CHAPTER- 9

GENERATION AND PROPAGATION OF WATER WAVE ON A RUNNING STREAM IN PRESENCE OF AN ICE SHEET DUE TO BOTTOM DISTURBANCE
Generation and propagation of water wave on a running stream in presence of an ice sheet due to bottom disturbance

The problem of water wave generation by a transient disturbance at the bottom or at the upper surface of the ocean is another interesting problem in water wave propagation. In general, waves at the surface of the ocean may be created or generated by various types of disturbances either at the upper surface or at the bottom e.g. sudden impact of objects, ocean surface displacement or deformation of the bottom floor, volcanic eruption, underwater landslides and several other naturally occurring phenomena. The wave generated may be of large wavelength compared to the depth of the ocean leading to tsunami waves. Once a wave is generated, it propagates throughout the ocean often with high amplitude and energy and impacts on the coastal region resulting catastrophic effects. The mathematical form of a disturbance at the upper surface or at the bottom may be modelled with the aid of generalised function. In general, the problem of water wave is formulated in mathematics as a non-linear problem which is apparently difficult to solve due to the non-linearity of the governing equations of motion. Within the framework of linearised theory these equations may be linearised in usual way by expanding the unknown function with respect to some small parameter characterising the geometry of the ocean floor upto certain order and then be reduced to a Cauchy-Poisson problem (cf. Debnath (1989)).

Over a few decades many scientists such as Lamb (1932), Stoker (1957),

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Keller (1949) and Ursell (1961, 1962) investigated the mathematical and physical aspects of problems of wave generation either by various types of disturbances in the form of depression or impulse at the upper surface or due to a fault on the bottom topography. A systematic description of this phenomenon is given in the work of Dutykh and Dias (2007). They considered the wave motion generated by a circular or rectangular fault at the bottom of the ocean and the bottom disturbance was modelled by a mathematical function of space coordinate and time. In the mathematical analysis, the time dependent part of the bottom variation was separated from spatial coordinate and for different time profiles, the elevation of the upper surface and the pressure distribution at the bottom was obtained along with numerical results. Maiti and Mandal (2007) investigated the problem of generation of water wave due to bottom disturbance when the upper surface was covered by a thin elastic ice sheet. The problem was solved by using combined Laplace-Fourier transform. The time profile of the dynamic source model of the bottom disturbance was taken as the Dirac delta function and both steady state component and transient part of the wave system was obtained by using Airy wave function. Mandal and Chakraborty (2013) investigated the problem of generation of water wave when the disturbance was applied at the bottom of a beach slopping at an angel and then solved by using Laplace and Mellin transform. The depression of the upper surface was obtained in terms of multiple infinite integrals. The study of wave generation problems becomes more interesting if there is a uniform running stream $U$ in a particular direction throughout the water region. Debnath and Rosenblatt (1968) studied the problem of generation and propagation of water wave in presence of a
shallow running stream and solved by using the integral transform technique. They also showed that depending on the critical velocity of the running stream, one or more component of the wave system may exist. Debnath and Basu (1978) considered the problem of generation of tsunami wave in a shallow running stream over dynamic bottom topography and by using integral transform in the mathematical solution, they obtained the depression of the upper surface in terms of infinite integrals. These integrals were evaluated by using classical stationary phase method which showed that there exist three different types of wave depending on the critical velocity of the running stream. These problems ultimately lead to the studies of Cauchy-Poisson problem in two or three-dimension. Mathematical analysis of the critical velocity component of the running stream reveals some important results.

In this chapter, we formulate an unsteady boundary value problem in a finite depth water region where the incompressible Euler equation for potential flow can be linearised. Apart from free surface, here the upper surface of the water is covered by a thin ice sheet which is often observed in the polar region. The ice sheet is modelled as a thin elastic plate and the fluid flows with a uniform velocity $U$ in a certain direction throughout the region (cf. Debnath and Basu (1978)). Apart from considering a static bottom undulation (cf. Okada (1985)), we consider that the wave motion is generated by transient bottom movement which naturally occurs due to some fault or deformation at the bottom. This bottom variation can be described by a regular mathematical function of space coordinate and time. The problem can be well-posed by using standard linear theory of water wave and then it is
solved by using combined Laplace and Fourier transform (cf. Erdelyi (1954)) with respect to time and horizontal spatial coordinate respectively. In the mathematical analysis a physical situation of bottom variation is approximately taken into account and we follow the same linearised model as described by Dutykh and Dias (2007). Without using the radiation condition of the motion the depression of the upper surface is obtained in terms of infinite integrals. These integrals are oscillatory in nature and can be approximated by using classical stationary phase method to incorporate the far field behaviour of the motion. This also leads to the transient component of the wave system. The dominant contribution of the wave motion comes from this approximate form and the pole of the integrand and the behaviour of the solution around these poles can be evaluated by using residue calculus method of complex function theory. The main emphasis in this chapter is given to analyse theoretical evolution of the effect of the running stream $U$ on the entire wave system in presence of the moving bottom and some conclusions are made.

