Appendices
In this appendix, we give the details of canonical transformations and their inverse transformations in $2 + 1$D on a finite lattice for $Z_2$ (section A.1) and $SU(N)$ (section A.2) lattice gauge theory. Canonical transformations in $3 + 1$D is illustrated on a single cube in section A.3 for both $Z_2$ as well as $SU(N)$ lattice gauge theory.

A.1 $Z_2$ lattice gauge theory in $2 + 1$ dimensions

A.1.1 From links to loops & strings

In this section, we describe the canonical transformation involved in the construction of the loop/spin formulation of $Z_2$ lattice gauge theory on a finite lattice. The net canonical transformation leads to the duality relation between the basic operators of $Z_2$ gauge theory and Ising model in $2 + 1$ dimensions. Construction of this duality relation on a single plaquette was described in section 3.1.2. We now generalize the duality transformation relation to a finite lattice by iterating the plaquette canonical transformation (C.T) ((3.20), (3.21)) all over the two dimensional lattice starting from the top left plaquette of the lattice and systematically repeating it from top to bottom and left to right. We will illustrate this procedure on a $2 \times 2$ lattice which contains all the essential features of the construction on any finite lattice. The sites of the lattice are labelled as $O \equiv (0, 0), A \equiv (0, 1), B \equiv (0, 2), C \equiv (1, 0), D \equiv (1, 1), E \equiv (1, 2), F \equiv (2, 0), G \equiv (2, 1), H \equiv (2, 2)$ and the plaquettes are numbered from top to bottom and left to right (see Figure A.1) for convenience. The dual spin operators are constructed on a $2 \times 2$ lattice in 4 steps.

1. We begin by performing the plaquette canonical transformation (3.20), (3.21) on plaquette 1. The spin conjugate operators $\{\mu_1(1); \mu_3(1)\}$ on plaquette 1 are

$$\mu_1(1) \equiv \mu_1(E) = \sigma_3(A, \hat{1}) \sigma_3(D, \hat{2}) \sigma_3(B, \hat{1}) \sigma_3(A, \hat{2}),$$

$$\mu_3(1) \equiv \mu_3(E) = \sigma_1(B, \hat{1}).$$

(A.1)
Figure A.1: The ‘plaquette’ canonical transformations involved in the construction of the duality transformation between $Z_2$ lattice gauge theory and $Z_2$ spin model on a $2 \times 2$ lattice. The steps (a), (b), (c) and (d) are plaquette CTs on plaquettes 1, 2, 3 and 4 respectively. The electric field $\sigma_1(l)$ corresponding to the vertical and horizontal links are denoted by • and ♦ respectively.

The redefined link and string operators around plaquette 1 are

$$
\sigma_{3[x]}(D, \hat{2}) = \sigma_3(D, \hat{2}), \quad \sigma_{1[x]}(D, \hat{2}) = \sigma_1(D, \hat{2}) \sigma_1(B, \hat{1}) \\
\sigma_{3[x]}(A, \hat{2}) = \sigma_3(A, \hat{2}), \quad \sigma_{1[x]}(A, \hat{2}) = \sigma_1(A, \hat{2}) \sigma_1(B, \hat{1}) \\
\sigma_{3[x]}(A, \hat{1}) = \sigma_3(A, \hat{1}), \quad \sigma_{1[x]}(A, \hat{1}) = \sigma_1(A, \hat{1}) \sigma_1(B, \hat{1}).
$$

Our notation is such that $\sigma_3(A, \hat{1})$ denotes the $\sigma_3$ operator corresponding to the link which starts at site A and is in the $\hat{1}$ direction. The subscript $[x]$ on $\sigma_{3[x]}(A, \hat{1})$ indicates that the electric field $\sigma_1(A, \hat{1})$ absorbs the electric field of the vanishing horizontal link $(B, \hat{1})$ to become $\sigma_{1[x]}(A, \hat{1})$ during the plaquette canonical transformation. Note that by our convention, the plaquette or spin operators are labelled by the top right corner of the plaquette. This plaquette canonical transformation is illustrated in Figure A.1 (a).

As a result of Gauss law at B:

$$
\sigma_{1[x]}(A, \hat{2}) \equiv g(B) \approx 1.
$$

Therefore, $\{\sigma_{1[x]}(A, \hat{2}); \sigma_{3[x]}(A, \hat{2})\} \equiv \{\tilde{\sigma}_1(B); \tilde{\sigma}_3(B)\}$ are frozen and hence decouple from the physical Hilbert space. Again, as in the main text, the string operators are labelled by their right/top endpoints. We are now left with the conjugate spin opera-
tors \( \{ \mu_1(1); \mu_3(1) \} \) and the two link conjugate pair operators \( \{ \sigma_{1|x}(D, \hat{2}); \sigma_{3|x}(D, \hat{2}) \}, \{ \sigma_{1|x}(A, \hat{1}); \sigma_{3|x}(A, \hat{1}) \} \). These link operators undergo further canonical transformations.

