algebraic structure (1).

Let \( \Gamma : \Delta \) be valid in the class of all algebraic structure (1)-models.

Then \( v(\Gamma) \leq v(\Delta) \) holds for every member of that class with valuation function \( v \).

In particular, \( \Gamma : \Delta \) is valid in the Lindenbaum algebra \( F/R \) with the canonical valuation i.e., when \( \alpha \) is mapped to its equivalence class \([\alpha]\). Thus \([\Gamma] \leq [\Delta]\). With the help of axioms and rules of this sequent calculus, it can be proved that the sequent \( \Gamma : \Delta \) is derivable.

Thus completeness is established.
Chapter 2

Algebraic structures weaker than pre-rough algebra with rough arrow operator and their logics

In this chapter a set of algebras will be defined, all weaker than the pre-rough algebra. The main focus is on the arrow operator $\Rightarrow_R$. It is already mentioned that this operator is not available in topological quasi Boolean algebra 5. So our intention is to look for algebraic systems prior to the pre-rough algebra in which the operator may be available. As a result, Hilbert type logical systems for these algebras will be available too. [20]

2.1 System0 Algebra

Definition 2.1.1. An algebra $\langle A, \leq, \land, \lor, \sim, I, 0, 1 \rangle$ is a System0 algebra if and only if

1'. $\langle A, \leq, \land, \lor, \sim, 0, 1 \rangle$ is a quasi Boolean algebra.
2′. $I_1 = 1$.

7′. $a \leq b$ implies $I_a \leq I_b$, for all $a, b \in A$.

2.1.1 Sequent Calculus for System0 algebra

All axioms and rules of qBl are taken to obtain sequent calculus sq0 for the algebra System0. Also, two more rules are taken.

Rule 1: $\frac{\alpha : \beta}{l\alpha : l\beta}$  \hspace{2cm} (RL)$^r$ : $\frac{\alpha}{l\alpha}$

Theorem 2.1.2. The sequent calculus is sound and complete with respect to the class of all System0 algebras.

2.2 SystemI Algebra

Definition 2.2.1. An algebra $\langle A, \leq, \land, \lor, \sim, I, 0, 1 \rangle$ is a SystemI algebra if and only if

1′. $\langle A, \leq, \land, \lor, \sim, 0, 1 \rangle$ is a quasi Boolean algebra.

2′. $I_1 = 1$.

5′. $\sim I_a \lor I_a = 1$, for all $a \in A$.

6′. $Ca \leqCb, Ia \leq Ib$ imply $a \leq b$, for all $a, b \in A$.

7′. $a \leq b$ implies $I_a \leq I_b$, for all $a, b \in A$.

Example 2.2.2. It can be verified from the lattice whose Hasse diagram is shown in Fig.

6 that $\langle A = \{0, a, b, c, d, 1\}, \leq, \land, \lor, \sim, I, 0, 1 \rangle$ is a SystemI algebra.
Where $\sim, I, C$ are defined as follows

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sim$</td>
<td>1</td>
<td>0</td>
<td>b</td>
<td>a</td>
<td>d</td>
<td>c</td>
</tr>
</tbody>
</table>

I

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>d</th>
<th>d</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>d</td>
<td>c</td>
</tr>
</tbody>
</table>

Through this example we can verify that $a \lor \sim a \neq 1, Ia \nleq a, CIa \nleq a, Ib \nleq Iib, CIb \nleq Ib$.

**Note 2.2.3.** The above example is minimal in the sense that if the number of elements of a SystemI algebra is reduced then it possesses at least one of the properties

$Ia \leq a, CIa \leq a, Ia \leq IIa, CIa \leq Ia$.

**Observation 2.2.4.** In this algebra an implication operator $\Rightarrow_R$ can be defined in the same way as in pre-rough algebra by $a \Rightarrow_R b := (\sim Ia \lor Ib) \land (\sim Ca \lor Cb)$.

**Proposition 2.2.5.** In systemI algebra $a \Rightarrow_R b = 1$ if and only if $a \leq b$.

**Proof.** Follows from the proof of the Proposition 1.3.24.

**Proposition 2.2.6.** A SystemI algebra satisfying the condition $Ia \leq a$ is a pre-rough algebra.

**Proof.** By our assumption, $I \sim (a \lor b) \leq \sim (a \lor b) = \sim a \land \sim b \leq \sim a$, then using Axiom 5′ we have $\sim I(\sim a \land \sim b) \lor \sim a = 1$. Now by axioms of qBa $a \land I(\sim a \land \sim b) = 0$, then substituting $Ia$ for $a$ and $\sim a$ for $b$ we have
\(Ia \land I(\sim Ia \land a) = 0\). Now \(I(\sim Ia \land a) = Ia \land I(\sim Ia \land a) = 0\) [since \(\sim Ia \land a \leq a\) then by Axiom 7' \(I(\sim Ia \land a) \leq Ia\)]. Hence \(I(\sim Ia \land a) = 0\).

It may be noted that \(I(\sim Ia \land a) = 0\) is already proved from the assumption that \(I(a \land b) = Ia \land Ib\) in (1.3.21.1) of Theorem 1.3.21. Here we proved it from the assumption \(a \leq b\) implies \(Ia \leq Ib\) instead of \(I(a \land b) = Ia \land Ib\).

As we obtain \(I(\sim Ia \land a) = 0\) here, we also have \(CIa = Ia\). The proof is exactly same as the proof given in the proof of (1.3.21.12) of Theorem 1.3.21.

Now from this result and \(Ia \leq a\), we have \(CIa = Ia \leq a\), hence \(CIa \leq a\), which means this algebra is a SystemIB (vide Definition 2.4.1 of the Section 2.4) algebra hence \(I(a \land b) = Ia \land Ib\) holds [no circularity arises as we shall first prove that in SystemIB algebra, \(I(a \land b) = Ia \land Ib\) holds, then prove this proposition]. So we can conclude that if we add \(Ia \leq a\) with SystemI algebra then it will become a pre-rough algebra (modified form).

In the next subsection we present the Hilbert system for the SystemI algebra and call it \(LI\).

### 2.2.1 The Hilbert System \(LI\)

The language of \(LI\) is same as that of \(L\). Here we consider the first six axioms and all rules of \(L\) (see 1.3.1 of Chapter 1).

**Definition 2.2.7.** A System \(LI\)-model is a SystemI algebra \((A, \leq, \land, \lor, \sim, l, 0, 1)\), with a valuation \(v\) assigning a value \(v(p) \in A\) to each atomic formula \(p\) of the language.

\(v\) is extended to arbitrary formulae by \(v(\alpha \land \beta) = v(\alpha) \land v(\beta)\), \(v(\sim \alpha) = \sim v(\alpha)\) and \(v(l\alpha) = lv(\alpha)\). It follows that \(v(\alpha \lor \beta) = v(\alpha) \lor v(\beta)\), \(v(\alpha \rightarrow \beta) =\)
\( v(\alpha) \rightarrow v(\beta) \) and \( v(m\alpha) = mv(\alpha) \).

A formula \( \alpha \) is said to be true in a model if and only if \( v(\alpha) = 1 \). \( \alpha \) is said to be SystemI-valid if and only if it is true in every System \( LI \)-model. So, a wff \( \alpha \rightarrow \beta \) is valid if and only if \( v(\alpha) \leq v(\beta) \) for all models \( \langle A, \leq, \land, \lor, \neg, 0, 1, v \rangle \).

By \( \vdash \alpha \) we mean that \( \alpha \) is a theorem in the usual sense.

**Theorem 2.2.8.** (Soundness) If \( \vdash \alpha \) in \( LI \) then \( \alpha \) is valid in the class of all System \( LI \)-models.

*Proof.* Soundness is established by the following usual method:

1. All axioms are valid.

2. All rules preserve validity.

3. Applying induction on the length of the proof of \( \alpha \).

\( \square \)

**Theorem 2.2.9.** (Completeness) If \( \alpha \) is valid in the class of all System \( LI \)-models then \( \vdash \alpha \) in \( LI \).

*Proof.* To prove completeness we first construct the corresponding Lindenbaum algebra.

Let us define a relation \( R \) in \( LI \) by \( \alpha R \beta \) holds if and only if \( \vdash \alpha \rightarrow \beta \) and \( \vdash \beta \rightarrow \alpha \).

(Now onward by \( LI \) we mean set of all well formed formuale only.)

Let \( \alpha \in LI \) then we have \( \vdash \alpha \rightarrow \alpha \) (a proved theorem in \( LI \)). So \( \alpha R \alpha \) holds for all \( \alpha \in LI \).

Thus \( R \) is reflexive.

\( R \) is symmetric by definition.

Let \( \alpha, \beta, \gamma \in LI \) be such that \( \alpha R \beta \) and \( \beta R \gamma \) hold then we can write \( \vdash \alpha \rightarrow \beta \), \( \vdash \beta \rightarrow \alpha \)
and ⊢ β → γ, ⊢ γ → β. Now by rule 2 ⊢ α → γ and ⊢ γ → α which mean αRγ holds.

Thus R is transitive.

Therefore R is an equivalence relation in LI.

Now consider the quotient set LI/R and define a relation ≤ in LI/R by [α] ≤ [β] if and only if ⊢ α → β.

First we have to show this definition is independent of the choice of representatives.

Let α′ ∈ [α], β′ ∈ [β] then α′Rα and β′Rβ hold. So we have ⊢ α′ → α, ⊢ α → α′, ⊢ β′ → β
and ⊢ β → β′ hold.

We also have ⊢ α → β then using rule 2 on ⊢ α′ → α and ⊢ α → β we get ⊢ α′ → β.

Again applying rule 2 on ⊢ α′ → β and ⊢ β → β′ we get ⊢ α′ → β′ which means [α′] ≤ [β′].

Hence the proof of independence.

Lemma 2.2.10. The relation ≤ in LI/R is a partial ordered relation.

Proof. Since ⊢ α → α for all α ∈ LI then [α] ≤ [α] for all [α] ∈ LI/R.

Thus ≤ is reflexive.

Let [α], [β] ∈ LI/R be such that [α] ≤ [β] and [β] ≤ [α] hold then ⊢ α → β and ⊢ β → α
hold. So αRβ holds which implies [α] = [β].

Thus ≤ is antisymmetric.

Let [α], [β], [γ] ∈ LI/R be such that [α] ≤ [β] and [β] ≤ [γ] hold then ⊢ α → β and ⊢ β → γ
hold. Now by rule 2 we get ⊢ α → γ which implies [α] ≤ [γ].

Thus ≤ is transitive.

Therefore ≤ is a partial ordered relation in LI/R.

□

Let us define:
(i) $\sim [\alpha] = [\sim \alpha]$  

(ii) $[\alpha] \rightarrow [\beta] = [\alpha \rightarrow \beta]$  

(iii) $l[\alpha] = [l\alpha]$  

Now we have to show these definitions are independent of the choice of representatives.

(i) Let $[\alpha] = [\alpha']$ then $\vdash \alpha \rightarrow \alpha'$ and $\vdash \alpha' \rightarrow \alpha$. So by rule 4 we have $\vdash \sim \alpha' \rightarrow \sim \alpha$ and $\vdash \sim \alpha \rightarrow \sim \alpha'$ which mean $\sim \alpha R \sim \alpha'$ holds. Thus $[\sim \alpha] = [\sim \alpha']$.

Therefore $\sim [\alpha] = [\sim \alpha]$ is well defined.

(ii) Let $[\alpha'] = [\alpha]$ and $[\beta'] = [\beta]$ then $\vdash \alpha' \rightarrow \alpha$, $\vdash \alpha \rightarrow \alpha'$, $\vdash \beta' \rightarrow \beta$ and $\vdash \beta \rightarrow \beta'$.

