x)^{m^{\alpha_n}-k} f \left( \frac{k}{m^{\alpha_n}} \right),
\]

where \( \alpha_n \) is a non decreasing sequence.

For more examples the reader is referred to [48].
1.4 Quantitative approximation

1.4.1 Ditzian-Totik modulus of smoothness

In 1911, D. Jackson introduced the modulus of continuity in his PhD thesis [54]. In 1987, Ditzian and Totik introduced another modulus of smoothness which is found a better tool to study the rate of approximation, inverse theorems and embedding theorems (see [31], p. 1-4). The Ditzian-Totik modulus of smoothness is given by

$$
\omega_\nu^r(f, \delta)_p = \sup_{0 < h \leq \delta} \| \Delta_{h, \varphi}^r f \|_{L^p}
$$

(1.4.1)

where the function $\varphi(x)$ and the interval in question are related to the problem at hand and $\Delta_{h, \varphi}^r f$ denotes the $r$-th order of forward difference of $f$ with step length $h\varphi$.

\textbf{Remark 1.17.} A vital feature of (1.4.1) is that the increment $h\varphi(x)$ varies with $x$. For $\varphi(x) \equiv 1$, (1.4.1) is reduced to the classical modulus.

1.4.2 Lipschitz class of functions

Let $M > 0$, $0 < \nu \leq 1$ and $f \in C[0, \infty)$. Then the Lipschitz class $Lip_M(\nu)$ is defined by

$$
Lip_M(\nu) = \{ f \in C[0, \infty) : |f(\zeta_1) - f(\zeta_2)| \leq M |\zeta_1 - \zeta_2|^\nu \ (\zeta_1, \zeta_2 \in [0, \infty)) \}. 
$$

(1.4.2)

1.4.3 Modulus of smoothness of first and second order

The first and second order moduli of smoothness of $f \in C[a, b]$ are defined by

$$
\omega_1(f, \delta) = \sup \{ |f(x + h) - f(x)| : x, x + h \in [a, b], \ 0 \leq h \leq \delta \} 
$$

(1.4.3)

$$
\omega_2(f, \delta) = \sup \{ |f(x + h) - 2f(x) + f(x - h)| : x, x \pm h \in [a, b], \ 0 \leq h \leq \delta \}, 
$$

(1.4.4)

or, we can write it as

$$
\omega_1(f, \delta) := \sup_{0 < h \leq \sqrt{\delta}} \sup_{x \in [a, b]} |f(x + h) - f(x)| \ \text{or} 
$$

$$
\omega_2(f, \delta) := \sup_{x, y \in [a, b]} \sup_{|x - y| \leq \delta} |f(x) - f(y)|, 
$$

(1.4.5)

$$
\omega_2(f, \delta) := \sup_{0 < h \leq \sqrt{\delta}} \sup_{x \in [a, b]} |f(x + h) - 2f(x) + f(x - h)|. 
$$

(1.4.6)
where $\delta \geq 0$.

$\omega_1 = \omega$ inherits its name from the first part of the following proposition.

**Proposition 1.18.** Let $f \in C[a, b]$ and $\delta > 0$.

(i) If $f$ is uniformly continuous on $(a, b)$, then it is necessary and sufficient that $\lim_{\delta \to 0} \omega(f, \delta) = 0$.

(ii) For every $\delta > 0$ we have

$$|f(y) - f(x)| \leq \left(1 + \frac{|y - x|}{\delta}\right)\omega(f, \delta), \quad (1.4.7)$$

and

$$|f(y) - f(x)| \leq \left(1 + \frac{(y - x)^2}{\delta^2}\right)\omega(f, \delta). \quad (1.4.8)$$

(iii) The following implication holds: $f \in \text{Lip}_M\alpha$ if $\omega(f, \delta) \leq M\delta^\alpha$, where $0 < \alpha \leq 1$ and $M > 0$.

**Definition 1.19.** For $k \in \mathbb{N}$, $\delta \in \mathbb{R}^+$ and $f \in C[a, b]$ the modulus of smoothness of order $k$ is defined by

$$\omega_k(f, \delta) := \sup\{|\Delta_k^h f(x)| : 0 \leq h \leq \delta, x, x + kh \in [a, b]\}.$$ 

where $\Delta_k^h f$ is the $k$-th order of forward difference of $f(x)$ with step length $h$.