2. We now iterate the plaquette canonical transformation. on plaquette 2 to construct the spin or plaquette conjugate operators \( \{ \mu_1(2); \mu_3(2) \} \) and the link conjugate operators

\[
\begin{align*}
\sigma_{1|x}(C, \hat{2}); & \; \sigma_{3|x}(C, \hat{2}) \\
\{ \sigma_{1|x}(O, \hat{1}); & \; \sigma_{3|x}(O, \hat{1}) \}, \{ \sigma_{1|x}(O, \hat{2}); & \; \sigma_{3|x}(O, \hat{2}) \}
\end{align*}
\]

as illustrated in Figure A.1. The spin operators are

\[
\begin{align*}
\mu_1(2) & \equiv \mu_1(D) = \sigma_3(A, \hat{1})\sigma_3(O, \hat{2})\sigma_3(O, \hat{1})\sigma_3(C, \hat{2}), \\
\mu_3(2) & \equiv \mu_3(D) = \sigma_1(A, \hat{1}) = \sigma_1(A, \hat{1})\sigma_1(B, \hat{1})
\end{align*}
\]

The redefined link and new string operators around plaquette 2 are

\[
\begin{align*}
\sigma_{3|x}(C, \hat{2}) & = \sigma_3(C, \hat{2}), & \sigma_{1|x}(C, \hat{2}) & = \sigma_1(C, \hat{2})\sigma_{1|x}(A, \hat{1}) = \sigma_1(C, \hat{2})\sigma_1(A, \hat{1})\sigma_1(B, \hat{1}) \\
\sigma_{3|x}(O, \hat{1}) & = \sigma_3(O, \hat{1}), & \sigma_{1|x}(O, \hat{1}) & = \sigma_1(O, \hat{1})\sigma_{1|x}(A, \hat{1}) = \mathcal{G}(O)\mathcal{G}(A)\mathcal{G}(B) \approx 1 \\
\sigma_{3|x}(O, \hat{2}) & = \sigma_3(O, \hat{2}), & \sigma_{1|x}(O, \hat{2}) & = \sigma_1(O, \hat{2})\sigma_{1|x}(A, \hat{1}) = \mathcal{G}(A)\mathcal{G}(B) \approx 1.
\end{align*}
\]

Thus the string conjugate pairs \( \{ \sigma_{1|x}(O, \hat{1}); \sigma_{3|x}(O, \hat{1}) \} \equiv \{ \sigma_1(C); \sigma_3(C) \} \) and \( \{ \sigma_{1|x}(O, \hat{2}); \sigma_{3|x}(O, \hat{2}) \} \equiv \{ \sigma_1(A); \sigma_3(A) \} \) are frozen due to Gauss law at O, A and B.

3. The third step involves iterating the plaquette canonical transformation. on plaquette 3 as shown in Figure A.1(c). This leads to decoupling of \( \{ \sigma_{3|x}(G, \hat{2}); \sigma_{3|x}(G, \hat{2}) \} \equiv \{ \sigma_1(H); \sigma_3(H) \}, \{ \sigma_{1|x}(D, \hat{2}); \sigma_{3|x}(D, \hat{2}) \} \equiv \{ \sigma_1(E); \sigma_3(E) \} \) due to the \( \mathbb{Z}_2 \) Gauss laws at E and H. The canonical transformations on plaquette 3 defining the spins are

\[
\begin{align*}
\mu_1(3) & \equiv \mu_1(H) = \sigma_3(E, \hat{1})\sigma_3(G, \hat{2})\sigma_3(D, \hat{1})\sigma_3(D, \hat{2}), \\
\mu_3(3) & \equiv \mu_3(H) = \sigma_3(E, \hat{1})
\end{align*}
\]

The redefined links and strings around plaquette 3 are

\[
\begin{align*}
\sigma_{3|x}(D, \hat{2}) & = \sigma_{3|x}(D, \hat{2}) = \sigma_3(D, \hat{2}), & \sigma_{1|x}(D, \hat{2}) & = \sigma_{1|x}(D, \hat{2})\sigma_1(E, \hat{1}) = \mathcal{G}(E) \approx 1 \\
\sigma_{3|x}(G, \hat{2}) & = \sigma_3(G, \hat{2}), & \sigma_{1|x}(G, \hat{2}) & = \sigma_1(G, \hat{2})\sigma_1(E, \hat{1}) = \mathcal{G}(H) \approx 1 \\
\sigma_{3|x}(D, \hat{1}) & = \sigma_3(D, \hat{1}), & \sigma_{1|x}(D, \hat{1}) & = \sigma_1(D, \hat{1})\sigma_1(E, \hat{1})
\end{align*}
\]
4. Finally, we iterate the plaquette canonical transformation, on plaquette 4 which are shown in Figure A.1(d). The conjugate spin operators \( \{ \mu_1(4); \mu_3(4) \} \) on plaquette 4 are