So by rule 6 we have $\vdash (\alpha' \rightarrow \beta') \rightarrow (\alpha \rightarrow \beta)$ and $\vdash (\alpha \rightarrow \beta) \rightarrow (\alpha' \rightarrow \beta')$ which mean $(\alpha' \rightarrow \beta') R (\alpha \rightarrow \beta)$ holds. Thus $[\alpha' \rightarrow \beta'] = [\alpha \rightarrow \beta]$.

Therefore $[\alpha] \rightarrow [\beta] = [\alpha \rightarrow \beta]$ is well defined.

(iii) Let $[\alpha'] = [\alpha]$ then $\vdash \alpha' \rightarrow \alpha$ and $\vdash \alpha \rightarrow \alpha'$. So by rule 7 we have $\vdash l\alpha' \rightarrow l\alpha$ and $\vdash l\alpha \rightarrow l\alpha'$ which mean $l\alpha' R l\alpha$ holds. Thus $[l\alpha'] = [l\alpha]$.

Therefore $l[\alpha] = [l\alpha]$ is well defined.

**Lemma 2.2.11.** The partial ordered set $\langle LI/R, \leq \rangle$ is a lattice.

*Proof.* From Lemma 2.2.10 we have $\langle LI/R, \leq \rangle$ is a partial ordered set.

Now we have to show any two elements of $LI/R$ have least upper bound (lub) and greatest lower bound (glb).

Let $[\alpha], [\beta] \in LI/R$ then we shall show $[\alpha \wedge \beta]$ is the glb of $[\alpha], [\beta]$.

First we shall show if $[\alpha] = [\alpha']$ and $[\beta] = [\beta']$ then $[\alpha \wedge \beta] = [\alpha' \wedge \beta']$.

Let $[\alpha] = [\alpha']$ and $[\beta] = [\beta']$ which mean $\alpha R \alpha'$ and $\beta R \beta'$ hold. So $\vdash \alpha \rightarrow \alpha'$, $\vdash \alpha' \rightarrow \alpha$, $\vdash \beta \rightarrow \beta'$ and $\vdash \beta' \rightarrow \beta$.

Now $\vdash \alpha \wedge \beta \rightarrow \alpha$ (by an axiom)
\[ \vdash \alpha \rightarrow \alpha' \]
\[ \vdash \alpha \land \beta \rightarrow \alpha' \text{ (by rule 2)}. \]

Again \[ \vdash \alpha \land \beta \rightarrow \beta \text{ (by the Axiom 3, 4 and rule 2)} \]
\[ \vdash \beta \rightarrow \beta' \]
\[ \vdash \alpha \land \beta \rightarrow \beta' \text{ (by rule 2)}. \]

Then applying rule 5 on \[ \vdash \alpha \land \beta \rightarrow \alpha' \] and \[ \vdash \alpha \land \beta \rightarrow \beta' \] we have \[ \vdash \alpha \land \beta \rightarrow \alpha' \land \beta'. \]

Similarly we can get \[ \vdash \alpha' \land \beta' \rightarrow \alpha \land \beta. \] Hence \((\alpha \land \beta)R(\alpha' \land \beta')\) holds.

Therefore \([\alpha \land \beta] = [\alpha' \land \beta']\).

Now from an axiom we can write \[ \vdash \alpha \land \beta \rightarrow \alpha \] and \[ \vdash \alpha \land \beta \rightarrow \beta. \] Then \([\alpha \land \beta] \leq [\alpha]\) and \([\alpha \land \beta] \leq [\beta]\). Thus \([\alpha \land \beta]\) is a lower bound of \([\alpha], [\beta]\).

Let \([\delta]\) be another lower bound of \([\alpha], [\beta]\) then \([\delta] \leq [\alpha]\) and \([\delta] \leq [\beta]\) which imply \[ \vdash \delta \rightarrow \alpha \] and \[ \vdash \delta \rightarrow \beta. \] So by rule 5 \[ \vdash \delta \rightarrow \alpha \land \beta. \] Hence \([\delta] \leq [\alpha \land \beta]\).

Therefore \([\alpha \land \beta]\) is the glb of \([\alpha], [\beta]\).

We shall denote glb of \([\alpha], [\beta]\) by \([\alpha] \cap [\beta]\) which means \([\alpha] \cap [\beta] = [\alpha \land \beta]\).

Let \([\alpha], [\beta] \in LI/\text{R} \) then we shall show \([\alpha \lor \beta]\) is the lub of \([\alpha], [\beta]\).

First we shall show if \([\alpha] = [\alpha']\) and \([\beta] = [\beta']\) then \([\alpha \lor \beta] = [\alpha' \lor \beta']\).

Let us try to show that for any \(\alpha, \beta, \gamma \in LI\)
\[ \alpha \rightarrow \gamma \]
\[ \beta \rightarrow \gamma \]
\[ \underbrace{\alpha \lor \beta \rightarrow \gamma} \quad (2.2.11.1) \]

Let \[ \vdash \alpha \rightarrow \gamma, \vdash \beta \rightarrow \gamma \] hold then applying rule 4 we have \[ \vdash \sim \gamma \rightarrow \sim \alpha \] and \[ \vdash \sim \gamma \rightarrow \sim \beta. \]

Again by rule 5 \[ \vdash \sim \gamma \rightarrow (\sim \alpha \land \sim \beta) \]
\[ \vdash \sim (\sim \alpha \land \sim \beta) \rightarrow \sim \sim \gamma \text{ (by rule 4)} \]
\[ \vdash \neg \neg \gamma \rightarrow \gamma \text{ (by an axiom)} \]

\[ \vdash \neg (\neg \alpha \land \neg \beta) \rightarrow \gamma \text{ (by rule 2)}, \]

which means \[ \vdash \alpha \lor \beta \rightarrow \gamma. \]

Hence the proof of rule (2.2.11.1).

Let \([\alpha] = [\alpha']\) and \([\beta] = [\beta']\) which mean \(\alpha R \alpha'\) and \(\beta R \beta'\) hold. So \(\vdash \alpha \rightarrow \alpha', \vdash \alpha' \rightarrow \alpha,\)

\(\vdash \beta \rightarrow \beta'\) and \(\vdash \beta' \rightarrow \beta.\)

Now \(\vdash \neg \neg \alpha \land \neg \beta \rightarrow \neg \alpha\) (by an axiom)

\[ \vdash \neg \neg \alpha \rightarrow \neg \neg (\neg \alpha \land \neg \beta)\text{ (by rule 4)} \]

\[ \vdash \alpha \rightarrow \neg \neg \alpha \text{ (by an axiom)} \]

\[ \vdash \alpha \rightarrow \neg \neg (\neg \alpha \land \neg \beta) \text{ (by rule 2)} \]

\[ \vdash \alpha \rightarrow \alpha \lor \beta \text{ (by the definition of 'v') (2.2.11.2)} \]

\[ \vdash \alpha' \rightarrow \alpha \text{ (by the assumption)} \]

\[ \vdash \alpha' \rightarrow \alpha \lor \beta \text{ (by rule 2).} \]

Again \(\vdash \beta \rightarrow \alpha \lor \beta\) (by the Axiom 4, rule 4, definition of 'v', (2.2.11.2) and rule 2)

\[ \vdash \beta' \rightarrow \beta \text{ (by the assumption)} \]

\[ \vdash \beta' \rightarrow \alpha \lor \beta \text{ (by rule 2).} \]

Then applying rule (2.2.11.1) on \(\vdash \alpha' \rightarrow \alpha \lor \beta\) and \(\vdash \beta' \rightarrow \alpha \lor \beta\) we get

\[ \vdash \alpha' \lor \beta' \rightarrow \alpha \lor \beta. \]

Similarly we can get \(\vdash \alpha \lor \beta \rightarrow \alpha' \lor \beta'.\) Hence \((\alpha \lor \beta)R(\alpha' \lor \beta')\) holds.

Therefore \([\alpha \lor \beta] = [\alpha' \lor \beta'].\)

Now from the Axiom 4, rule 4, definition of 'v', (2.2.11.2) and rule 2 we can write

\(\vdash \alpha \rightarrow \alpha \lor \beta\) and \(\vdash \beta \rightarrow \alpha \lor \beta.\) Then \([\alpha] \leq [\alpha \lor \beta]\) and \([\beta] \leq [\alpha \lor \beta].\) Thus \([\alpha \lor \beta]\)

is an upper bound of \([\alpha], [\beta].\)
Let $\delta$ be another upper bound of $[\alpha], [\beta]$ then $[\alpha] \leq [\delta]$ and $[\beta] \leq [\delta]$ which imply $\vdash \alpha \rightarrow \delta$ and $\vdash \beta \rightarrow \delta$. So by rule (2.2.11.1) $\vdash \alpha \lor \beta \rightarrow \delta$. Hence $[\alpha \lor \beta] \leq [\delta]$.

Therefore $[\alpha \lor \beta]$ is the lub of $[\alpha], [\beta]$.

We shall denote lub of $[\alpha], [\beta]$ by $[\alpha] \cup [\beta]$ which means $[\alpha] \cup [\beta] = [\alpha \lor \beta]$.

Lemma 2.2.12. The lattice $(LI/R, \leq)$ is a bounded distributive lattice with the properties

(i) $\sim [\alpha] = [\alpha]$

(ii) $\sim ([\alpha] \cup [\beta]) = \sim [\alpha] \cap \sim [\beta]$.

In other words $(LI/R, \leq)$ is a quasi-Boolean algebra.

Proof. First we shall show that the set of all theorems forms an equivalence class.

Let $\vdash \alpha$ and $\vdash \beta$ then by rule 3 $\vdash \beta \rightarrow \alpha$ and $\vdash \alpha \rightarrow \beta$. So $\alpha R \beta$ holds which implies $[\alpha] = [\beta]$.

Let $\delta \in [\alpha]$ then $\delta R \alpha$ holds which means $\vdash \delta \rightarrow \alpha$ and $\vdash \alpha \rightarrow \delta$.

Now using rule 1 on $\vdash \alpha$ and $\vdash \alpha \rightarrow \delta$ we get $\vdash \delta$.

Thus the set of all theorems forms an equivalence class. We shall denote this equivalence class by $\mathbf{1}$.

Next we shall show for any $[\alpha] \in LI/R$, $[\alpha] \leq \mathbf{1}$.

Let $\beta \in \mathbf{1}$ then $\vdash \beta$. So by rule 3 $\vdash \alpha \rightarrow \beta$ which means $[\alpha] \leq [\beta] = \mathbf{1}$.

Hence $[\alpha] \leq \mathbf{1}$ for all $[\alpha] \in LI/R$.

Thus $\mathbf{1}$ is the greatest element of the lattice $(LI/R, \leq)$.

Now we shall show the set of all antitheorems forms an equivalence class.

Let us define an antitheorem.
An well formed formula $\alpha$ is said to be an antitheorem if and only if there is a $\vdash \beta$ such that $\vdash \alpha \rightarrow \neg \beta$ and $\vdash \neg \beta \rightarrow \alpha$.

Let $\alpha, \gamma$ be two antitheorems then there are $\vdash \beta$ and $\vdash \delta$ such that $\vdash \alpha \rightarrow \neg \beta$, $\vdash \neg \beta \rightarrow \alpha$, $\vdash \gamma \rightarrow \neg \delta$ and $\vdash \neg \delta \rightarrow \gamma$. Also we have $[\beta] = [\delta]$ then $\vdash \beta \rightarrow \delta$ and $\vdash \delta \rightarrow \beta$.

Now applying rule 4 on $\vdash \delta \rightarrow \beta$ we get $\vdash \neg \beta \rightarrow \neg \delta$. Again applying rule 2 on $\vdash \alpha \rightarrow \neg \beta$, $\vdash \neg \beta \rightarrow \alpha$ and $\vdash \neg \beta \rightarrow \neg \delta$ we have $\vdash \alpha \rightarrow \neg \delta$. Next applying rule 2 on $\vdash \alpha \rightarrow \neg \delta$ and $\vdash \neg \delta \rightarrow \gamma$ we get $\vdash \alpha \rightarrow \gamma$. Similarly we have $\vdash \gamma \rightarrow \alpha$. So $\alpha R \gamma$ holds which implies $[\alpha] = [\gamma]$.