1. Formulation of the problem

We consider the rectangular Cartesian coordinate system to describe the problem mathematically in which origin is at the free surface and the direction of the $y$-axis is taken vertically downwards into the water region. The upper surface of the ocean is covered by an ice sheet which is modelled as a thin elastic plate. In the undisturbed state of the liquid the upper surface and the bottom of the ocean remains horizontal, defined by $y = 0$ and $y = h$ respectively and the fluid flows with constant speed $U$ throughout water in the positive $x$-direction. For time $t > 0$, the bottom of the ocean moves in a specific manner which is described by $y = h + \zeta(x, t)$. From the resulting
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motion, the form of the elevation of the upper surface \( \eta(x,t) \) must be computed. By assuming that the fluid is incompressible, inviscid and the flow is irrotational, there exists a potential function \( \phi(x,y,t) \) which satisfies the two-dimensional Laplace’s equation

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad \text{in the fluid region:} \quad (1)
\]

Besides (1), the potential function \( \phi(x,y,t) \) must satisfy the kinematical and dynamical boundary conditions at the upper surface of the ocean

\[
\frac{\partial \phi}{\partial y} = \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \eta(x,t) \quad \text{on } y = 0, t > 0, \quad (2)
\]

\[
\left(D \frac{\partial^4}{\partial x^4} + 1 \right) \eta(x,t) - \frac{1}{g} \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \phi - \epsilon \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{on } y = 0, t > 0. \quad (3)
\]

Now (2) and (3) together can be written as

\[
\left(D \frac{\partial^4}{\partial x^4} + 1 \right) \frac{\partial \phi}{\partial y} - \frac{1}{g} \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \phi - \epsilon \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{on } y = 0, t > 0. \quad (4)
\]

Here, \( D = \frac{E h_0^3}{12(1-\nu^2) \rho g} \), \( \epsilon = \frac{\rho_0 h_0}{\rho} \) are flexural rigidity and the stiffness parameter of the ice sheet. \( E \) and \( \nu \) are the Young modulus and Poisson ratio of the elastic ice sheet. \( h_0 \) being the thickness and \( \rho_0, \rho \) are the density of ice and water respectively. Incorporating the time dependencies of the bottom variation, the boundary condition at the bottom of the ocean is given by

\[
\frac{\partial \phi}{\partial y} = \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \zeta(x,t) \quad \text{on } y = h, \quad (5)
\]

where \( h \) is the uniform finite depth below the mean free surface and \( \zeta(x,t) \) is a regular function representing the profile of the moving bottom. The initial conditions of the motion are prescribed as
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\[ \phi(x, 0, 0) = 0, \]
\[ \eta(x, 0) = 0, \]
\[ \zeta(x, 0) = 0. \] (6)

2. **Method of solution**

The wave motion described by (1) – (6) is formulated as an unsteady initial boundary value problem (IBVP) so it is convenient to use Laplace-Fourier transform. The Fourier transform and Laplace transform of a general function \( f(x, y, t) \) with respect to the spatial variable \( x \) and time \( t \) is given by

\[
\tilde{f}(k, y, s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \left( \int_{0}^{\infty} e^{-st} f(x, y, t) dt \right) dx
\] (7)

Assuming that the bottom profile function \( \zeta(x, t) \) and elevation of the upper surface \( \eta(x,t) \) have all the properties to compute Laplace and Fourier transform with respect to \( t \) and \( x \), the IBVP described by (1) – (6) is then reduced to

i. \[ \frac{d^2 \tilde{\phi}}{dy^2} - k^2 \tilde{\phi}(k, y, s) = 0, \quad \text{on} \quad 0 < y = h + \zeta(x, t), \]

ii. \[ \left(Dk^4 + 1 + \frac{\epsilon(s+ik)}{g}\right) \frac{d\tilde{\phi}}{dy} - \frac{(s+ik)^2}{g} \tilde{\phi} = 0 \quad \text{on} \quad y = 0, \]

iii. \[ \frac{d\tilde{\eta}}{dy} = (s + iUk) \tilde{\zeta}(k, s) \quad \text{on} \quad y = h, \]

iv. \[ (\tilde{\phi} - \epsilon \tilde{\phi}_y)(k, 0, 0) = 0, \]
\[ \tilde{\eta}(k, 0) = 0, \]
\[ \tilde{\zeta}(k, 0) = 0. \]