\[
\begin{align*}
\mu_1(4) &\equiv \mu_1(G) = \sigma_3(D, \hat{i})\sigma_3(F, \hat{j})\sigma_3(C, \hat{i})\sigma_3(C, \hat{j}), \\
\mu_3(4) &\equiv \mu_3(G) = \sigma_{1[\bar{x}]}(D, \hat{i}) = \sigma_1(D, \hat{i})\sigma_1(E, \hat{i})
\end{align*}
\] (A.5)

The remaining string operators are

\[
\begin{align*}
\sigma_{3[\bar{x}]}(C, \hat{j}) &= \sigma_{3[\bar{x}]}(C, \hat{j}) = \sigma_3(C, \hat{j}), \\
\sigma_{1[\bar{x}]}(C, \hat{j}) &= \sigma_{1[\bar{x}]}(C, \hat{j}) = \sigma_1(C, \hat{j}) = G(D)G(E) \approx 1 \\
\sigma_{3[\bar{x}]}(C, \hat{i}) &= \sigma_3(C, \hat{i}) \\
\sigma_{1[\bar{x}]}(C, \hat{i}) &= \sigma_1(C, \hat{i}) = G(C)G(O)G(A)G(D)G(B)G(E) \approx 1 \\
\sigma_{3[\bar{x}]}(F, \hat{j}) &= \sigma_3(F, \hat{j}) \\
\sigma_{1[\bar{x}]}(F, \hat{j}) &= \sigma_1(F, \hat{j}) = G(G)G(H) \approx 1.
\end{align*}
\] (A.6)

Gauss laws at O, A, B, C, D, E, G and H implies that the remaining string operators \( \{ \sigma_{3[\bar{x}]}(C, \hat{j}); \sigma_{3[\bar{x}]}(C, \hat{j}) \} \equiv \{ \tilde{\sigma}_1(D); \tilde{\sigma}_3(D) \} \), \( \{ \sigma_{1[\bar{x}]}(C, \hat{i}); \sigma_{3[\bar{x}]}(C, \hat{i}) \} \equiv \{ \tilde{\sigma}_1(F); \tilde{\sigma}_3(F) \} \) and \( \{ \sigma_{1[\bar{x}]}(F, \hat{j}); \sigma_{3[\bar{x}]}(F, \hat{j}) \} \equiv \{ \tilde{\sigma}_1(G); \tilde{\sigma}_3(G) \} \) are frozen. As a result, after the series of 4 plaquette canonical transformations, all the Gauss law constraints are solved. Only the plaquette/spin variables \( \{ \mu_1(1); \mu_3(1) \}, \{ \mu_1(2); \mu_3(2) \}, \{ \mu_1(3); \mu_3(3) \} \) and \( \{ \mu_1(4); \mu_3(4) \} \) remains in the physical Hilbert space. This leads to a dual \( Z_2 \) spin model. These results can be directly generalized to any finite lattice without any new issues, to give the duality relations (3.26a),(3.26b), (3.28a)-(3.28b).

### A.1.2 From loops & strings to links (Inverse transformations)

In this section we will invert the above transformations to write down the link operators \( \{ \sigma_1(n, \hat{i}); \sigma_2(n, \hat{i}) \} \) in terms of the plaquette and string variables \( \{ \mu_1(p), \mu_3(p) \} \) and \( \{ \tilde{\sigma}_1(n, \hat{i}), \tilde{\sigma}_3(n, \hat{i}) \} \) respectively. We will consider a 2 \( \times \) 2 lattice and explicitly construct these inverse relations by inverting the steps 4-1 involved in the construction of the loop formulation described in the previous section.
1. We start by inverting step (4) in section A.1.1 by the inverse transformation:
\[
\begin{bmatrix}
\mu(4), \sigma_\text{x}[F, \hat{2}], \sigma_\text{x}[C, \hat{1}], \sigma_\text{xx}[C, \hat{2}]
\end{bmatrix} \rightarrow
\begin{bmatrix}
\sigma(F, \hat{2}), \sigma(C, \hat{1}), \sigma_\text{x}[C, \hat{2}], \sigma_\text{x}[D, \hat{1}]
\end{bmatrix}.
\]
\[
\sigma_\text{x}[D, \hat{1}] = \mu_1(4)\sigma_\text{x}[F, \hat{2}]\sigma_\text{x}[C, \hat{1}]\sigma_\text{xx}[C, \hat{2}]
= \mu_1(4)\bar{\sigma}_3(G)\bar{\sigma}_3(F)\bar{\sigma}_3(D);
\]
\[
\sigma_\text{x}[D, \hat{1}] = \mu_3(4).
\]
\[
\sigma_3(F, \hat{2}) = \sigma_\text{x}[F, \hat{2}]; \quad \overline{\sigma}_1(F, \hat{2}) = \sigma_\text{x}[F, \hat{2}]\mu_3(4) = \overline{\sigma}_1(G)\mu_3(4)
\]
\[
\sigma_3(C, \hat{1}) = \sigma_\text{x}[C, \hat{1}]; \quad \overline{\sigma}_1(C, \hat{1}) = \sigma_\text{x}[C, \hat{1}]\mu_3(4) = \overline{\sigma}_1(F)\mu_3(4)
\]
\[
\sigma_3(C, \hat{2}) = \sigma_\text{x}[C, \hat{2}] = \sigma_3(D); \quad \sigma_1(C, \hat{2}) = \sigma_\text{x}[C, \hat{2}]\mu_3(4) = \sigma_1(D)\mu_3(4)
\]