Let $\theta \in [\alpha]$ where $\alpha$ is an antitheorem then there is a $\vdash \delta$ such that $\vdash \alpha \rightarrow \neg \beta$, $\vdash \neg \beta \rightarrow \alpha$ and also $\theta R \alpha$ holds. So $\vdash \theta \rightarrow \alpha$ and $\vdash \alpha \rightarrow \theta$ hold.

Now applying rule 2 on $\vdash \theta \rightarrow \alpha$ and $\vdash \alpha \rightarrow \neg \beta$ we get $\vdash \theta \rightarrow \neg \beta$.

Again applying rule 2 on $\vdash \neg \beta \rightarrow \alpha$ and $\vdash \alpha \rightarrow \theta$ we have $\vdash \neg \beta \rightarrow \theta$.

Hence $\theta$ is an antitheorem.

Thus the set of all antitheorems forms an equivalence class. We shall denote this equivalence class by $0$.

Let $\beta \in 0$ then there is a $\vdash \delta$ such that $\vdash \beta \rightarrow \neg \delta$ and $\vdash \neg \delta \rightarrow \beta$.

Now $\vdash \delta$

$\vdash \neg \alpha \rightarrow \delta$ (by rule 3)

$\vdash \neg \delta \rightarrow \neg \neg \alpha$ (by rule 4)

$\vdash \neg \neg \alpha \rightarrow \alpha$ (by an axiom)

$\vdash \neg \delta \rightarrow \alpha$ (by rule 2)

$\vdash \beta \rightarrow \neg \delta$ (definition of $\beta$)

$\vdash \beta \rightarrow \alpha$ (by rule 2)

which implies $0 = [\beta] \leq [\alpha]$. 

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Hence $0 \leq [\alpha]$ for all $[\alpha] \in LI/R$.

Thus $0$ is the least element of the lattice $\langle LI/R, \leq \rangle$.

Therefore $\langle LI/R, \leq \rangle$ is a bounded lattice.

Now $\vdash \alpha \land (\beta \lor \gamma) \to (\alpha \land \beta) \lor (\alpha \land \gamma)$ (by an axiom)

$\vdash (\alpha \land \beta) \lor (\alpha \land \gamma) \to \alpha \land (\beta \lor \gamma)$ (by an axiom)

which mean $\alpha \land (\beta \lor \gamma) \leq_R (\alpha \land \beta) \lor (\alpha \land \gamma)$ holds.

Then we have $[\alpha \land (\beta \lor \gamma)] = [(\alpha \land \beta) \lor (\alpha \land \gamma)]$

i.e., $[\alpha] \cap [\beta \lor \gamma] = [\alpha \land \beta] \cup [\alpha \land \gamma]$

i.e., $[\alpha] \cap ([\beta] \cup [\gamma]) = ([\alpha] \cap [\beta]) \cup ([\alpha] \cap [\gamma])$.

Thus $\langle LI/R, \leq \rangle$ is a bounded distributive lattice.

Again $\vdash \alpha \to \sim \sim \alpha$ (by an axiom)

$\vdash \sim \sim \alpha \to \alpha$ (by an axiom)

which mean $\alpha R \sim \sim \alpha$ holds. Then we have $[\alpha] = [\sim \sim \alpha]$.

In other words $\sim \sim [\alpha] = [\alpha]$.  \stepcounter{equation}(2.2.12.1)

Let $[\alpha] \leq [\beta]$ which implies $\vdash \alpha \to \beta$ then by rule 4 $\vdash \sim \sim \beta \to \sim \sim \alpha$. So $[\sim \sim \beta] \leq [\sim \sim \alpha]$ which means $\sim [\beta] \leq \sim [\alpha]$.

Hence for any $[\alpha], [\beta] \in LI/R$ if $[\alpha] \leq [\beta]$ then $\sim [\beta] \leq \sim [\alpha]$.  \stepcounter{equation}(2.2.12.2)

Now we can write $[\alpha] \leq [\alpha] \cup [\beta]$ and $[\beta] \leq [\alpha] \cup [\beta]$. Then using (2.2.12.2) we have

$\sim ([\alpha] \cup [\beta]) \leq \sim [\alpha]$ and $\sim ([\alpha] \cup [\beta]) \leq \sim [\beta]$.

So $\sim ([\alpha] \cup [\beta]) \leq \sim [\alpha] \cap \sim [\beta]$.  \stepcounter{equation}(2.2.12.3)

Let $[\gamma] \in LI/R$ be such that $[\gamma] \leq \sim [\alpha]$ and $[\gamma] \leq \sim [\beta]$. Then using (2.2.12.2) we have

$\sim \sim [\alpha] \leq \sim [\gamma]$ and $\sim \sim [\beta] \leq \sim [\gamma]$. Now using (2.2.12.1) we can write $[\alpha] \cup [\beta] \leq \sim [\gamma]$.

Hence using (2.2.12.1) and (2.2.12.2) we have $[\gamma] \leq \sim ([\alpha] \cup [\beta])$ which implies
\[ \sim [\alpha] \cap \sim [\beta] \leq \sim ([\alpha] \cup [\beta]). \quad (2.2.12.4) \]

Thus from (2.2.12.3) and (2.2.12.4) we get \( \sim ([\alpha] \cup [\beta]) = \sim [\alpha] \cap \sim [\beta] \).

Therefore \( \langle LI/_{R}, \leq \rangle \) is a quasi-Boolean algebra.

\begin{proof}

Lemma 2.2.13. In the quasi-Boolean algebra \( \langle LI/_{R}, \leq \rangle \) the following properties are satisfied

(i) \( l1 = 1 \)

(ii) \( \sim l[\alpha] \cup l[\alpha] = 1 \)

(iii) \( l[\alpha] \cup l[\beta] = l[\beta], m[\alpha] \cup m[\beta] = m[\beta] \) imply \( [\alpha] \cup [\beta] = [\beta] \)

(iv) \( [\alpha] \cup [\beta] = [\beta] \) implies \( l[\alpha] \cup l[\beta] = l[\beta] \).

Proof. Let \( \alpha \in 1 \) then \( \vdash \alpha \) in \( LI \). Now by rule 8 \( \vdash l[\alpha] \) in \( LI \). So \( [l[\alpha]] = [\alpha] = 1 \) then \( l[\alpha] = 1 \).

Hence \( l1 = 1 \).

In \( LI \) \( \vdash \alpha \rightarrow \alpha \) (a proved theorem)

i.e., \( \vdash (\sim l[\alpha] \lor l\alpha) \land (\sim m[\alpha] \lor m\alpha) \) (definition of ‘→’)

\( \vdash (\sim l[\alpha] \lor l\alpha) \land (\sim m[\alpha] \lor m\alpha) \rightarrow (\sim l\alpha \lor l\alpha) \) (by an axiom)

\( \vdash \sim l\alpha \lor l\alpha \) (by rule 1)

Now we can write \( [\sim l\alpha \lor l\alpha] = 1 \).

Hence \( l[\alpha] \cup l[\alpha] = 1 \).

Let \( l[\alpha] \cup l[\beta] = l[\beta], m[\alpha] \cup m[\beta] = m[\beta] \) then we have \( [l\alpha \lor l\beta] = [l\beta], [m\alpha \lor m\beta] = [m\beta] \).

So we can write \( \vdash l\alpha \lor l\beta \rightarrow l\beta, l\beta \rightarrow l\alpha \lor l\beta \) and \( \vdash m\alpha \lor m\beta \rightarrow m\beta, m\beta \rightarrow m\alpha \lor m\beta \).

Now \( \vdash l\alpha \rightarrow l\alpha \lor l\beta \) (by (2.2.11.2))

\[ 63 \]
⊢ lα ∨ lβ → lβ

⊢ lα → lβ (by rule 2)

Similarly ⊢ mα → mβ

⊢ α → β (by rule 9)

⊢ β → β (proved theorem)

⊢ α ∨ β → β (by 2.2.11.1))

⊢ β → α ∨ β (by 2.2.11.2))

Therefore [α ∨ β] = [β] which means [α] ∪ [β] = [β].

Hence l[α] ∪ l[β] = l[β], m[α] ∪ m[β] = m[β] imply [α] ∪ [β] = [β].

Let [α] ∪ [β] = [β] then we have [α ∨ β] = [β]. So we can write ⊢ α ∨ β → β and β → α ∨ β.

Now ⊢ α → α ∨ β (by 2.2.11.2))

⊢ α ∨ β → β

⊢ α → β (by rule 2)

⊢ lα → lβ (by rule 7)

⊢ lβ → lβ (proved theorem)

⊢ lα ∨ lβ → lβ (by 2.2.11.1))

⊢ lβ → lα ∨ lβ (by 2.2.11.2))

Therefore [lα ∨ lβ] = [lβ] which means [lα] ∪ [lβ] = [lβ].

Hence [α] ∪ [β] = [β] implies l[α] ∪ l[β] = l[β].

Therefore from the Lemma 2.2.10, 2.2.11, 2.2.12 and 2.2.13 we can conclude that the Lindenbaum algebra ⟨LI/R, ≤⟩ is a SystemI algebra with respect to the corresponding operations defined on it.
Proof of the main theorem

It can be proved that the above Lindenbaum algebra \( \langle LI/R, \cap, \cup, \sim, l, 1, 0 \rangle \) along with the canonical valuation, which means when \( \alpha \) is mapped to its equivalence class \([\alpha]\), is a member of the class of all System LI-models. This proves the completeness since if \( \alpha \) is valid in the class of all System LI-models which means \( v(\alpha) = 1 \) holds for every member of that class with valuation function \( v \), it holds in \( LI/R \) with the canonical valuation. Thus \([\alpha] = 1\) which implies that \( \vdash \alpha \).

\[
\square
\]

2.2.2 Sequent Calculus for SystemI algebra

All the axioms and rules of sq0 are present in the sequent calculus sqI for the algebra SystemI. In addition, the rules \( L \sim, R \sim \) and \( lmR \) stated in 1.3.2 of Chapter 1 are taken in this calculus.

Theorem 2.2.14. The sequent calculus is sound and complete with respect to the class of all SystemI algebras.

Proof. Following the usual procedure we present the sketch of the theorem.

To prove soundness, we shall use induction on the depth of the derivation of the derivable sequent.

For completeness, we first construct the Lindenbaum algebra. Let \( F \) be the set of all well-formed formulae. Let us define a binary relation \( R \) in \( F \) by \( \alpha R \beta \) if and only if \( \alpha : \beta \) and \( \beta : \alpha \) are derivable sequents.

Due to the presence of the axioms

\[
Ax1 \quad \alpha : \sim \sim \alpha \quad \text{and} \quad Ax2 \quad \sim \sim \alpha : \alpha
\]

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and Cut rule, \( R \) turns out to be an equivalence in \( F \).

In the quotient set, let us define a binary relation \( \leq \) by \([\alpha] \leq [\beta]\) holds if and only if \( \alpha \vdash \beta \) is a derivable sequent, for \([\alpha], [\beta] \in F/R\).

Then the quotient algebra \( F/R \) with respect to suitable operations is verified to be the algebraic structure SystemI algebra.

Let \( \Gamma : \Delta \) be valid in the class of all SystemI algebra-models.

Then \( v(\Gamma) \leq v(\Delta) \) holds for every member of that class with valuation function \( v \).

In particular, \( \Gamma : \Delta \) is valid in the Lindenbaum algebra \( F/R \) with the canonical valuation i.e., when \( \alpha \) is mapped to its equivalence class \([\alpha]\). Thus \([\Gamma] \leq [\Delta]\). With the help of axioms and rules of this sequent calculus, it can be proved that the sequent \( \Gamma : \Delta \) is derivable.

Thus completeness is established.