The potential function \( \phi(x, y, t) \) and the elevation of the upper surface \( \eta(x, t) \) can now be obtained by using inverse Fourier and Laplace integral formula

\[
\phi(x, y, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \left( \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \tilde{\phi}(k, y, s) ds \right) dk
\] (8)

\[
\eta(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \left( \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \tilde{\eta}(k, s) ds \right) dk
\] (9)

where
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\[
\phi(k, y, s) = \frac{(s+iuk)\xi(k,s)\left(Dk^4 + 1 + \frac{(s+iuk)^2}{g}\cosh k\gamma + \frac{(s+iuk)^2}{g}k\sinh k\gamma\right)}{\Delta(k)}
\]

\[
\eta(k, s) = \frac{(s+iuk)^2\xi(k,s)}{g\Delta(k)}
\]

and

\[
\Delta(k) = k\left(Dk^4 + 1 + \frac{(s+iuk)^2}{g}\right)\sinh kh + \frac{(s+iuk)^2}{g}\cosh kh.
\]

3. Depression of the upper surface

The depression of the upper surface can be obtained as

\[
\eta(x, t) = \frac{1}{(2\pi)^{\frac{3}{2}}i} \int_{-\infty}^{\infty} e^{-ikx} \left(\int_{c-i\infty}^{c+i\infty} \frac{(s+iuk)^2}{(s+iuk)^2 + b^2} \xi(k,s)e^{st} ds\right) dk
\]

where

\[
b^2 = \frac{gk(Dk^4 + 1)\sinh kh}{\cosh kh + \epsilon\sinh kh}
\]

Instead of static axisymmetric deformation at seabed (cf. Okada (1985)) of uniform finite depth water, we consider dynamic bottom boundary condition specified by (5) where \(\zeta(x, t)\) describing the profile of the moving bottom. The time dependent part can be separated from \(\zeta(x, t)\) apart from the spatial coordinate (cf. Dutykh and Dias (2007)) as

\[
\zeta(x, t) = \xi(x)T(t).
\]

There are two reasons for doing this. First of all, we can able to invert analytically the Laplace transform with time variable \(t\) and secondly, one can assess the bottom variation with respect to space coordinate and time separately by \(\xi(x)\) and \(T(t)\) such that various types of time dependency can be considered according to the physical source model available. Here, we take

\[
\xi(x) = e^{-\alpha|x-1|} \text{ and } T(t) = H(t)
\]
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where \( a > 0 \) and \( H(t) \) being the Unit step function. \( \xi(x) \) and \( T(t) \) are supposed to have all the properties to possess Fourier and Laplace transform respectively. Therefore, we obtain

\[
\xi(k) = \frac{2}{\sqrt{\pi}} \frac{a}{a^2+k^2} e^{ik} \quad \text{and} \quad T(t) = \frac{1}{s}
\]  

(13)

Using (13) in (10), we obtain

\[
\eta(x, t) = \frac{a}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-ik(x-1)}}{(a^2+k^2)(\cosh kh + e \sinh kh)} \left( \int_{c-i\infty}^{c+i\infty} \frac{(s+iUk)^2}{s(s+iUk)^2+b^2} e^{st} ds \right) dk
\]

(14)

The integral within bracket in (14)

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(s+iUk)^2}{s(s+iUk)^2+b^2} e^{st} ds
\]

is to be evaluated by using convolution theorem as

\[
L^{-1}(F(s)G(s)) = f * g = \int_{0}^{t} f(u)g(t-u)du.
\]

Therefore, we obtain,

\[
I = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{b}{b^2-k^2} e^{it(b-kU-k(x-1))} \frac{e^{it(b-kU-k(x-1))}}{b-kU} dk,
\]

\[
J = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{b}{b^2-k^2} e^{-it(b+kU+k(x-1))} \frac{e^{-it(b+kU+k(x-1))}}{b+kU} dk,
\]

\[
L = \frac{-1}{\pi} \int_{-\infty}^{\infty} \frac{k^2U^2}{(a^2+k^2)(\cosh kh + e \sinh kh)} \frac{e^{-ik(x-1)}}{b^2-k^2U^2} dk.
\]

(15)

It is convenient to write the depression of the upper surface \( \eta(x, t) \) as

\[
\eta(x, t) = I + J + L
\]