In the above, the \(\sigma\) without a subscript 1 or 3 denotes the conjugate pair \(\{\sigma_1, \sigma_3\}\). Same is true for \(\mu\) also.

2. Inverting step (3) in section A.1.1 by the inverse transformation:
\[
\begin{bmatrix}
\mu(3), \sigma_\text{x}[D, \hat{1}], \sigma_\text{x}[G, \hat{2}], \sigma_\text{xx}[D, \hat{2}]
\end{bmatrix} \rightarrow
\begin{bmatrix}
\sigma(D, \hat{1}), \sigma(G, \hat{2}), \sigma(E, \hat{1}), \sigma_\text{x}[D, \hat{2}]
\end{bmatrix}.
\]
\[
\sigma_3(D, \hat{1}) = \sigma_\text{x}[D, \hat{1}] = \mu_1(4)\sigma_3(G)\sigma_3(F)\sigma_3(D);
\]
\[
\sigma_1(D, \hat{1}) = \sigma_\text{x}[D, \hat{1}]\mu_3(3) = \mu_3(4)\mu_3(4).
\]
\[
\sigma_3(G, \hat{2}) = \sigma_\text{x}[G, \hat{2}]; \quad \overline{\sigma}_1(G, \hat{2}) = \sigma_\text{x}[G, \hat{2}]\mu_3(3) = \overline{\sigma}_1(H)\mu_3(3).
\]
\[
\sigma_3(D, \hat{2}) = \sigma_\text{x}[D, \hat{2}] = \sigma_3(E); \quad \sigma_1(D, \hat{2}) = \sigma_\text{x}[D, \hat{2}]\mu_3(3) = \sigma_1(E)\mu_3(4).
\]
\[
\sigma_3(E, \hat{1}) = \mu_1(3)\sigma_\text{x}[G, \hat{2}]\sigma_3(D, \hat{1})\sigma_\text{x}[D, \hat{2}]
= \mu_1(3)\sigma_3(H)\mu_1(4)\sigma_3(G)\sigma_3(F)\sigma_3(D)\sigma_3(E);
\]
\[
\sigma_1(E, \hat{1}) = \mu_3(3).
\]

3. Inverting step (2) in section A.1.1 by the inverse transformation:
\[
\begin{bmatrix}
\mu(2), \sigma_\text{x}[C, \hat{2}], \sigma_\text{x}[O, \hat{1}], \sigma_\text{x}[O, \hat{2}]
\end{bmatrix} \rightarrow
\begin{bmatrix}
\sigma(O, \hat{1}), \sigma(O, \hat{2}), \sigma(C, \hat{2}), \sigma(A, \hat{1})
\end{bmatrix}.
\]
\[
\sigma_3(O, \hat{1}) = \sigma_\text{x}[O, \hat{1}] = \overline{\sigma}_3(C); \quad \sigma_1(O, \hat{1}) = \sigma_\text{x}[O, \hat{1}]\mu_3(2) = \overline{\sigma}_1(C)\mu_3(2).
\]
\[
\sigma_3(O, \hat{2}) = \sigma_\text{x}[O, \hat{2}] = \sigma_3(A); \quad \sigma_1(O, \hat{2}) = \sigma_\text{x}[O, \hat{2}]\mu_3(2) = \sigma_1(A)\mu_3(2).
\]
\[
\sigma_3(C, \hat{2}) = \sigma_\text{x}[C, \hat{2}] = \overline{\sigma}_3(D); \quad \sigma_1(C, \hat{2}) = \sigma_\text{x}[C, \hat{2}]\mu_3(2) = \overline{\sigma}_1(D)\mu_3(2).
\]
\[
\sigma_3[3][A, \hat{1}] = \mu_1(2)\sigma_\text{x}[C, \hat{2}]\sigma_3[3][O, \hat{2}]\sigma_3[3][O, \hat{1}]
= \mu_1(2)\overline{\sigma}_3(D)\overline{\sigma}_3(C)\overline{\sigma}_3(A);
\]
\[
\sigma_1(A, \hat{1}) = \mu_3(2).
\]
4. Inverting step (i) in section A.1.1 by the inverse transformation:
\[ \left[ \mu(1), \sigma_{[x]}(D, \hat{2}), \sigma_{[x]}(A, \hat{1}), \sigma_{[x]}(A, \hat{2}) \right] \rightarrow \left[ \sigma(A, \hat{1}), \sigma(D, \hat{2}), \sigma(B, \hat{1}), \sigma(A, \hat{2}) \right]. \]