**Remark 1.20.** For clarity sometimes we will write $\omega_k(f, \delta; [a, b])$. It is obvious that for $\delta \geq \frac{b-a}{k}$ one has $\omega_k(f, \delta) = \omega_k(f, \frac{b-a}{k})$.

### 1.4.4 $K$-functionals and their relationship with the moduli of smoothness

Another important tool to measure the accuracy of approximation of a function by positive linear operators is the Peetre’s $K$-functional. It was introduced by J. Peetre [102] in 1968. It is defined as follows.

**Definition 1.21.** Let $f \in C[a, b]$ and $\delta \geq 0$. Then the Peetre’s $K$-functional of $f$ of order $s$ is defined by

$$K_s(f; \delta) = \inf\{|f - g|_\infty + \delta \cdot \|g^{(s)}\|_\infty : g \in C^s[a, b]\}. \quad (1.4.9)$$

where $s \geq 1$.

In the following lemma we collect some important properties of $K_s(f; \cdot)$ [18, 30, 43, 113].
Lemma 1.22. (referred to the Proposition 3.2.3 in [14]) Let $K_s(f, \cdot)$ be defined as in (1.4.9). Then

(i) the mapping $K_s(f, \delta) : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous especially at $\delta = 0$, i.e.,

$$\lim_{\delta \to 0^+} K_s(f, \delta) = 0 = K_s(f, 0).$$

(ii) for each $f \in C[a, b]$ the mapping $K_s(f, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is monotonically increasing and concave function.

(iii) for arbitrary $\lambda, \delta \geq 0$, and fixed $f \in C[a, b]$, one has the inequality.

$$K_s(f, \lambda \cdot \delta) \leq \max \{1, \lambda\} \cdot K_s(f, \delta).$$

(iv) for arbitrary $f_1, f_2 \in C[a, b]$ we have $K_s(f_1 + f_2, \delta) \leq K_s(f_1, \delta)K_s(f_2, \delta)$, $\delta \geq 0$.

(v) for each $\delta \geq 0$ fixed, $K_s(\cdot, \delta)$ is a seminorm on $C[a, b]$, such that

$$K_s(f, \delta) \leq \|f\|_\infty,$$

for all $f \in C[a, b]$.

The $K$-functionals and the moduli of smoothness are connected as in the following theorem [57].

Theorem 1.23. There exist constants $C_1$ and $C_2$, depending only on $s$ and $[a, b]$ such that

$$C_1 \cdot \omega_s(f, \delta) \leq K_s(f, \delta^s) \leq C_2 \cdot \omega_s(f, \delta),$$  \hspace{1cm} (1.4.10)

for all $f \in C[a, b]$ and $\delta > 0$.

In the above theorem, the sharp constants are not determined in general but the case of $s = 1, 2$ [103]. From applications point of view, the case of $s = 2$ is sufficient for us. For $s = 2, \delta > 0$, the Peetre’s $K$-functional is defined by

$$K_2(f, \delta) = \inf_{h \in C^2[a, b]} \{ \|f - h\| + \delta \|h''\| \},$$  \hspace{1cm} (1.4.11)

where $C^2[a, b] = \{ h \in C[a, b] : h', h'' \in C[a, b] \}$.

Details can be found in [119].
1.5 Functions on unbounded interval

We denote the interval \([0, \infty)\) by \(\mathbb{R}^+\) and define the following classes of functions.

\(C_B(0, \infty)\): the space of all bounded and continuous functions on \(\mathbb{R}^+\) equipped with the norm

\[
\| g \|_{C_B(\mathbb{R}^+)} = \sup_{x \in \mathbb{R}^+} |g(x)|. \tag{1.5.1}
\]

and the space

\[C_B^2(\mathbb{R}^+) = \{ g \in C_B(\mathbb{R}^+) : g', g'' \in C_B(\mathbb{R}^+) \} \tag{1.5.2}\]

with the norm

\[
\| g \|_{C_B^2(\mathbb{R}^+)} = \| g \|_{C_B(\mathbb{R}^+)} + \| g' \|_{C_B(\mathbb{R}^+)} + \| g'' \|_{C_B(\mathbb{R}^+)}. \tag{1.5.3}
\]