The nature of the wave motion and the depression of the upper surface is dependent on these integrals and therefore \( I, J \) and \( L \) are to be evaluated separately. As the integral \( J \) does not contribute anything to the wave motion due to its non-wave like form, therefore the dominant contribution to \( \eta(x, t) \) comes from the integral \( I \) and \( L \). The integral \( I \) is oscillatory in nature and
therefore it is to be evaluated for large $x$ and $t$ to determine the asymptotic approach of the wave system. The contribution to the integral $L$ comes from its poles at the points given by the following equations

\[
\begin{align*}
   b - kU &\equiv \frac{gk(Dk^4+1)\sinh kH}{\cosh kH + \epsilon \sinh kH} - kU = 0, \quad (16.a) \\
   b + kU &\equiv \frac{gk(Dk^4+1)\sinh kH}{\cosh kH + \epsilon \sinh kH} + kU = 0. \quad (16.b)
\end{align*}
\]

For computational purpose, we take the values of flexural rigidity and stiffness parameter $D = 0.1 \text{ cm}^4$, $\epsilon = 0.01 \text{ cm}$ of the ice sheet. The location of the poles of the integral $L$ can be found from fig-1, where the curve

\[
\frac{gk(Dk^4+1)\sinh kH}{\cosh kH + \epsilon \sinh kH}
\]

intersects the line $\pm U k$ at these points

\[
\begin{align*}
   k &= 0, \quad k = \sigma \\
   k &= 0, \quad k = -\sigma
\end{align*}
\]

respectively.
In all possible cases, it is observed that the point \( k = 0 \) is always present but the poles at \( k = \pm \sigma \) occurs only when (cf. Stoker (1957))

\[
U > U_c (= \sqrt{gh})
\]

Furthermore, in case of \( U = U_c \), it is observed that the lines \( \pm Uk \) becomes asymptote to the curve \( \frac{gk(Dk^4+1)\sinh kh}{\cosh kh+ek\sinh kh} \) and the wave system becomes degenerate.
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The dominant contribution to the integral $L$ comes from its poles at $k = \pm \sigma$ which are necessarily present only when $U > U_c$. Following Copson (1935), the integral $L$ can now be evaluated by using Cauchy’s residue theorem which states that for a complex function $f(z)$ having singularity at the points $a_1, a_2, a_3, \ldots, a_n$

$$\oint f(z)dz = 2\pi i \left[ \sum_n \text{Residue of } f(z) \text{ at } z = a_n \right] \quad (18)$$

Incorporating the condition given by (17) and by using (18), the integral $L$ is now of the form

$$L = iH(U - U_c)[\varphi_1(-\sigma) - \varphi_2(\sigma)] \quad (19)$$

where,

$$\varphi_1(-\sigma) = e^{i\sigma(x-1)} \sigma U \left[ \frac{1}{(\alpha^2+\sigma^2)(\cosh \sigma + \sigma \sinh \sigma)} \right]$$

$$\varphi_2(\sigma) = e^{-i\sigma(x-1)} \sigma U \left[ \frac{1}{(\alpha^2+\sigma^2)(\cosh \sigma + \sigma \sinh \sigma)} \right] \quad (20)$$

The remaining part of this section is focused on the evolution of the integral $I$ which has two parts:

a. The contribution of $I$ to the wave motion from its poles which is given by (16.a)

b. For large $x$ and $t$, the integral $I$ is to be evaluated by utilising the method of stationary phase so that $\frac{x}{t}$ remains finite.

Now, the stationary point of the integral $I$ can be obtained from the equation

$$\frac{d}{dk} \left[ b - kU - \frac{k(x-1)}{t} \right] = 0 \quad (21)$$

which gives

$$\frac{db}{dk} = U + \frac{(x-1)}{t} \quad (22)$$
It is reasonable to assume that for large \( x \) and \( t \) so that \( \frac{x}{t} \leq U_c \) and the (22) can be approximate as

\[
\frac{db}{dk} = U \tag{23}
\]

The location of the stationary point can be found from fig-2, which shows that the gradient of the curve \( \sqrt{\frac{gk(Dk^4+1)\sinh k\hbar}{\cosh k\hbar + \epsilon k_0 \sinh k\hbar}} \) decreases as \( k \) varies from \(-\infty\) to \( \infty \) and intersects the line \( y = U \) at a point \( k = k_0 \) only when the condition given by (17) holds. So the stationary point at \( k = k_0 \) exists for \( U > U_c \) along with another two poles of the integrand of \( I \) which are present at \( k = 0 \) and \( k = \sigma \) given by (16.a). If, however, \( U < U_c \) then the line \( y = U \) does not intersects the curve \( \sqrt{\frac{gk(Dk^4+1)\sinh k\hbar}{\cosh k\hbar + \epsilon k_0 \sinh k\hbar}} \) and therefore the stationary point \( k_0 \) do not occur.