\( \square \)

## 2.3 SystemII Algebra

**Definition 2.3.1.** An algebra \( \langle A, \leq, \land, \lor, \sim, I, 0, 1 \rangle \) is said to be a **SystemII algebra** if and only if

1. \( \langle A, \leq, \land, \lor, \sim, 0, 1 \rangle \) is a quasi Boolean algebra.

2. \( I1 = 1 \).

4. \( I(a \land b) = Ia \land Ib \), for all \( a, b \in A \).

5. \( \sim Ia \lor Ia = 1 \), for all \( a \in A \).

6. \( Ca \leq Cb, Ia \leq Ib \) imply \( a \leq b \), for all \( a, b \in A \).
Proposition 2.3.2. Any SystemII algebra is a SystemI algebra.

Proof. For any \(a, b \in A\), let \(a \leq b\). So, \(a \wedge b = a\). Then \(I(a \wedge b) = Ia\). So by Axiom 4', \(Ia \wedge Ib = Ia\), i.e., \(Ia \leq Ib\).

\[\square\]

Observation 2.3.3. In this algebra an implication operator \(\Rightarrow_R\) can be defined in the same way as in pre-rough algebra by \(a \Rightarrow_R b := (\sim Ia \lor Ib) \land (\sim Ca \lor Cb)\).

Proposition 2.3.4. In SystemII algebra \(a \Rightarrow_R b = 1\) if and only if \(a \leq b\).

Remark 2.3.5. The converse of the above proposition 2.3.2 is still an open question.

Remark 2.3.6. It follows from the modified definition of pre-rough algebra that if the condition \(Ia \leq a\) is added to SystemII axioms it reduces to a pre-rough algebra.

2.3.1 The Hilbert Systems LII

The language of the system LII corresponding to SystemII is the same as that of LI. We take all axioms and rules of LI, together with an extra axiom viz.,

\(l(\alpha \wedge \beta) \rightarrow l(\alpha) \land l(\beta)\).

Theorem 2.3.7. The Hilbert system is sound and complete with respect to the class of all SystemII algebras.

2.3.2 Sequent Calculus for SystemII algebra

With sqI one more axiom

\(Ax4 \quad l\alpha, l\beta : l(\alpha \wedge \beta)\)

is added to get the sequent calculus sqII.
Theorem 2.3.8. The sequent calculus is sound and complete with respect to the class of all SystemII algebras.

Relationship-diagram 1

The relationship between the algebras defined so far and the existing nodal algebras are shown in the following diagram (Fig. 7).

If \( A \) and \( B \) are two algebraic structures then \( A \Rightarrow B \) means that one extra operator and some axioms for the new operator are added with \( A \) to obtain \( B \).

\[ A \rightarrow B \] means that \( A \) and \( B \) have the same operators and if an algebra is \( B \) then it is \( A \).

2.4 System Algebras with modal axioms \( T, B, S_4, S_5 \)

In SystemII algebra and hence in SystemI algebra too, none of the following properties hold

\[ (T) : Ia \leq a. \]
(B) : \( CIa \leq a. \)

(S4) : \( Ia \leq IHa. \)

(S5) : \( CIa \leq Ia. \)

This may be verified by the lattice in Fig. 6 which is a SystemI algebra as well as a SystemII algebra. But all these properties are available in topological quasi Boolean algebra 5. The properties are counterparts of the modal axioms \( T, B, S_4 \) and \( S_5 \) respectively.

Since it is not yet settled whether SystemI is equivalent to SystemII, we add axioms \( B, S_4 \) and \( S_5 \) to SystemI and II separately. It is already established that if \( T \) is added with SystemI or II we get pre-rough algebra straightway.

**Definition 2.4.1.** An algebra \( \langle A, \leq, \wedge, \lor, \sim, I, 0, 1 \rangle \) is a **SystemIB algebra** if and only if

- \( \langle A, \leq, \wedge, \lor, \sim, I, 0, 1 \rangle \) is a SystemI algebra and
- \( CIa \leq a, \) for all \( a \in A. \)

**Example 2.4.2.** It can be verified from the lattice whose Hasse diagram is shown in Fig. 8 that \( \langle A = \{0, a, b, 1\}, \leq, \wedge, \lor, \sim, I, 0, 1 \rangle \) is a SystemIB algebra.

![Fig. 8 An example of SystemIB algebra](image)

Where \( \sim, I, C \) are defined as follows

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Definition 2.4.3. An algebra $\langle A, \leq, \land, \lor, \sim, \bot, 1 \rangle$ is a **SystemI4 algebra** if and only if

- $\langle A, \leq, \land, \lor, \sim, \bot, 1 \rangle$ is a SystemI algebra and
- $Ia \leq IIa$, for all $a \in A$.

Example 2.4.4. It can be verified from the lattice whose Hasse diagram is shown in Fig. 9 that $\langle A = \{0, a, 1\}, \leq, \land, \lor, \sim, \bot, 1 \rangle$ is a **SystemI4 algebra**.

Where $\sim, I, C$ are defined as follows

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>~</td>
<td>1</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>I</td>
<td>0</td>
<td>1</td>
<td>b</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>1</td>
<td>b</td>
</tr>
</tbody>
</table>

Definition 2.4.5. An algebra $\langle A, \leq, \land, \lor, \sim, \bot, 1 \rangle$ is a **SystemI5 algebra** if and only if

- $\langle A, \leq, \land, \lor, \sim, \bot, 1 \rangle$ is a SystemI algebra and
- $CIa \leq Ia$, for all $a \in A$.

Note 2.4.6. Replacing the condition $\leq$ of axioms $S_4, S_5$ by equality $(=)$ we get stronger systems.
Definition 2.4.7. An algebra \( \langle A, \leq, \land, \lor, \sim, I, 0, 1 \rangle \) is a \textbf{SystemI4E} algebra if and only if

- \( \langle A, \leq, \land, \lor, \sim, I, 0, 1 \rangle \) is a SystemI algebra and
- \( Ia = IIa \), for all \( a \in A \).

Definition 2.4.8. An algebra \( \langle A, \leq, \land, \lor, \sim, I, 0, 1 \rangle \) is a \textbf{SystemI5E} algebra if and only if

- \( \langle A, \leq, \land, \lor, \sim, I, 0, 1 \rangle \) is a SystemI algebra and
- \( CIa = Ia \), for all \( a \in A \).

Note 2.4.9. The lattice in Example 2.4.4 is also an example of all the systems SystemI5, SystemI4E and SystemI5E.

Similarly SystemIIB, SystemII4, SystemII5, SystemII4E and SystemII5E are defined.

Proposition 2.4.10. A SystemIB algebra is a SystemII algebra.

Proof. We want to show that SystemIB is SystemII, i.e., \( I(a \land b) = (Ia \land Ib) \) holds in SystemIB.

First we shall prove that

- IB.1 \( a \lor Ib = 1 \) if and only if \( \sim Ib \leq a \).
- IB.2 \( a \lor \sim Ib = 1 \) if and only if \( Ib \leq a \).
- IB.3 \( ICIa = Ia \).
- IB.4 \( IIa \land \sim CIa = 0 \).
- IB.5 If \( IIa = 0 \) then \( Ia = 0 \).
- IB.6 \( II(a \land \sim IIa \land b) = 0 \).
IB.7 $IIIa = Ia$.

IB.8 $I \sim Ia = \sim IIa$.

IB.9 $I(a \lor \sim b) \lor I(a \lor \sim IIb) = I(a \lor \sim IIb)$.

IB.10 $\sim IIa \lor I(b \lor \sim Ia) = I(b \lor \sim Ia)$.

IB.11 $I(IIa \land b) = I(a \land b)$.

IB.12 $a \lor I(\sim Ia \lor b) = 1$.

IB.13 $\sim I(b \lor I(a \lor Ib)) \leq a$.

IB.14 $a \lor I(b \lor Ia) = a \lor Ib$.

IB.15 $I(a \lor \sim IIb) \lor Ib = 1$.

IB.16 $I(a \lor \sim IIb) = \sim Ib \lor Ia$ and

IB.17 $I(b \lor \sim Ia) = \sim IIa \lor Ib$.

Proof of IB.1

Let $a \lor Ib = 1$. Then $\sim Ib \land (a \lor Ib) = \sim Ib \land 1$, i.e.,

$(\sim Ib \land a) \lor (\sim Ib \land Ib) = \sim Ib$. So, $(\sim Ib \land a) \lor 0 = \sim Ib$ [by Axiom 5' and an axiom of qBa] i.e., $\sim Ib \land a = \sim Ib$.

Conversely, $\sim Ib \leq a$ implies that $\sim Ib \lor Ib \leq Ib \lor a$, i.e., $Ib \lor a = 1$ [by Axiom 5'].

Proof of IB.2

Let $a \lor \sim Ib = 1$. Then $Ib \land (a \lor \sim Ib) = Ib \land 1$, i.e.,

$(Ib \land a) \lor (Ib \land \sim Ib) = Ib$. So by Axiom 5' and an axiom of qBa, $(Ib \land a) \lor 0 = Ib$, i.e.,
\[ Ib \land a = Ib. \]

Conversely, \( Ib \leq a \) implies that \( \sim Ib \lor Ib \leq \sim Ib \lor a \), i.e., \( \sim Ib \lor a = 1 \) [by Axiom 5'].

Proof of IB.3

As \( CIa \leq a \), \( ICIa \leq Ia \) [by Axiom 7'].

Again, \( CI \sim Ia \leq \sim Ia \) holds in SystemIB. So, \( Ia \leq \sim CI \sim Ia = ICIa. \)

Hence, \( ICIa = Ia. \)

Proof of IB.4

In SystemIB, we have \( IIa \leq CIa \). [Here we use Proposition 2.5.4. Although the proof is given later, there is no circularity. We first prove Proposition 2.5.4 then proved the present theorem.]

So, \( IIa \land \sim CIa \leq CIa \land \sim CIa = 0 \) [by Axiom 5' and an axiom of qBa].

Hence, \( IIa \land \sim CIa = 0. \)

Proof of IB.5

Let \( IIa = 0. \) From IB.4 it follows that \( IIa \land \sim CIa = IIa \), i.e.,

\[ IIa \leq \sim CIa = I \sim Ia. \]

Also, \( C \sim Ia = \sim IIa = \sim 0 = 1. \) So,

\[ CIa \leq C \sim Ia. \] Thus we have proved that \( IIa \leq I \sim Ia \) and \( CIa \leq C \sim Ia. \) Hence by Axiom 6', we have \( Ia \leq \sim Ia \), i.e., \( Ia \land \sim Ia = Ia. \) In other words, \( Ia = 0 \) [by Axiom 5' and an axiom of qBa].

Hence if \( IIa = 0 \) then \( Ia = 0. \)

Proof of IB.6

As \( b \land I \sim a \leq I \sim a \), \( I(b \land I \sim a) \leq II \sim a \leq CI \sim a \leq \sim a \), by Axiom 7' and Proposition 2.5.4. So, \( a \leq \sim I(b \land I \sim a). \)

Hence, \( a \land I(b \land I \sim a) \leq \sim I(b \land I \sim a) \land I(b \land I \sim a) = 0 \) [by Axiom 5' and an axiom of
So, \(a \land I(b \land I \sim a)\) = 0. Substituting \(Ia\) for \(a\) and \(a\) for \(b\), we get \(Ia \land I(a \land I \sim Ia)\) = 0. As, \(a \land I \sim Ia \leq a\), \(I(a \land I \sim Ia) \leq Ia\) [by Axiom 7']. So, \(Ia \land I(a \land I \sim Ia) = I(a \land I \sim Ia)\).

Then \(I(a \land I \sim Ia) = 0\).

Again, \(\sim Ia \land b \leq \sim Ia\). So, \(I(\sim Ia \land b) \leq I \sim Ia\) [by Axiom 7'].