1.6 Weighted space of functions

In this section we present the weighted space of functions. The weight function will be denoted by \(\rho(x) = 1 + x^2\). Let \(B_{x^2}[0, \infty)\) be the linear space of all functions \(h\) satisfying the condition \(|h(x)| \leq K_h(1 + x^2)\), where \(K_h\) is a constant connected with \(h\). We denote the subspace of all continuous functions of \(B_{x^2}[0, \infty)\) by \(C_{x^2}[0, \infty)\). Also, we denote by \(C^*_x[0, \infty)\), the subclass of \(C_{x^2}[0, \infty)\) of those functions \(h\) for which \(\lim_{x \to \infty} \frac{h(x)}{1 + x^2}\) is finite, say \(k\). The norm on the space \(C^*_x[0, \infty)\) is defined by

\[
\| h \|_{x^2} = \sup_{x \in [0, \infty)} \frac{|h(x)|}{1 + x^2}. \tag{1.6.1}
\]

We shall denote this space by \(Q^k_x(\mathbb{R}^+)\).

1.7 Korovkin-type approximation

In 1953, P.P. Korovkin gave an easy and elegant criterion for the uniform convergence of a sequence of positive linear operators. It gave a real impetus to the development of approximation theory. The criterion guarantees the uniform convergence of a sequence of positive linear operators on \([a, b]\) provided it converges uniformly for three functions \(1, t, t^2\) on \([a, b]\). Thus it reduces a considerable amount of work for the process of testing the
uniform convergence of a sequence of positive linear operators on \([a, b]\). The trigonometric version of this criterion has also been obtained [64, 65].

The discovery of this useful criterion attracted the attention of many mathematicians to obtain its analogues in many other generalized spaces: abstract Banach spaces, function spaces, Banach algebras etc. The applications of these became so significant that today it is commonly known as the Korovkin-type theory. It finds applications in several areas such as functional analysis, real analysis, harmonic analysis, measure theory and probability theory, summability theory and in partial differential equations. We remark here that the Weierstrass approximation theorem [122], can be obtained from the Korovkin’s theorem.

In the following we state the Bohman-Korovkin theorem.

**Theorem 1.24.** Let \(P_n : C[a, b] \to C[a, b]\) be a sequence of positive linear operators. If \(\lim_{n \to \infty} P_n f = f\), for \(f(t) = 1, t, t^2\) uniformly on \([a, b]\), then \(\lim_{n \to \infty} P_n f = f\) uniformly on \([a, b]\) for every \(f \in C[a, b]\).

We point out that from the Weierstrass theorem it is possible to obtain a special version of Korovkin’s theorem which involves only positive linear operators \(P_n, n \geq 1\), such that \(P_n(C([0, 1])) \subset B([0, 1])\) for every \(n \geq 1\). This special version will be referred to as the restricted version of the Korovkin’s theorem.

**Theorem 1.25.** The restricted version of Korovkin’s theorem and the Weierstrass approximation theorem are equivalent.

The trigonometric version of the Korovkin’s theorem is stated below [65].

**Theorem 1.26.** Let \(f\) be a bounded, \(2\pi\)-periodic and continuous function on the interval \([a, b]\). Let \(\alpha_n(x), \beta_n(x)\) and \(\gamma_n(x)\) be three sequences converging uniformly to zero in the interval \([a, b]\). If the three conditions

\[
P_n(1; x) = 1 + \alpha_n(x),
\]

\[
P_n(\sin t; x) = \sin x + \beta_n(x)
\]

\[
P_n(\cos t; x) = \cos x + \gamma_n(x),
\]

are satisfied for the sequence of positive linear operators \(P_n(f; x)\), then the sequence \(P_n(f; x)\) converges uniformly to the function \(f\) in the interval \([a, b]\).
1.8 Quantum calculus

The $q$-calculus is the generalization as well as the modification of the usual calculus. Recently the $q$-calculus has been applied in the approximation theory. It finds important applications in various branches of Mathematics such as Number Theory, Combinatorics, Orthogonal Polynomials, Mechanics and Theory of Relativity.

We are familiar with the derivative of a function $f$ at $x = x_0$ in the usual calculus which is obtained as the limit $x$ approaches $x_0$ of the quotient

$$\frac{f(x) - f(x_0)}{x - x_0},$$

whenever the limit exists. When we substitute $x = qx_0$, where $q > 0$ is any fixed number ($\neq 1$) and do not consider the limit, then we enter into the theory of quantum calculus. Most of the notions of the usual calculus have been rephrased in the language of the $q$-calculus. A generalization of the $q$-calculus with one more parameter has also been introduced. It is known as the $(p, q)$-calculus.