\[
\begin{align*}
\sigma_3(A, \hat{1}) &= \sigma_{[x]}(A, \hat{1}) = \mu_1(2)\sigma_3(D)\sigma_3(C)\sigma_3(A); \\
\sigma_1(A, \hat{1}) &= \sigma_{[x]}(A, \hat{1})\mu_3(1) = \mu_3(4)\mu_3(1). \\
\sigma_3(O, \hat{2}) &= \sigma_{[x]}(O, \hat{2}) = \sigma_3(C); \\
\sigma_1(O, \hat{2}) &= \sigma_{[x]}(O, \hat{2})\mu_3(1) = \sigma_1(E)\mu_3(3)\mu_3(1). \\
\sigma_3(B, \hat{1}) &= \mu_1(1)\sigma_{[x]}(D, \hat{2})\sigma_{[x]}(A, \hat{1})\sigma_{[x]}(A, \hat{2}) = \mu_1(1)\sigma_3(E)\sigma_3(D)\sigma_3(B); \\
\sigma_1(B, \hat{1}) &= \mu_3(1). \\
\sigma_3(A, \hat{2}) &= \sigma_{[x]}(A, \hat{2}) = \sigma_3(B); \\
\sigma_1(A, \hat{2}) &= \sigma_{[x]}(A, \hat{2})\mu_3(1) = \sigma_1(B)\mu_3(1).
\end{align*}
\]

(A.10)

We now show that the local Gauss laws at all the sites are redundant in terms of the new dual variables. Consider the 4 links meeting at a site \((x, y)\).

\[
\begin{align*}
\sigma_1(x, y, \hat{1}) &= \mu_3(x + 1, y)\mu_3(x + 1, y + 1) \\
\sigma_1(x, y, \hat{2}) &= \mu_3(x, y + 1)\mu_3(x + 1, y + 1) \\
\sigma_1(x - 1, y, \hat{1}) &= \mu_3(x, y + 1)\mu_3(x, y) \\
\sigma_1(x, y - 1, \hat{2}) &= \mu_3(x, y)\mu_3(x + 1, y)
\end{align*}
\]

Therefore, \( G^a(x, y) = \sigma_1(x, y, \hat{1})\sigma_1(x, y, \hat{2})\sigma_1(x - 1, y, \hat{1})\sigma_1(x, y - 1, \hat{2}) = 1. \)

A.2 SU(N) LATTICE GAUGE THEORY IN 2 + 1 DIMENSIONS

A.2.1 From links to loops & strings

In this section, we generalize the three canonical transformations (3.36), (3.39) and (3.40) in the single plaquette case to the entire lattice in two dimension. We define a comb shaped maximal tree with its base along the X axis and make a series of canonical transformations along the maximal tree to construct the string operators \( T_{[xy]}(x, y) \) attached to each lattice site \((x, y)\) away from the origin. This is similar to the construction of string operators \( T_{[xy]}(x, y) \) attached to the points \( A \equiv (1, 0), B \equiv (1, 1) \) and \( C \equiv (0, 1) \) in the simple single plaquette example illustrated in Figure 3.10-a,b,c. The gauge covariant loop operators \( \mathcal{W}(x, y) \) are constructed by fusing the string operators with the horizontal link operators \( U(x, y; \hat{1}) \) again through canonical transformations. As expected, all string operators \( T_{[xy]}(x, y) \) decouple as a consequence of SU(N) Gauss laws \( G^a(x, y) = 0. \) Thus only the fundamental physical loop operators are left at the end. The iterative canonical transformations are performed in 6 steps. These 6 steps are also illustrated graphically for the sake of clarity.
We now iterate the above canonical transformations to extend \( T_{[x]}(x, 0) \) along the \( x \) axis. They transform the \( N \) link operators \( U(x, 0; \hat{1}) \) into \( N \) string operators \( T_{[xx]}(x, 0) \). The canonical transformations are defined iteratively as:

\[
T_{[x]}(x + 1, y = 0) \equiv T_{[x]}(x, 0) \quad U(x, 0; \hat{1}), \quad T_{[xx]}(x, 0) \equiv T_{[x]}(x, 0), \quad (A.11)
\]

\[
E^a_{[x]++}(x + 1, 0) = E^a_- (x + 1, 0; \hat{1}), \quad E^a_{[xx]++}(x, 0) = E^a_- (x, 0; \hat{1}) + E^a_+ (x, 0; \hat{1}).
\]