Thus, \(a \land I(\sim Ia \land b) \leq a \land I \sim Ia\) and so by Axiom 7',

\[I(a \land I(\sim Ia \land b)) \leq I(a \land I \sim Ia) = 0.\]

Hence \(I(a \land I(\sim Ia \land b)) = 0\). Substituting \(Ia\) for \(a\) and \(a\) for \(b\), \(I(Ia \land I(\sim IIa \land a)) = 0\).

As \(\sim IIa \land a \leq a\), \(I(\sim IIa \land a) \leq Ia\) [by Axiom 7']. Thus \(II(\sim IIa \land a) = 0\).

Hence \(II(\sim IIa \land a \land b) = 0\) [as \(\sim IIa \land a \land b \leq \sim IIa \land a\)]

**Proof of IB.7**

From IB.5 and IB.6 we have \(I(\sim IIa \land a \land b) = 0\). Substituting

\(I(I \sim IIa \land b)\) for \(b\), \(I(\sim IIa \land a \land I(I \sim IIa \land b)) = 0\). As \(I \sim IIa \land b \leq I \sim IIa\), then by Axiom 7' \(I(I \sim IIa \land b) \leq II \sim IIa \leq CI \sim IIa \leq \sim IIa\) [by Proposition 2.5.4 and the fact that for all \(a\), \(CIa \leq a\) holds in SystemIB]. So, \(I(a \land I(I \sim IIa \land b)) = 0\). Substituting \(Ia\) for \(a\) and \(a\) for \(b\), \(I(Ia \land I(I \sim IIIa \land a)) = 0\).

As \(I \sim IIIa \land a \leq a\), \(I(I \sim IIIa \land a) \leq Ia\) [by Axiom 7']. Then \(II(I \sim IIIa \land a) = 0\) which implies that \(II(I(\sim IIIa \land b) \land a) = 0\), as \(\sim IIIa \land b \leq \sim IIIa\). Again by IB.5,

\(I(I(\sim IIIa \land b) \land a) = 0\). Substituting \(Ia\) for \(a\) and \(a\) for \(b\), \(I(I(\sim IIIIa \land a) \land Ia) = 0\). So \(II(\sim IIIIa \land a) = 0\), as \(\sim IIIIa \land a \leq a\). Then from IB.5 it follows that \(I(\sim IIIIa \land a) = 0\).

Now, \(IIIa \leq IIIa \lor b\). Then by Axiom 7', \(IIIa \leq I(IIIa \lor b)\) which implies that \(\sim I(IIIa \lor b) \leq \sim IIIa\). Thus \((\sim I(IIIa \lor b)) \land a \leq \sim IIIa \land a\) which implies that \(I((\sim I(IIIa \lor b)) \land a) \leq I(\sim IIIa \land a)(= 0)\).
Hence $\overline{I(\sim I(IIIa \lor b) \land a)} = 0$. Substituting $\sim Ia$ for $a$ and $a$ for $b$,

$I(\sim I(III \sim Ia \lor a) \land \sim Ia) = 0$. Now $a \leq III \sim Ia \lor a$.

So, $Ia \leq I(III \sim Ia \lor a)$ which implies that $\sim I(III \sim Ia \lor a) \leq \sim Ia$.

Thus $I(\sim I(III \sim Ia \lor a)) = 0$, i.e., $CI(III \sim Ia \lor a) = 1$. But

$CI(III \sim Ia \lor a) \leq III \sim Ia \lor a$. So $III \sim Ia \lor a = 1$. Substituting $\sim Ia$ for $a$,

$III \sim Ia \lor \sim Ia = 1$ i.e., $III \sim CIIa \lor Ia = 1$. So by IB.3, $III \sim CIIa \lor Ia = 1$. Then by

IB.2, $Ia \leq IIIa$.

Also we know that $IIIa \leq CIIa \leq Ia$, by Proposition 2.5.4.

Thus $IIIa = Ia$.

**Proof of IB.8**

As $CIa \leq a$, $\sim a \leq CIa = I \sim Ia$. Substituting $IIa$ for $a$,

$\sim IIa \leq I \sim IIIa = I \sim Ia$, by IB.7.

Also we know that $IIa \leq CIa$ holds in SystemIB [by Proposition 2.5.4]. So, $I \sim Ia \leq IIa$.

Hence $I \sim Ia \sim IIa$.

From IB.8 it follows that $IIa = CIa$.

**Proof of IB.9**

As $IIb \leq CIb \leq b$ (by Proposition 2.5.4 and the fact that for all $a$, $CIa \leq a$ holds in

SystemIB), $\sim b \leq IIb$.

So, $a \lor \sim b \leq a \lor IIb$.

Then $I(a \lor \sim b) \leq I(a \lor IIb)$.

**Proof of IB.10**

As $\sim Ia \leq b \lor \sim Ia$, $I \sim Ia \leq I(b \lor \sim Ia)$ [by Axiom 7’].

Then by IB.8, $\sim IIa \leq I(b \lor \sim Ia)$. 

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Proof of IB.11

As $CIa \leq b \lor CIa$ and $CIa \leq a$, $CIa \leq a \land (b \lor CIa)$. Then

$ICIa \leq I(a \land (b \lor CIa))$. Using IB.3, we get $Ia \leq I(a \land (b \lor CIa))$.

Again $I(a \land (b \lor CIa)) \leq Ia$, as $a \land (b \lor CIa) \leq a$. Thus $I(a \land (b \lor CIa)) = Ia$. From IB.8 we get $I(a \land (b \lor IIa)) = Ia$. Substituting $a \land b$ for $a$ and $IIa$ for $b$, $I(a \land b \land (IIa \lor II(a \land b))) = I(a \land b)$. Then $I(a \land b \land IIa) = I(a \land b)$, as $a \land b \leq a$. Also $IIa \leq CIa \leq a$.

Hence $I(b \land IIa) = I(a \land b)$.

Proof of IB.12

$1 = IIa \lor \sim IIa \leq CIa \lor \sim IIa$ (by Axiom 5’ and Proposition 2.5.4) \leq a \lor \sim IIa$ (as in SystemIB $CIa \leq a$) = a \lor I \sim Ia \ (by IB.8) \leq a \lor I(\sim Ia \land b)$, as $\sim Ia \leq \sim Ia \land b$.

Hence $a \lor I(\sim Ia \land b) = 1$.

Proof of IB.13

Substituting $a \lor Ib$ for $a$ in IB.12, we get $(a \lor Ib) \lor I(\sim I(a \lor Ib) \land b) = 1$. As $b \leq \sim I(a \lor Ib) \lor b$, $Ib \leq I(\sim I(a \lor Ib) \land b)$ [by Axiom 7’]. So, $a \lor I(\sim I(a \lor Ib) \land b) = 1$.

Then by IB.1, $\sim I(\sim I(a \lor Ib) \land b) \leq a$.

Proof of IB.14

Substituting $a \lor \sim I(b \lor Ia)$ for $a$ in IB.12,

$a \lor \sim I(b \lor Ia) \lor I(\sim I(a \lor \sim I(b \lor Ia)) \lor b) = 1$. Then by IB.13 (interchanging $a$ and $b$), $a \lor \sim I(b \lor Ia) \lor Ib = 1$. From IB.2, it follows that $I(b \lor Ia) \leq a \lor Ib$. Thus $a \lor Ib \lor I(b \lor Ia) = a \lor Ib$, i.e., $a \lor I(b \lor Ia) = a \lor Ib$, as $b \leq b \lor Ia$.

Proof of IB.15

$\sim Ib = \sim IIb$ (by IB.7) = $I \sim IIb$ (by IB.8) \leq I(a \lor \sim IIb)$, as

$\sim IIb \leq a \lor \sim IIb$. 76
Then by IB.1, \( I(a \lor \sim IIb) \lor Ib = 1 \).

**Proof of IB.16**

\((a \lor Ib) \land (a \lor \sim Ib) = a \lor (Ib \land \sim Ib) = a \lor 0 = a \) [by Axiom 5' and an axiom of qBa]. Thus 
\((a \lor Ib) \land (a \lor \sim Ib) = a \). Substituting \(I(a \lor I \sim Ib)\) for \(a\), \((I(a \lor I \sim Ib) \land (I(a \lor I \sim Ib) = I(a \lor I \sim Ib)\). So by IB.14, \((I(a \lor I \sim Ib) \land (Ia \lor \sim Ib) = I(a \lor I \sim Ib)\).

From IB.8, we get \((I(a \lor \sim IIb) \lor Ib) \land (Ia \lor \sim Ib) = I(a \lor \sim IIb)\). Using IB.15, we get 
\(Ia \lor \sim Ib = I(a \lor \sim IIb)\).

**Proof of IB.17**

From IB.9 and IB.16, we get \((Ia \lor \sim b) \lor Ib \lor Ia \equiv Ib \lor Ia\). Then \((Ia \lor \sim b) \lor Ib =\sim Ib \lor Ia\), as \(a \leq a \lor \sim b\). Substituting \(b\) for \(a\) and \(Ia\) for \(b\), \((Ib \lor \sim Ib) \lor Ia \equiv Ib \lor Ia\).

Then by IB.10, \((Ib \lor \sim Ia) =\sim IIa \lor Ib\).

Now we are in a position to prove the main proposition, viz.,

\(I(a \land b) = (Ia \land Ib)\) holds in SystemIB.

\(~ II(b \lor \sim Ia) = I \sim (Ib \lor \sim Ia) \) (by IB.8) \(= I \sim (\sim IIa \lor Ib) \) (by IB.17)

\(= I(a \land \sim Ib)\), by IB.11. So, \(I(a \land \sim Ia) = I(a \land \sim Ib)\).

\(~ I(\sim IIa \lor Ib)( \) (by IB.17) \(= I(a \land \sim Ib)\).

Thus \(I(a \land \sim Ib) = Ia \land \sim IIb\). Substituting \(\sim Ib\) for \(b\), \(I(a \land \sim I \sim Ib) = Ia \land \sim II \sim Ib\).

Then \(I(a \land IIb) = Ia \land IIIb\) [by IB.8]

i.e., \(I(a \land b) = Ia \land IIIb\) [by IB.11]

i.e., \(I(a \land b) = Ia \land Ib\) [by IB.7].

\(\square\)

**Remark 2.4.11.** Since SystemII algebra is a SystemI algebra, when \(B\) is added to SystemII,
we get that a SystemII 5B algebra is a SystemIB algebra. Also by above proposition SystemIB algebra is a SystemII 5B algebra. Thus SystemIB and SystemII 5B are equivalent algebras.

**Proposition 2.4.12.** A SystemII 5E algebra is a SystemII 4E algebra.

**Proof.** In SystemII 5E, $CIa = Ia$, for all $a \in A$, i.e., $I \sim Ia = Ia$.

So, $Ia \sim I \sim Ia = I \sim Ia$. 

(2.4.12.1)

Then $I \sim I \sim Ia = \sim I \sim Ia$ and so by (2.4.12.1), $I \sim Ia = \sim Ia$. Then $IIa = Ia$, for all $a \in A$. Hence the algebra becomes a SystemII 4E algebra.

\[ \square \]

**Remark 2.4.13.** So, SystemII 5E algebra is a SystemII 4 algebra.

**Proposition 2.4.14.** The following algebraic systems are equivalent.

1. SystemII 4 algebra
2. SystemII 5 algebra
3. SystemII 4E algebra
4. SystemII 5E algebra

**Proof.** We shall first show that SystemII 4 algebra is a SystemII 5E algebra.

To prove this first we shall prove the following.

e.1 $II0 \land I \sim Ia = I0$.

e.2 If $I \sim I \sim a \land I \sim Ia = \sim I \sim a$ then $a \land Ia = a$.

e.3 If $II0 = I0$ and $\sim IIa \land I \sim I0 = \sim I \sim I0$ then $I0 = 0$. 