1.8.1 Rudiments of $q$-calculus

The application of $q$-calculus has accelerated research in approximation theory. It has produced generalizations and modifications of a number of well known operators. The classical Bernstein polynomials (1.3.1) were the first to meet their $q$-companions and after that the use of the $q$-calculus in approximation theory grew very rapidly. The $q$-analogues find significant applications in many areas such as computer-aided geometric design, numerical analysis and solutions of differential equations. In the year of 1987, A. Lupaş [72] gave the first $q$-analogue of the classical Bernstein polynomials. G.M. Phillips obtained another $q$-analogue of the same in 1997. This attracted many researchers to construct new operators of well known operators using the $q$-calculus. For example, $q$-analogue of Baskakov and Baskakov-Kantorovich operators [74], Bernstein-Kantorovich operators in [108], Stancu-Beta operators [10, 91]; Szász-Kantorovich operators [76]; Bleimann-Butzer and Hahn operators [9, 35]; and generalized Bernstein-Schurer operators [92] are a few to mention. Below we review the basics of $q$-calculus [45, 58].
The $q$-analogues of an integer, the factorial and the binomial coefficient are defined by

$$ [n]_q := \begin{cases} \frac{1-q^n}{1-q}, & \text{if } q \in \mathbb{R}^+ \setminus \{1\}, \text{ for } n \in \mathbb{N} \text{ and } [0]_q = 0, \\ n, & \text{if } q = 1, \end{cases} $$

$$ [n]_q! := \begin{cases} [n]_q[n-1]_q \cdots [1]_q, & n \geq 1, \\ 1, & n = 0, \end{cases} $$

$$ \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!} $$

respectively.

The following recurrence relations hold:

$$ \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \quad \text{and} $$

$$ \begin{bmatrix} n \\ k \end{bmatrix}_q = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q. $$

The $q$-companion of $(1+x)^n$ is defined by

$$ (1+x)_q^n := \begin{cases} (1+x)(1+qx) \cdots (1+q^{n-1}x), & n = 1, 2, 3, \cdots \\ 1, & n = 0. \end{cases} $$

For the common Pochhammer symbol the $q$-analogue, also called a $q$-shifted factorial, is defined by

$$ (x; q)_0 = 1, \quad (x; q)_n = \prod_{j=0}^{n-1} (1-q^jx), \quad (x; q)_\infty = \prod_{j=0}^{\infty} (1-q^jx). $$

The $q$-analogue of the Gauss binomial formula is given by

$$ (x+a)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} a^k x^{n-k}. $$

The Heine’s binomial formula:

$$ \frac{1}{(1-x)_q^n} = 1 + \sum_{k=0}^{\infty} \frac{[n]_q[n+1]_q \cdots [n+k-1]_q}{[k]_q!} x^k. $$
The $q$-derivative $D_q f$ of a function $f$ is given by

$$(D_q f)(x) := \frac{f(x) - f(qx)}{(1-q)x}, \text{ if } x \neq 0$$

and $(D_q f)(0) := f'(0)$, provided $f'(0)$ exists.

Note that $\lim_{q \to 1} D_q f(x) = \frac{df(x)}{dx}$, if $f$ is differentiable. Clearly, for $n \geq 1$, we can show that

$$D_q (1 + x)^n_q = [n]_q (1 + qx)^{n-1}_q \text{ and } D_q \left\{ \frac{1}{(1+x)_q^n} \right\} = -\frac{[n]_q}{(1+x)_{q+1}^n}.$$

The $q$-derivative of a function exhibits the following properties.

(i) The $q$-derivative of a function is a linear operator.

(ii) The $q$-derivative of a product at $x \neq 0$ is

$$D_q \left( f(x)g(x) \right) = f(x)D_q g(x) + D_q f(x)g(qx),$$

$$= f(qx)D_q g(x) + D_q f(x)g(x).$$

(iii) The Leibnitz rule for the $q$-derivative is defined as

$$D_q^{(n)} (fg)(x) = \sum_{k=0}^{n} \binom{n}{k}_q D_q^{(k)} f(q^{n-k}x) D_q^{(n-k)} g(x).$$

(iv) $D_q \left\{ \frac{f(x)}{g(x)} \right\} = \frac{g(qx)D_q f(x) - f(qx)D_q g(x)}{g(x)g(qx)}$.