Above \( x = 1, \cdots, N_s - 1 \) and the starting input for the first equation in \((A.11)\) is \( T_{[x]}(1, 0) \equiv U(1, 0; \hat{1}) \). The canonical transformations \((A.11)\) iteratively transform the flux operators \([T_{[x]}(x, 0), U(x, 0; \hat{1})] \) and their electric fields into \([T_{[xx]}(x, 0), T_{[x]}(x + 1, 0)] \) and their electric fields as shown in Figure A.2. At the boundary \( x = N_s - 1 \), we define \( T_{[xx]}(N_s - 1, 0) \equiv T_{[x]}(N, 0) \) for later convenience. As is also clear from Figure A.2, the subscript \([xx]\) on the string flux operator \( T_{[xx]}(x, 0) \) encodes the structure of its right electric field \( E^a_{[xx]++}(x, 0) \) in \((A.11)\). More explicitly, the last equation in \((A.11)\) states that \( E^a_{[xx]++}(x, 0) \) is the sum of two adjacent Kogut-Susskind electric fields in \( x \) direction. Note that if we were in one dimension with open boundary conditions, the Gauss law \((2.18)\) would imply \( \mathcal{G}^a(x) \equiv T^a_{[xx]++}(x, 0) = 0 \); \( \forall x \) making all string operators \( T_{[xx]}(x, 0) \) unphysical and irrelevant as expected.

**A.2.1.1 Strings along \( x \) axis**

We start by defining iterative canonical transformation along the \( x \) axis. They transform the \( N \) link operators \( U(x, 0; \hat{1}) \) into \( N \) string operators \( T_{[xx]}(x, 0) \). These string operators start at the origin and end at \( x = 1, 2, \cdots, N_s - 1 \) along the \( x \) axis as shown in the Figure A.2. The canonical transformations are defined iteratively as:

\[
T_{[x]}(x + 1, y = 0) \equiv T_{[x]}(x, 0) \quad U(x, 0; \hat{1}), \quad T_{[xx]}(x, 0) \equiv T_{[x]}(x, 0), \quad (A.11)
\]

\[
E^a_{[x]++}(x + 1, 0) = E^a_- (x + 1, 0; \hat{1}), \quad E^a_{[xx]++}(x, 0) = E^a_- (x, 0; \hat{1}) + E^a_+ (x, 0; \hat{1}).
\]

Above \( x = 1, \cdots, N_s - 1 \) and the starting input for the first equation in \((A.11)\) is \( T_{[x]}(1, 0) \equiv U(1, 0; \hat{1}) \). The canonical transformations \((A.11)\) iteratively transform the flux operators \([T_{[x]}(x, 0), U(x, 0; \hat{1})] \) and their electric fields into \([T_{[xx]}(x, 0), T_{[x]}(x + 1, 0)] \) and their electric fields as shown in Figure A.2. At the boundary \( x = N_s - 1 \), we define \( T_{[xx]}(N_s - 1, 0) \equiv T_{[x]}(N, 0) \) for later convenience. As is also clear from Figure A.2, the subscript \([xx]\) on the string flux operator \( T_{[xx]}(x, 0) \) encodes the structure of its right electric field \( E^a_{[xx]++}(x, 0) \) in \((A.11)\). More explicitly, the last equation in \((A.11)\) states that \( E^a_{[xx]++}(x, 0) \) is the sum of two adjacent Kogut-Susskind electric fields in \( x \) direction. Note that if we were in one dimension with open boundary conditions, the Gauss law \((2.18)\) would imply \( \mathcal{G}^a(x) \equiv T^a_{[xx]++}(x, 0) = 0 \); \( \forall x \) making all string operators \( T_{[xx]}(x, 0) \) unphysical and irrelevant as expected.