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e.4 If $\sim I \sim a \land (\sim I \sim b \lor \sim I \sim Ia) \Rightarrow I \sim a$ then $a \land (b \lor Ia) = a$.

e.5 $\sim Ia \lor I \sim Ia \Rightarrow \sim I0$.

e.6 $IIa \land I \sim Ia = I0$.

e.7 $I \sim I \sim I0 = I0$.

e.8 If $II0 \land (II0 \lor I \sim Ia) = I0$ then $II0 = I0$.

e.9 $(II0 \lor I \sim Ia) \land I \sim I0 = I \sim Ia$.

e.10 If $II0 = I0$ then $I0 = 0$.

e.11 $\sim Ia \land I \sim Ia = I \sim Ia$.

Proof of e.1

Since $0 \leq a$ then $I0 \leq Ia$ [by Proposition 2.3.2]. So we can write

$\sim I0 \lor I0 \leq \sim I0 \lor Ia$. Now by using Axiom 5', $1 \leq \sim I0 \lor Ia$ which means $\sim I0 \lor Ia = 1$. Then by using axioms of qBa we have $I0 \land \sim Ia = 0$.

Hence by using Axiom 4', $II0 \land I \sim Ia = I0$. (e.1)

Proof of e.2

From Axiom 6' we have if $Ia \land Ila = Ia$ and $Ca \land CIta = Ca$ hold then $a \land Ia = a$. Now in SystemII4 $Ia \land Ila = Ia$ is always true. Thus

if $\sim I \sim a \land \sim I \sim Ia \Rightarrow \sim I \sim a$ then $a \land Ia = a$. (e.2)

Proof of e.3

From Axiom 6' we have if $I \sim Ia \land II0 = II0$ and

$\sim I \sim Ia \land \sim I \sim I0 \Rightarrow \sim I \sim I0$ then $I0 = \sim Ia \land I0 = 0$ [as $\sim Ia \land I0 = 0$ is already proved within the proof of (e.1)]. Now using (e.1) we have
if \(II_0 = I_0\) and \(\sim IIa \land \sim I \sim I_0 = \sim I \sim I_0\) then \(I_0 = 0\).  \hfill (e.3)

Proof of e.4

We know that \(Ia \leq Ia \lor b\). So by Proposition 2.3.2, \(IIa \leq I(Ia \lor b)\). Thus \(Ia \land I(Ia \lor b) = Ia \land (IIa \lor I(Ia \lor b)) = (Ia \land IIa) \lor (Ia \land I(Ia \lor b)) = Ia \lor (Ia \land I(Ia \lor b))\) (since \(Ia \leq IIa\))

= \(Ia\) [using absorption law].

Now \(Ia \land I(Ia \lor b) = Ia\) is true always, so from Axiom 6′ we have

if \(\sim I \sim a \land \sim I \sim (b \lor Ia) = \sim I \sim a\) then \(a \land (b \lor Ia) = a\). Using axiom of qBa,

Proposition 1.1.2(i) and Axiom 4′ we have

if \(\sim I \sim a \land (\sim I \sim b \lor \sim I \sim Ia) = \sim I \sim a\) then \(a \land (b \lor Ia) = a\). \hfill (e.4)

Proof of e.5

Using Axiom 5′ and axioms of qBa we have \(Ia \land \sim Ia = 0\), again using Axiom 4′, \(IIa \land I \sim Ia = Ia = I0\) \hfill (e.A)

Now since \(Ia \land IIa = Ia\) then \(Ia \land (IIa \land b) = Ia \land b\).

Substituting \(I \sim Ia\) for \(b\) we have \(Ia \land (IIa \land I \sim Ia) = Ia \land I \sim Ia\), so using (e.A)

\(Ia \land I_0 = Ia \land I \sim Ia\). As \(0 \leq a\), Proposition 2.3.2 yields \(I0 \leq Ia\) which means \(Ia \land I \sim Ia = I0\). Now using Proposition 1.1.2(i), we have

\(\sim Ia \lor \sim I \sim Ia = \sim I0\). \hfill (e.5)

Proof of e.6

Using (e.A) and Axiom 4′ we have \(IIIa \land II \sim Ia = II0\). \hfill (e.B)

Again since \(Ia \land IIa = Ia\) then \(Ia \land (b \land IIa) = b \land Ia\).

Now substituting \(\sim Ia\) for \(a\) and \(IIIa\) for \(b\), we have

\(I \sim Ia \land (IIIa \land II \sim Ia) = IIIa \land I \sim Ia\) or, \(I \sim Ia \land II0 = IIIa \land I \sim Ia\) [using (e.B)].

Now using (e.1), \(IIIa \land I \sim Ia = I0\). \hfill (e.6)
Proof of e.7

As $0 \leq a$, $I0 \leq Ia$ which means $I0 \lor Ia = Ia$. So by axioms of qBa we have $\sim I0 \land \sim Ia = \sim Ia$. Hence using Axiom 4', we get

$$I \sim I0 \land I \sim Ia = I \sim Ia.$$  \hfill (e.C)

Again using axioms of qBa in (e.5) we have $Ia \land I \sim Ia = I0$. \hfill (e.D)

Now substituting $\sim I0$ for $a$ in both (e.C) and (e.D) we have

$I \sim I0 \land I \sim I0 = I \sim I \sim I0 \sim I = I \sim I0 = I0$ respectively, which imply

$I \sim I \sim I0 = I0$. \hfill (e.7)

Proof of e.8

Let $IIa \land (Ia \lor b) = IIa \lor b$. So $(IIa \land Ia) \lor (IIa \land b) = IIa \land b$. Then $(IIa \land Ia) \land (IIa \land b) = IIa \land Ia$ which implies $Ia \land b = Ia$ [since $Ia \leq IIa$]. So, if $IIa \land (Ia \lor b) = IIa \land b$ then

$Ia \land b = Ia$. \hfill (e.E)

In SystemII4 algebra, we have $Ia \leq IIa$. Then by Proposition 2.3.2, $IIa \leq IIIa$, so $Ia \leq IIIa$ i.e., $Ia \land IIIa = Ia$ which implies that

$Ia \land (b \land IIIa) = b \land Ia$. \hfill (e.F)

Now substituting $\sim Ia$ for $a$ and $IIIa$ for $b$ in (e.F), we have

$I \sim Ia \land (IIIa \land III \sim Ia) = IIIa \land I \sim Ia$

i.e., $I \sim Ia \land (IIIa \land III \sim Ia) = I0$ [using (e.6)]

i.e., $I \sim Ia \land III0 = I0$ [using (e.D) and Axiom 4']. \hfill (e.G)

Again substituting $I0$ for $a$ and $I \sim Ia$ for $b$ in (e.E) we have

if $III0 \land (I0 \lor I \sim Ia) = IIII0 \land I \sim Ia$ then $I0 \land I \sim Ia = II0$. Hence using (e.G) and (e.1), we have

if $III0 \land (I0 \lor I \sim Ia) = I0$ then $II0 = I0$. \hfill (e.8)
Proof of e.9

We have \(0 \leq a\) then \(I0 \leq Ia\) then \(\sim Ia \leq \sim I0\) implies \(I \sim Ia \leq I \sim I0\)

i.e. \(I \sim Ia \lor I \sim I0 = I \sim I0\). \hspace{1cm} (e.H)

Again using Axiom 5', axioms of qBa and Axiom 4' we have

\[(a \lor IIb) \land (a \lor I \sim I) = a \lor I0.\] \hspace{1cm} (e.I)

Now substituting 0 for \(b\) and \(I \sim Ia\) for \(a\) in (e.I) we have

\[(I \sim Ia \lor II0) \land (I \sim Ia \lor I \sim I0) = I \sim Ia \lor I0,\] then using (e.H) we have \((I \sim Ia \lor II0) \land I \sim I0 = I \sim Ia \lor I0\). Now since \(0 \leq \sim Ia\) implies \(I0 \leq I \sim Ia\),

so \((II0 \lor I \sim Ia) \land I \sim I0 = I \sim Ia\). \hspace{1cm} (e.9)

Proof of e.10

Substituting \(\sim I \sim I0\) for \(a\) in (e.2) we get

if \(\sim II \sim I0 \land \sim I \sim I \sim I \sim I0 = \sim II \sim I0\)

then \(\sim I \sim I0 \land \sim I \sim I \sim I0 = \sim I \sim I0\). Now using (e.7) we have,

if \(\sim II \sim I0 \land \sim I \sim I \sim I0 = \sim II \sim I0\) then \(\sim I \sim I0 \land I0 = \sim I \sim I0\). \hspace{1cm} (e.J)

Again in SystemII4 we have \(Ia \lor IIa = IIa\) then using axioms of qBa we can write

\(\sim Ia \land \sim IIa = \sim IIa\). Now if we substitute \(\sim I0\) for \(a\) then we can conclude \(\sim I \sim I0 \land \sim II \sim I0 \sim I \sim I0 = \sim II \sim I0\) holds always. So from (e.J), we have \(\sim I \sim I0 \land I0 = \sim I \sim I0\) holds always. \hspace{1cm} (e.K)

Again from (e.5) we have \(\sim Ia \lor \sim I \sim Ia = \sim I0\)

i.e., \(Ia \land (\sim Ia \lor \sim I \sim Ia) = Ia \land \sim I0\)

i.e., \((Ia \land \sim Ia) \lor (Ia \land \sim I \sim Ia) = Ia \land \sim I0\). Hence by Axiom 5' and axioms of qBa, we get \(Ia \land \sim I \sim Ia = Ia \land \sim I0\). \hspace{1cm} (e.L)

By Axiom 5' and axioms of qBa, it follows from (e.L) (substituting 0 for \(a\)) that \(I0 \land \sim
\[ I \sim I0 = 0. \quad \text{(e.M)} \]

So, from (e.K) and (e.M) we have \( \sim I \sim I0 = 0. \quad \text{(e.N)} \)

Hence we can say that \( \sim IIa \land \sim I \sim I0 = \sim I \sim I0 \) holds always. Thus from (e.3) we have, if \( II0 = I0 \) then \( I0 = 0. \quad \text{(e.10)} \)

**Proof of e.11**

From (e.9) and (e.N) we have \( II0 \lor I \sim I0 = I \sim I0. \quad \text{(e.O)} \)

Now from (e.8) and (e.O) we have, if \( III0 \land I \sim I0 = I0 \) then \( II0 = I0 \).

Hence using (e.G) we can say \( II0 \sim I0 \), so from (e.10) we can conclude that \( I0 = 0 \) holds always. \( \quad \text{(e.P)} \)

Again from (e.5) and (e.P) we have \( \sim Ia \lor \sim I \sim Ia = 1 \)

i.e., \( I \sim Ia \land (\sim Ia \lor \sim I \sim Ia) = I \sim Ia \land 1 \)

i.e., \( (I \sim Ia \land \sim Ia) \land (I \sim Ia \land \sim I \sim Ia) = I \sim Ia \land I \sim Ia \).

Hence by Axiom 5' and axioms of qBa, \( \sim Ia \land I \sim Ia = I \sim Ia. \quad \text{(e.11)} \)

Finally we shall show that \( CIa = Ia. \)

From (e.5) and (e.P) we have \( \sim Ia \lor \sim I \sim Ia = 1 \) and then substituting \( \sim a \) for \( a \) we have, \( \sim I \sim a \land \sim I \sim I \sim a = 1 \quad \text{(e.Q)} \)

Again substituting \( \sim a \) for \( a \) and \( a \) for \( b \) in (e.4) we have

if \( \sim Ia \land (\sim I \sim a \lor \sim I \sim I \sim a) = \sim Ia \) then \( \sim a \land (a \lor I \sim a) = \sim a. \)

Since from (e.Q) \( \sim Ia \land (\sim I \sim a \lor \sim I \sim I \sim a) = \sim Ia \) holds always, hence \( \sim a \land (a \lor I \sim a) = \sim a \) holds always. Now substituting \( Ia \) for \( a \), we have \( \sim Ia \land (Ia \lor I \sim Ia) = \sim Ia. \)

So \( (\sim Ia \land Ia) \lor (\sim Ia \land I \sim Ia) = \sim Ia. \) Then by Axiom 5' and axioms of qBa, \( \sim Ia \land I \sim Ia = \sim Ia. \) Hence using (e.11) we have \( I \sim Ia = \sim Ia \), so \( \sim I \sim Ia = Ia \) i.e. \( CIa = Ia. \)

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Thus we have proved that a SystemII4 algebra is a SystemII5E algebra.