Eventually it can be concluded that \( 0 < k_0 < \sigma \) which can also be verified from the above two figures. By using the classical method of stationary phase, the approximate form of the integral \( I \) can be obtained as

\[
I = I_{\text{Transient}} \cong H(U - U_c) \left( \frac{2\pi}{F(k_0)} \right)^{\frac{1}{2}} \frac{b_0}{(a^2 + k_0^2)(\cosh k_0 h + \epsilon k_0 \sinh k_0 h)} \frac{e^{it\left(k_0 - kU - k_0 \frac{(x-1)t}{4}\right)}}{b - k_0 U} \tag{24}
\]

where,

\[
b_0 = \sqrt{\frac{gk_0(Dk_0^4 + 1)\sinh k_0 h}{\cosh k_0 h + \epsilon k_0 \sinh k_0 h}}
\]

\[
F(k) = b - kU - k(x - 1) \frac{t}{t}
\]

It remains to calculate the polar contribution to the integral \( I \) which comes from its poles given by (16.a). The analysis is similar to the case of the integral \( L \). Assuming that the figure-1 is relevant and by using Cauchy’s residue theorem given by (18), we obtain
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\[ I = I_{polar} = iH(U - U_c)\varphi_3(\sigma) \]  \hspace{1cm} (25)

\[ \varphi_3(\sigma) = \frac{b(\sigma)e^{it(b(\sigma)-\sigma(U - U_c))}}{(a^2+\sigma^2)(\cosh h + \varepsilon \sinh h)} \left[ \frac{1}{kb-U} \right]_{k=\sigma} \]  \hspace{1cm} (26)

4. Observations and conclusions

The depression of the upper surface of the ocean given by (15) can now be written with its essential components. Therefore, we write

\[ \eta(x, t) = \eta_S + \eta_{Tr} \]  \hspace{1cm} (27)

where \( \eta_S \) is the steady state component and \( \eta_{Tr} \) is the transient part. Here the steady state component comes from the polar part of the integrals of (15) and it is given by (19) and (25) as

\[ \eta_S = I_{polar} + L = iH(U - U_c)(\varphi_3(\sigma) + \varphi_1(-\sigma) - \varphi_2(\sigma)) \]  \hspace{1cm} (28)

The transient part is obtained by using the classical method of stationary phase to the integral \( I \) and it is given by (24) as

\[ \eta_{Tr} = \eta_{Transient} \approx H(U - U_c) \left( \frac{2\pi}{iF'(k_0)} \right)^{1/2} \frac{p_0}{(a^2+k_0^2)(\cosh k_0 h + \varepsilon k_0 \sinh k_0 h)} e^{it(k_0-k_0 U-k_0(x-1)/t)} \cdot \left[ \frac{1}{b-k_0 U} \right]. \]  \hspace{1cm} (29)

Here \( \varphi_1(-\sigma), \varphi_2(\sigma) \) and \( \varphi_3(\sigma) \) are given by (20) and (26) respectively. It is observed that from (27)-(29), in the steady state condition of the wave system there are two waves downstream the origin with the wave number \( \sigma_1 \) and \( \sigma_2 \) respectively in the positive and negative \( x \) direction respectively. The wave motion become degenerate in case of \( U = U_c \).

Most of the analytical problems in linearised water wave theory are solved by using integral transform which may leads to complex integral solutions. Here the depression of the upper surface involving infinite integrals is being approximated by stationary phase method. These also give flexibility
for the numerical computation of the results for different time and spatial coordinate. However, this is not pursued here. Throughout the analysis, the dependence of the ultimate wave system on the critical value of the uniform stream \( U \) can be observed. In many practical situations it is really difficult to avail any information about the exact characteristic of the dynamic source model and therefore it is impossible to conclude which time profile will give the best description of the bottom variation. Here, we take the Heaviside step function \( H(t) \) to describe the time dependence \( T(t) \). The reason behind this can be stated as the actual disturbance at the surface of earthquake lasts for a very short interval of time and thus it is widely used in different experimental and theoretical studies regarding transient disturbances. Other time dependences such as instantaneous, linear and trigonometric profiles can also be considered according to physical scenario.