There are defined two $q$-analogues of the classical exponential function $e^x$:

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!} = \frac{1}{(1 - (1-q)x)_q^\infty},$$

$$E_q(x) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{x^k}{[k]_q!} = (1 + (1-q)x)_q^\infty.$$

We note the following relations for the $q$-exponential function.

(i) $D_q e_q(x) = e_q(x), \quad D_q E_q(x) = E_q(qx)$,

(ii) $e_q(x)E_q(-x) = E_q(x)e_q(-x) = 1$. 

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1.8.2 Rudiments of \((p, q)\)-calculus

Recently, Mursaleen et al. [85, 86, 87, 93, 96, 97] have applied \((p, q)\)-calculus in Approximation Theory. They constructed and investigated approximating properties of a number of operators. The \((p, q)\)-integers were introduced to generalize or unify several forms of \(q\)-oscillator algebras well known in the early physics literature related to the representation theory of single parameter quantum algebras [22]. We recollect some basic notions of the \((p, q)\)-calculus as follows.

The \((p, q)\)-integer \([n]_{p,q}\), and the \((p, q)\)-factorial are defined by

\[
[n]_{p,q} := \frac{p^n - q^n}{p - q}, \quad n = 0, 1, 2, \ldots, \quad 0 < q < p \leq 1,
\]

\[
[n]_{p,q}! = \prod_{k=1}^{n} [k]_{p,q}!, \quad n \geq 1, \quad [0]_{p,q}! = 1
\]

respectively.

The \((p, q)\)-binomial coefficients are defined as

\[
\binom{n}{k}_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}, \quad 0 \leq k \leq n.
\]

The \((p, q)\)-companion of binomial expansion \((ax + by)^n\) is defined as

\[
(ax + by)^n_{p,q} := \sum_{k=0}^{n} \binom{n}{k}_{p,q} p^{(n-k)}q^{(k)}a^{n-k}b^ky^k,
\]

\[
(x - y)^n_{p,q} := (x - y)(px - qy)(p^2x - q^2y) \cdots (p^{n-1}x - q^{n-1}y).
\]

Let \(m\) and \(n\) be two non-negative integers. Then the following assertion is valid

\[
(x - y)^{m+n}_{p,q} := (x - y)^m_{p,q}(p^mx - q^my)^n_{p,q}.
\]

The derivative of a function \(f\) in the \((p, q)\)-calculus, denoted by \(D_{p,q}f\), is defined by

\[
(D_{p,q}f)(x) := \frac{f(px) - f(qx)}{(p - q)x}, \quad x \neq 0, \quad (D_{p,q}f)(0) := f'(0),
\]
if \( f \) is differentiable at 0. The \((p, q)\)-derivative operator obeys the following product and quotient rules

\[
D_{p,q}(f(x)g(x)) := f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x),
\]

\[
D_{p,q}\left(\frac{f(x)}{g(x)}\right) := \frac{g(qx)D_{p,q}f(x) - f(px)D_{p,q}g(x)}{g(px)g(qx)},
\]

and

\[
D_{p,q}\left(\frac{f(x)}{g(x)}\right) := \frac{g(px)D_{p,q}f(x) - f(px)D_{p,q}g(x)}{g(px)g(qx)}.
\]

We observe that the two integers, the \( q \)-integers and the \((p, q)\)-integers are not same. The \((p, q)\)-integer can not be obtained from the \( q \)-integer by just substituting \( q \) for \( \frac{q}{p} \). But by putting \( p = 1 \), all the notions of the \((p, q)\)-integers reduce to those of the \( q \)-integers. So the \((p, q)\)-calculus is a consequence of the \( q \)-calculus. We refer to [55, 62, 75, 111, 112] for more details of \((p, q)\)-calculus.

### 1.9 Approximation of Volterra operator

#### 1.9.1 Volterra operator

The Volterra operators were introduced by the Italian mathematician and physicist V. Volterra. Later, Romanian mathematician T. Lalescu studied these operators in his PhD thesis titled *Sur le èquation de Volterra* in 1908. The Volterra operators correspond to the Volterra integral equations which find important applications in Physics, Acturial Science, Finance, Demography and in several branches of Pure and Applied Mathematics. They are used to provide mathematical models of several phenomena. For example, all viscoelastic models can be represented through a Volterra integral equation connecting *stress* and *strain*:

\[
\epsilon(t) = \frac{\sigma(t)}{E_{\text{inst, creep}}} + \int_0^t k(t - \xi)\sigma'(\xi)d\xi,
\]

where \( \sigma(t) \) is the stress the body undergoes on application of an external force, \( \epsilon(t) \) is the strain which measures the amount of deformation between the particles of the body with respect a reference point, \( E_{\text{inst, creep}} \) is the instantaneous elastic moduli for creep, \( k(t) \) is the creep function. In demography the Volterra integral equations provide models for
population dynamics [78]. Also, The Volterra integral equations are a subject of study in itself. In the simplest form the Volterra operator is defined by

\[ V(f)(x) := \int_0^x f(t) \, dt. \]