Now we shall show that a SystemII5 algebra is a SystemII4E algebra.

To prove it we first show that the following.

e.12 If $II0 \land Ia = II0$ then $I0 \land a = I0$.

e.13 $IIa \land (b \land I \sim Ia) = 0$.

e.14 $\sim IIa \lor I \sim Ia = 1$.

e.15 $I \sim Ia \land II \sim Ia = I \sim Ia$.

e.16 $IIa \lor I \sim Ia = 1$.

e.17 $I(a \lor I \sim Ia) = 1$.

e.18 $IIa \lor I \sim Ia = IIa$.

Proof of e.12

We have in SystemII5 $CIa \lor Ia = Ia$ which means $\sim I \sim Ia \lor Ia = Ia$ then we can write

$(I \sim Ia \lor I \sim Ia) \lor Ia = I \sim Ia \lor Ia$, hence using Axiom 5' we have

$I \sim Ia \lor Ia = 1$.  \hspace{1cm} (e.R)

Now $0 \leq \sim I0$ implies $I0 \leq I \sim I0$ (by Proposition 2.3.2 i.e.,

$I0 \lor I \sim I0 = I \sim I0$, hence using (e.R), $I \sim I0 = 1$ \hspace{1cm} (e.S)

From Axiom 6' we can write if $II0 \land Ia = II0$ and $\sim I \sim I0 \land \sim I \sim a =$
\(~ I \sim I0\) hold then \(I0 \land a = I0\).

Now using (e.S) we have \(~ I \sim I0 \land \sim I \sim a = \sim I \sim I0\) holds always, hence if \(II0 \land Ia = II0\) then \(I0 \land a = I0\).

\[\text{(e.12)}\]

**Proof of e.13**

Now using Axiom 5′, axioms of qBa and Axiom 4′ we have

\(IIa \land I \sim Ia = I0\). Substituting 0 for \(a\) and then using (e.S) we can write \(II0 = I0\). \(\text{(e.T)}\)

Now \(0 \leq a\) implies \(I0 \leq Ia\) (by Proposition 2.3.2) which means \(I0 \land Ia = I0\). Then using (e.T) we have \(II0 \land Ia = II0\) always holds, then by (e.12) \(I0 \land a = I0\) holds always, hence substituting 0 for \(a\), \(I0 = 0\).

\[\text{(e.U)}\]

Now \(IIa \land (b \land I \sim Ia) = I(IIa \land \sim Ia) \land b = I0 \land b = 0 \land b = 0\) [using Axiom 4′, Axiom 5′, axioms of qBa and (e.U)].

Hence, \(IIa \land (b \land I \sim Ia) = 0\).

\[\text{(e.13)}\]

**Proof of e.14**

Using axioms of qBa, Axiom 4′, Axiom 5′ and (e.U) we have

\(~ IIa \lor \sim I \sim Ia = \sim (IIa \land I \sim Ia) = \sim I(IIa \land \sim Ia) = \sim I0 = \sim 0 = 1\). Hence, \(~ IIa \lor \sim I \sim Ia = 1\).

\[\text{(e.14)}\]

**Proof of e.15**

In SystemII5 algebra we have \(~ I \sim Ia \lor Ia = Ia\) then using axioms of qBa and Axiom 4′, we can write \(I \sim Ia \land IIa \sim Ia = I \sim Ia\).

\[\text{(e.15)}\]

**Proof of e.16**

Now using Proposition 2.3.2 and the fact that \(~ Ia \leq \sim Ia \lor b\),

we have \(I(\sim Ia \lor b) = I \sim Ia \lor I(\sim Ia \lor b)\). So, \(Ia \lor I(\sim Ia \lor b) = Ia \lor (I \sim Ia \lor I(\sim Ia \lor b)) = 1 \lor I(\sim Ia \lor b) = 1\) [using (e.R)].
So $Ia \lor I(\sim Ia \lor b) = 1$. \hfill (e.V)

Now substituting $\sim Ia$ for $a$ and $Ia$ for $b$ in (e.V) we have

$I \sim Ia \lor I(\sim I \sim Ia \lor Ia) = 1$. Hence using the fact that $\sim I \sim Ia = Ia$ holds in SystemII5, we can write that $IIa \lor I \sim Ia = 1$. \hfill (e.16)

Proof of e.17

As $a \leq a \lor \sim Ia$, Proposition 2.3.2 yields $I(a \lor \sim Ia) = Ia \lor I(a \lor \sim Ia) = 1$ (using (e.V)). So $I(a \lor \sim Ia) = 1$. \hfill (e.17)

Proof of e.18

Now $(IIa \lor I \sim Ia) \land (IIa \lor I \sim Ia) = IIa \lor (I \sim Ia \land I \sim Ia) = IIa \lor 0 = IIa$, by Axiom 5' and axioms of qBa.

Hence using (e.16) we can write $IIa \lor I \sim Ia = IIa$. \hfill (e.18)

Proof of e.19

Also $(I \sim Ia \lor IIa) \land (I \sim Ia \lor IIa) = I \sim Ia \lor (IIa \land I \sim Ia) =$

$I \sim Ia \lor 0 = I \sim Ia$, by Axiom 5' and axioms of qBa. Hence using (e.16) we can write $I \sim Ia \lor IIa = I \sim Ia$. \hfill (e.19)

Proof of e.20

Now substituting $\sim Ia$ for $b$ in (e.13) we have $IIa \land (\sim Ia \land I \sim Ia) = 0$ which means $IIa \land Ia = 0$ [since in SystemII5 $I \sim Ia \leq Ia$, which implies $\sim Ia \leq I \sim Ia$]. Then using Proposition 1.1.2(i), (iii) $IIa \lor Ia = 1$, which implies $IIa \land (\sim IIa \lor Ia) = IIa \land 1$

i.e., $(IIa \land \sim IIa) \lor (IIa \land Ia) = IIa$. Hence by Axiom 5' and axioms of qBa we have $IIa \land Ia = IIa$. \hfill (e.W)

Now substituting $\sim Ia$ for $a$ in (e.W) we have

$II \sim Ia \land I \sim Ia = II \sim Ia$. Then using (e.15) we can write
II \sim Ia = I \sim Ia. \tag{e.20}

Proof of e.21

We have \((\sim IIa \lor I \sim Ia) \land (\sim IIa \lor \sim I \sim Ia)\)

\(=\ IIa \lor (I \sim Ia \land \sim I \sim Ia) =\ IIa \lor 0 =\ IIa,\) by Axiom 5' and axioms of qBa.

Thus \((\sim IIa \lor I \sim Ia) \land (\sim IIa \lor \sim I \sim Ia) =\ IIa.\) Hence using (e.14) we can write

\(\sim IIa \lor I \sim Ia =\ IIa,\) so by (e.19), \(\sim IIa = I \sim Ia.\) \(\tag{e.21}\)

Finally we shall show that \(IIa = Ia.\)

Substituting \(\sim Ia\) for \(a\) in (e.17) we have \(I(\sim Ia \lor \sim I \sim Ia) = 1,\) then with the help of (e.21) we can write \(I(IIa \lor \sim Ia) = 1.\)

Hence \(Ia \land I(IIb \lor \sim Ib) = Ia\) holds always. \(\tag{e.X}\)

Again \(\sim I \sim a \land \sim I \sim (IIb \lor \sim Ib) =\ \sim I \sim a \land \sim I(\sim IIb \land Ib)\)

\(=\ \sim I \sim a \land \sim I(I \sim Ib \land Ib) \ [\text{using (e.21)}]\)

\(=\ \sim I \sim a \land \sim (II \sim Ib \land IIb) \ [\text{using Axiom 4'}]\)

\(=\ \sim I \sim a \land \sim (I \sim Ib \land IIb) \ [\text{using (e.20)}]\)

\(=\ \sim I \sim a \land (\sim I \sim Ib \lor \sim IIb) =\ \sim I \sim a \land 1 \ [\text{using (e.14)}] =\ \sim I \sim a.\)

So \(\sim I \sim a \land \sim I \sim (IIb \lor \sim Ib) =\ \sim I \sim a\) holds always. \(\tag{e.Y}\)

Hence from (e.X), (e.Y) and Axiom 6' we can conclude that \(a \land (IIb \lor \sim Ib) = a,\) so we have \((a \land IIb) \lor (a \land \sim Ib) = a.\) Now substituting \(Ia\) for \(a\) and then \(a\) for \(b\) we can write \((Ia \land IIa) \lor (Ia \land \sim Ia) = Ia,\) which means

\(Ia \land IIa = Ia.\) \(\tag{e.Z}\)

Hence from (e.W) and (e.Z) we can conclude \(IIa = Ia.\)

Thus it is proved that a SystemII5 algebra is a SystemII4E algebra.

It is also obvious that any SystemII4E algebra is a SystemII4 algebra and any SystemII5E
algebra is a SystemII5 algebra. Hence the the required equivalence is established.

We next show a few non-implications.

**Proposition 2.4.15.** *SystemIB and systemI4 are independent.*

*Proof.* SystemIB algebra is not necessarily a SystemI4 algebra.

In Example 2.4.2, $IIa = Ib = a$ and $Ia = b$, since $a$ and $b$ are non comparable elements, so $Ia \not\preceq IIa$. Hence this is not a SystemI4 algebra but this is an example of SystemIB algebra.

Again, SystemI4 algebra is not necessarily a SystemIB algebra. In Example 2.4.4, $CIa = C1 = 1 > a$, so $CIa \not\preceq a$. Hence this is not a SystemIB algebra but this is an example of SystemI4 algebra.

**Proposition 2.4.16.** *SystemIB and systemI5 are independent.*

*Proof.* SystemIB algebra is not necessarily a SystemI5 algebra and SystemI5 algebra is not necessarily a SystemIB algebra. These can be easily verified by the Example 2.4.2 and Example 2.4.4 respectively.

**Proposition 2.4.17.** *SystemIB and systemI4E are independent.*

*Proof.* The result follows from Proposition 2.4.15.

**Proposition 2.4.18.** *SystemIB and systemI5E are independent.*
Proof. The result follows from Proposition 2.4.16.

Remark 2.4.19. The following implications are not yet settled.

- If $A$ is a SystemI algebra then it is a SystemII algebra.
- If $A$ is a SystemI4 algebra then it is a SystemI4E algebra.
- If $A$ is a SystemI5 algebra then it is a SystemI5E algebra.
- If $A$ is a SystemI4 algebra then it is a SystemI5E algebra.
- If $A$ is a SystemI5 algebra then it is a SystemI4E algebra.
- If $A$ is a SystemI4E algebra then it is a SystemI5E algebra.

Remark 2.4.20. Examples 2.2.2, 2.4.2 and 2.4.4 are not pre-rough algebras.
Relationship-diagram 2

The following figure (Fig. 10) summarizes the relationship among the algebras discussed so far.

If \( A \) and \( B \) are two algebraic structures then \( A \Rightarrow B \) means that one extra operator and some axioms for the new operator are added with \( A \) to obtain \( B \). \( A \rightarrow B \) means that \( A \) and \( B \) have the same operators and if an algebra is \( B \) then it is \( A \). If \( A \) and \( B \) be two algebraic systems then \( A \cdots B \) means, \( A \) and \( B \) are independent.

2.5 Investigation of a few properties

We now discuss the following four properties viz.