**A.2.1.2 Strings along \( y \) axis**

We now iterate the above canonical transformations to extend \( T_{[xx]}(x, 0) \) in the \( y \) direction to get \( T_{[yy]}(x, y = 1) \) and the final unphysical and ignorable string operators \( T_{[xxyy]}(x, 0) \) along the \( x \) axis as illustrated in Figure A.3-a:

\[
T_{[y]}(x, 1) \equiv T_{[xx]}(x, 0) \quad U(x, 0; \hat{2}), \quad T_{[xxyy]}(x, 0) \equiv T_{[xx]}(x, 0), \quad (A.12)
\]

\[
E^a_{[y]++}(x, 1) = E^a_- (x, 1; \hat{2}), \quad E^a_{[xxyy]++}(x, 0) = E^a_- (x, 0; \hat{1}) + E^a_+ (x, 0; \hat{2}).
\]
In (A.12) we have defined $T_{[xx]}(0,0) \equiv 1$ and $T_{[xx]}(N_s - 1,0) \equiv T_{[x]}(N_s - 1,0)$ as mentioned above. Substituting $E^q_{[xx]+}(x,0)$ from (A.11), we get:

$$E^q_{[xx]+}(x,0) = \left( E^a_{-}(x,0; \hat{1}) + E^a_{+}(x,0; \hat{1}) + E^a_{\pm}(x,0; \hat{2}) \right) \equiv G^q(x,0) = 0. \quad (A.13)$$

Again the subscript $[xy]$ on the string operator $T^q_{[xy]}(x,0)$ denotes that its electric field at $(x,0)$ is sum of three Kogut-Susskind electric fields, two in $x$ direction and one in $y$ direction as in (A.13) and represented by three squares in Figure A.3-a. We ignore $T_{[xy]}(x,0)$ from now onwards and repeat the canonical transformations (A.11) to fuse the links in $y$ direction along the maximal tree at fixed $x(=0,1,\cdots,N_s - 1)$. For this purpose, we replace $T_{[x]}(x,0)$ and $U(x,0;\hat{1})$ in (A.11) by $T_{[y]}(x,y)$ and $U(x,y;\hat{2})$ respectively with $y = 1,2,\cdots,(N_s - 1 - 1)$ and define:

$$T_{[y]}(x,y + 1) \equiv T_{[y]}(x,y) \ U(x,y;\hat{2}), \hspace{1cm} T_{[yy]}(x,y) \equiv T_{[y]}(x,y), \quad \ (A.14)$$

$$E^q_{[y]+}(x,y + 1) = E^a_{-}(x,y + 1;\hat{2}), \hspace{1cm} E^q_{[yy]+}(x,y) = E^a_{[y]+}(x,y) + E^a_{\pm}(x,y;\hat{2}).$$

In (A.14), the initial string operator $T_{[y]}(x,y = 1)$ is given in (A.12). The transformations (A.14) are illustrated in Figure A.3-b. Again the subscript $[yy]$ on $T_{[yy]}(x,y)$ is to emphasize that its electric field is sum of two adjacent Kogut-Susskind electric fields in the $y$ direction:

$$E^q_{[yy]+}(x,y) = E^a_{[y]+}(x,y) + E^a_{\pm}(x,y;\hat{2}) = E^a_{-}(x,y;\hat{2}) + E^a_{\pm}(x,y;\hat{2}). \quad (A.15)$$

In (A.15) we have used (A.14) to replace $E^q_{[y]+}(x,y)$ in terms of Kogut-Susskind electric fields $E^a(x,y;\hat{2})$. We again define $T_{[yy]}(x,N_s - 1) = T_{[y]}(x,N_s - 1)$ at the boundary for notational convenience.
A.2.1.3 Plaquette loop operators

In order to remove all local SU(N) gauge or string degrees of freedom and simultaneously obtain SU(N) covariant loop flux operators, we now fuse the horizontal link operator $U(x, y \neq 0)$ with $T_{[yy]}(x, y \neq 0)$ through the canonical transformations:

$$T_{[x]}(x + 1, y) \equiv T_{[yy]}(x, y) \ U(x, y; \hat{1}), \quad T_{[yy]}(x, y) = T_{[yy]}(x, y) \tag{A.16}$$

$$E^a_{[x]+}(x + 1, y) = E^a_{[x]}(x + 1, y; \hat{1}), \quad E^a_{[yy]}(x, y) = E^a_{[yy]}(x, y) + E^a_{[x]}(x, y; \hat{1})$$

at $x = 0, 1, 2, \cdots, (N_s - 1)$ and $y = 1, 2, \cdots, N_s - 1$. The above transformations are illustrated in Figure A.4. Using (A.14), the right electric field of the string flux operator $T_{[yy]}(x, y)$ is:

$$E^a_{[yy]+} = E^a_{[yy]+}(x, y) + E^a_{[x]}(x, y; \hat{1}) = E^a_{-}(x, y; \hat{2}) + E^a_{+}(x, y; \hat{2}) + E^a_{+}(x, y; \hat{1}). \tag{A.17}$$