The Volterra operators have been considered and studied between the same spaces \( L^p([0, 1]) (p \geq 1) \) or \( C([0, 1]) \). In view of the Ascoli’s theorem ([63], p. 236), it is a compact operator in these cases. The same holds when it is considered from the spaces \( L^1([0, 1]) \) to \( L^p([0, 1]) \) with a finite value of \( p \). The interesting case arises when it is considered between the spaces \( L^1([0, 1]) \) and \( C([0, 1]) \). In the latter case the Volterra operator is noncompact. But the operator still has some interesting and well behaving properties. Also we note that the Volterra operator is an isometry.

### 1.9.2 Definitions and examples

In the following we define certain terms for a bounded linear operator from a Banach space \( X \) to a Banach space \( Y \).

**Definition 1.27.** An operator \( T : X \to Y \) is said to be weakly compact if it transforms a neighbourhood of 0 into a weakly relatively compact set.

**Definition 1.28.** An operator \( T : X \to Y \) is said to be completely continuous if every weakly convergent sequence in \( X \) is mapped to a strongly convergent sequence in \( Y \).

**Definition 1.29.** An operator \( T : X \to Y \) is said to be strictly singular if for each \( \epsilon > 0 \) and every infinitely dimensional subspace \( E \) of \( X \) there exists a vector \( v \) in the unit sphere of \( E \) such that \( \|T(v)\| \leq \epsilon \). \( T \) is said to be finitely strictly singular if for each \( \epsilon > 0 \) there exists \( N_\epsilon \geq 1 \) such that for every subspace \( E \) of \( X \) with \( \dim(E) \geq N_\epsilon \), there exists a vector \( v \) in the unit sphere of \( E \) such that \( \|T(v)\| \leq \epsilon \).

In operator theory the notions of strictly singular and finitely strictly singular have been intensively studied [23, 37, 69, 70, 106]. It is well known that compactness \( \iff \) finite strict singularity \( \iff \) strict singularity. The inverse implications are not true as justified in the following examples [80, 106].

**Example 1.30.** The inclusion map \( l_p \hookrightarrow l_q, 1 \leq p < q \leq \infty \) is finitely strictly singular but not compact.
Example 1.31. Consider the above spaces for $1 < p < q < \infty$. Let $U_n$ be a subspace of $l_p$ consisting of the sequences whose supports lie between $2^n$ and $2^{n+1}$. Consider the subspace $V_n$ of $U_n$ generated by the $2^n$-dimensional analogue of the Rademacher functions and $G_n$ be its complement spanned by the rest of the $2^n$-dimensional version of Walsh functions. Then there exist constants $\alpha$, $\beta$ and $c_n > 0$ such that for every $v \in V_n$,

$$\beta c_n \|v\|_p \geq \|v\|_q \geq \alpha c_n \|v\|_p.$$ 

Moreover, the projections in $U_n$ onto $V_n$ along $G_n$ are uniformly bounded in the $l_p$-norm and the $l_q$-norm both. Define the operator $T : l_p \rightarrow l_q$ as follows: For $v \in V_n$, put $Tv = c_n v$ and for $v \in G_n$ put $Tv = v$. Extending it to a linear and bounded operator from $l_p$ to $l_q$ provides us the strictly singular operator which is not finitely strictly singular.

A closely related with the concept of being “compact” is the concept of being of “finite rank”. An operator is said to be of finite rank if its range is a finite dimensional space. For a long time there were speculations that the class of compact operators and that of finite rank operators are same. It was known as approximation problem in Functional Analysis. It was a long standing problem for almost four decades untill Swedish mathematician P. Enflo solved it negatively [102]. The techniques used by Enflo in constructing the counter example proved very useful in solving several other long standing problems of Functional Analysis. The Enflo’s solution to the approximation property has been recorded as one of the twenty two major discoveries in Functional Analysis of the last century.