(i) \( I_0 = 0 \)
D: $Ia \leq Ca$

K: $I(a \Rightarrow b) \leq Ia \Rightarrow Ib$

A: $Ia = a$ if and only if $Ca = a$

We have already mentioned about the three properties (i) $(I0 = 0)$, D($Ia \leq Ca$) and A($Ia = a$ if and only if $Ca = a$). The property K($I(a \Rightarrow b) \leq Ia \Rightarrow Ib$) corresponds to the K-axiom of modal logic.

It is to note that in our case $\Rightarrow$ is not a Boolean operator, that means $a \Rightarrow b \neq \neg a \lor b$.

**Proposition 2.5.1.** K holds in SystemI5E, SystemII5, SystemII5E, SystemII4 and SystemII4E.

**Proof.** We want to show that K, i.e., $I(a \Rightarrow b) \Rightarrow (Ia \Rightarrow Ib) = 1$ holds in SystemI5E.

First we shall prove that

1. $ICa = Ca$

2. $IIa = Ia$

3. $Ia \lor \sim I(a \land b) = 1$ and

4. $CA \lor \sim I(A \land B) = 1$, where $A = \sim Ia \lor Ib$ and $B = \sim Ca \lor Cb$.

In SystemI5E, $CI \sim a = I \sim a$ holds, so $CI \sim a = \sim Ia \sim a$.

In other words, $ICa = Ca$. \hspace{1cm} (2.5.1.1)

Again $Ia = CIIa = IClIa = IIa$, by (2.5.1.1). So $IIa = Ia$. \hspace{1cm} (2.5.1.2)

As $a \land b \leq a$, $I(a \land b) \leq Ia$ [by Axiom 7']. So, $\sim Ia \leq I(a \land b)$.

But $Ia \lor \sim Ia = 1$. Hence $Ia \lor \sim I(a \land b) = 1$. \hspace{1cm} (2.5.1.3)

We have $Ia \land \sim Ib \leq Ia$ and $Ia \land \sim Ib \leq \sim Ib$. Then by Axiom 7', $I(Ia \land \sim Ib) \leq IIa$ and
\[ I(Ia \land \sim Ib) \leq I \sim Ib. \] So, \( I(Ia \land \sim Ib) \leq Ia \) and \( I(Ia \land \sim Ib) \leq \sim Ib \), by (2.5.1.2) and (2.5.1.1) respectively.

Hence, \( \sim Ia, Ib \leq \sim I(Ia \land \sim Ib) = C(\sim Ia \lor Ib) = CA \). Thus \( A \leq CA \) and then \( IA \leq ICA = CA \). So by (2.5.1.3), \( CA \lor \sim I(A \land B) = 1 \). (2.5.1.4)

Now, \( a \Rightarrow b = (\sim Ia \lor Ib) \land (\sim C_a \lor C_b) = A \land B \). Also, \( Ia \Rightarrow Ib \)

\[ = (\sim Ia \lor Ib) \land (\sim CIA \lor CIB) = \sim Ia \lor Ib = A \] [by (2.5.1.2) and \( CIa = Ia \)].

\( I(a \Rightarrow b) \Rightarrow (Ia \Rightarrow Ib) = 1 \) if and only if

\( (\sim I(Ia \Rightarrow b) \lor I(Ia \Rightarrow Ib)) \land (\sim C(I(a \Rightarrow b)) \lor C(Ia \Rightarrow Ib)) = 1 \) if and only if

\( (\sim I(a \Rightarrow b) \lor I(Ia \Rightarrow Ib)) = 1 = (\sim I(a \Rightarrow b) \lor C(Ia \Rightarrow Ib)) \) [by (2.5.1.2) and \( CIA = Ia \)].

So, we have to show that \( \sim I(A \land B) \lor IA = 1 = \sim I(A \land B) \lor CA \). We have already proved these in (2.5.1.3) and (2.5.1.4).

Now using Proposition 2.3.2 we can prove that any SystemII5E algebra is a SystemI5E algebra. So \( I(a \Rightarrow b) \Rightarrow (Ia \Rightarrow Ib) = 1 \) i.e., \( K \) also holds in SystemII5E algebra. So by Proposition 2.4.14, \( I(a \Rightarrow b) \Rightarrow (Ia \Rightarrow Ib) = 1 \) i.e., \( K \) also holds in SystemII5, SystemII4 and SystemII4E.

\[ \square \]

**Remark 2.5.2.** No information about \( K: I(a \Rightarrow b) \leq Ia \Rightarrow Ib \) with respect to any other system discussed in this section is available.

**Proposition 2.5.3.** In System0, D: \( Ia \leq Ca \) implies (i): \( I0 = 0 \) but the converse does not hold.

**Proof.** Since \( I1 = 1, C0 = \sim I \sim 0 = \sim I1 = \sim 1 = 0 \). Now, since \( Ia \leq Ca \), for all \( a \), \( I0 \leq C0 = 0 \), so \( I0 = 0 \).

That the converse part does not hold may be verified by the Example 2.4.4.
Proposition 2.5.4. In SystemIB, D: \( Ia \leq Ca \) holds. So, (i): \( I0 = 0 \) also holds.

Proof. We want to show that D, i.e., \( Ia \leq Ca \) holds in SystemIB.

As \( CIa \leq a \) holds in SystemIB, \( a \lor I \sim Ia = (a \lor CIa) \lor I \sim Ia = a \lor (CIa \lor I \sim Ia) = a \lor 1 = 1 \), by Axiom 5'.

So, \( a \lor I \sim Ia = 1 \). (2.5.4.1)

As \( 0 \leq a \) by Axiom 7' we have \( I0 \leq Ia \) for all \( a \).

So \( I \sim II0 = I0 \lor I \sim II0 = 1 \), by replacing \( a \) by \( I0 \) in (2.5.4.1).

Thus, \( I \sim II0 = 1 \). (2.5.4.2)

Now, \( II0 \leq ICI00 \) (as \( a \leq ICa \)) = \( I \sim I \sim II0 = I \sim 1 \) (by (2.5.4.2)) = \( I0 \). Also, \( CI0 \leq 0 = C0 \). So by Axiom 6', we get \( I0 \leq 0 \).

In other words, \( I0 = 0 \). (2.5.4.3)

We know that \( \sim Ia \leq Ia \lor b \), so by Axiom 7', \( I \sim Ia \leq I(\sim Ia \lor b) \). Now from (2.5.4.1) it follows that \( a \lor I(\sim Ia \lor b) = 1 \). Hence, \( I(\sim IIa \lor a) = Ia \lor I(\sim IIa \lor a) = 1 \), as \( a \leq \sim IIa \lor a \).

Again \( CI(\sim IIa \lor a) \leq \sim IIa \lor a \). So \( C1 \leq \sim IIa \lor a \).

But \( C1 = \sim I \sim 1 = \sim I0 = \sim 0 \) (by (2.5.4.3)) = 1.

So, \( \sim IIa \lor a = 1 \). (2.5.4.4)

\[
\begin{align*}
  a \lor IIa &= (a \lor IIa) \land 1 = (a \lor IIa) \land (a \lor \sim IIa) \quad \text{(by (2.5.4.4)}
  \\
  &= a \lor (IIa \land \sim IIa) = a \lor 0 \quad \text{(by axioms of qBa and Axiom 5')}
  \\
  &= a.
\end{align*}
\]

Hence \( IIa \leq a \). Replacing \( a \) by \( Ca \), we have \( IICa \leq Ca \).

Again in SystemIB, \( a \leq ICa \). Thus by Axiom 7', \( Ia \leq IICa \). Hence \( Ia \leq Ca \).

\( \square \)
Remark 2.5.5. That $D: I_a \leq C_a$ does not hold in any other system discussed in this section can be verified by Example 2.4.4.

Remark 2.5.6. $I_0 = 0$ holds in SystemIB, SystemII4, SystemII5, SystemII4E and SystemII5E. First one follows from Proposition 2.5.4. All the rest follow from the proof of Proposition 2.5.1.

Proposition 2.5.7. In topological quasi Boolean algebra, the property $A: I_a = a$ if and only if $C a = a$ holds if and only if $C I a = I a$.

Proof. This is the same as Proposition 1.2.6.

\[\square\]

Proposition 2.5.8. The property $A: I_a = a$ if and only if $C a = a$ holds in SystemI5E, SystemII5E and SystemIB.

Proof. From the converse part of Proposition 2.5.7, it is clear that $A: I_a = a$ if and only if $C a = a$ holds in SystemI5E and hence SystemII5E.

Now we have to show $A: I_a = a$ if and only if $C a = a$ holds in SystemIB.

Let $C I x \leq x$, for all $x$ and let $C a = a$, now $C I x \leq x$ means $\sim I \sim I x \leq x$ then replace $x$ by $\sim x$ we have, $\sim I C x \leq \sim x$, substituting $x = a$, then $\sim I C a \leq \sim a$ implies $\sim I a \leq \sim a$ implies $a \leq I a$. Now in SystemIB, $D: I_a \leq C a$ holds, so $I_a \leq C a \leq a$ implies $I_a \leq a$, hence $I a = a$.

Again, if we assume $C I x \leq x$, for all $x$ and $I a = a$ then it can be easily shown that $C a = a$.

\[\square\]

Remark 2.5.9. No conclusion in other cases has been arrived at.
**Observation 2.5.10.** In all of these algebras an implication operator $\Rightarrow_R$ can be defined in the same way as in pre-rough algebra by $a \Rightarrow_R b := (\sim Ia \lor Ib) \land (\sim Ca \lor Cb)$.

**Proposition 2.5.11.** In all of these algebras $a \Rightarrow_R b = 1$ if and only if $a \leq b$.

### 2.6 The Hilbert Systems LI4, LI4E, LI5, LI5E, LIB, LII4

The languages of all the systems LI4, LI4E, LI5, LI5E, LIB, LII4 corresponding to SystemI4, SystemI4E, SystemI5, SystemI5E, SystemIB, SystemII4 algebras respectively are the same as that of LI. In all cases, we take all axioms and rules of LI, together with some extra axiom(s) viz.,

- $l\alpha \rightarrow ll\alpha$ for LI4
- $l\alpha \rightarrow ll\alpha$ and $ll\alpha \rightarrow l\alpha$ for LI4E
- $ml\alpha \rightarrow l\alpha$ for LI5
- $ml\alpha \rightarrow l\alpha$ and $l\alpha \rightarrow ml\alpha$ for LI5E
- $ml\alpha \rightarrow \alpha$ for LIB.

The language of the system LII4 corresponding to SystemII4 algebra is the same as that of LI. We take all axioms and rules of LI, together with two extra axioms $l(\alpha \land \beta) \rightarrow l(\alpha) \land l(\beta)$ and $l\alpha \rightarrow ll\alpha$.

**Theorem 2.6.1.** All these Hilbert systems are sound and complete with respect to the corresponding algebras.
2.7 The Sequent Calculi

Sequent Calculus for SystemI4 algebra

sqI4 is the sequent calculus for the algebra SystemI4 obtained from sqI only by replacing the rule \((Rl)^r\) by the rule

\[
(Rl')^r \quad \frac{l\beta : \alpha}{l\beta : l\alpha}
\]

Sequent Calculus for SystemI5 algebra

In a similar way, the sequent calculus sqI5 is obtained from sqI by adding the \(Ax3\).

Sequent Calculus for SystemI4E algebra

sqI4 along with the rule

\[
Ll'' \quad \frac{\beta : l\alpha}{l\beta : l\alpha}
\]

forms the sequent calculus sqI4E corresponding to the algebra SystemI4E.

Sequent Calculus for SystemII4 algebra

sqII4 is the sequent calculus for the algebra SystemII4 obtained from sqII only by replacing the rule \((Rl)^r\) by the previously stated rule \(Rl\).

Sequent Calculus for SystemII5 algebra

In a similar way, the sequent calculus sqII5 is obtained from sqII by adding the \(Ax3\).

Sequent Calculus for SystemII4E algebra

sqII4 along with the rule

\[
Ll' \quad \frac{\Gamma : l\alpha}{l\Gamma : l\alpha}
\]