The initial loop operators $(W(x, y), E^a(x, y))$ shown in Figure A.5 are defined as:

$$W(x, y) \equiv T_{[x]}(x, y \neq 0) \ T_{[yy]}(x, y), \quad T_{[yy]}(x, y) \equiv T_{[yy]}(x, y), \tag{A.18}$$

$$E^a_{[x]}(x, y \neq 0), \quad E^a_{[yy]}(x, y) = E^a_{[yy]+}(x, y \neq 0) + E^a_{[yy]+}(x, y).$$

Above $(W(x, y), E^a(x, y))$ are canonically conjugate pairs. We note that the conjugate electric fields of the string operators $T_{[yy]}$ vanishes in $\mathcal{H}^P$ as:

$$E^a_{[yy]+}(x, y) = E^a_{[yy]+}(x, y) + E^a_{[x]+}(x, y \neq 0)$$

$$= \left( E^a_{-}(x, y; \hat{2}) + E^a_{+}(x, y; \hat{2}) + E^a_{+}(x, y; \hat{1}) \right) = G^a(x, y) = 0. \tag{A.19}$$

In (A.19), we have used (A.16) and (A.17) to replace $E^a_{[x]+}(x, y \neq 0)$ and $E^a_{[yy]+}(x, y)$ respectively in terms of Kogut-Susskind electric fields. The relationship (A.19) solving the SU(N) Gauss law at $(x, y)$ is graphically illustrated in Figure A.5 and also earlier in Figure 3.12-a.

At this stage all the local gauge degrees of freedom, contained in the string operators $T_{[yy]}(x, y)$, have been removed. We now relabel $T_{[yy]}(x, y)$ as $T(x, y)$ and $E^a_{[yy]}(x, y)$ as $E^a_{\pm}(x, y)$ for notational simplicity. To simplify the magnetic field terms in the Kogut-Susskind

Figure A.4: Graphical representation of the canonical transformation in (A.16).
Hamiltonian (3.59), we further make the last set of canonical transformations (A.20) which transform the loop operators \((W(x, y), E_\pm^a(x, y))\) in (A.18) into the final plaquette loop operators \((W(x, y), E_\pm^a(x, y))\) as shown in Figure A.6. We define:

\[ W(x, y) \equiv W(x, y - 1) \; \tilde{W}^+(x, y), \quad \tilde{W}(x, y - 1) \equiv W(x, y - 1); \]  
\[ E_\pm^a(x, y) = \tilde{E}_\mp^a(x, y), \quad \tilde{E}_\pm^a(x, y - 1) = E_\mp^a(x, y - 1) + E_\mp^a(x, y) \]

Above \([W(x, y), E_\pm^a(x, y)]\), \([\tilde{W}^+(x, y), \tilde{E}_\pm^a(x, y)]\) are canonically conjugate loop operators and 
\(y = N_y - 1, (N_y - 1), \cdots, 1\). The canonical transformation is initiated with the boundary operator \(\tilde{W}(x, y = N_y - 1) \equiv W(x, y = N_y - 1)\) and at the lower boundary \(W(x, 1) \equiv \tilde{W}^+(x, 1)\).

Having constructed plaquette loop operators and conjugate electric fields using the canonical transformations (A.16)-(A.20), we now use these relations to write the plaquette loop electric fields directly in terms of the Kogut-Susskind link electric fields. Using (A.20),

\[ E_\pm^a(x, y) = \tilde{E}_\mp^a(x, y) = -R_{ab}(\tilde{W}(x, y))\tilde{E}_\mp^b(x, y) = -R_{ab}(\tilde{W}(x, y)) \left\{ E_\mp^b(x, y) + \tilde{E}_\mp^b(x, y - 1) \right\} \]  

(A.21)
Iterating this relation and using the relation \( E^a_\pm(x, y') = -R_{bc}(W^\dagger(x, y'))E^c_\pm(x, y') \), we get

\[
E^a_\pm(x, y) = -R_{ab}(\bar{W}(x, y)) \sum_{y'=y}^{N-1} E^b_\pm(x, y') = -R_{ab}(W(x, y)) \sum_{y'=y}^{N-1} -R_{bc}(W^\dagger(x, y'))E^c_\pm(x, y')
\]

(A.22)

From eqn. (A.18) we have \( E^c_\pm(x, y') = E^c_{[\pm]}(x, y') = -R_{cd}(T_{[\pm]}(x, y')) E^d_{[\pm]}(x, y') \) and from (A.16), \( E^d_{[\pm]}(x, y') = E^d_\pm(x, y') \). Therefore,

\[
E^a_\pm(x, y) = -\sum_{y'=y}^{N-1} R_{ab} \left(W(x, y)W^\dagger(x, y')T_{[\pm]}(x, y')\right) E^b_\pm(x, y', \hat{1})
\]

(A.23)

\[
= -\sum_{y'=y}^{N-1} R_{ab} \left(T(x-1, y) U(x-1, y; \hat{1}) \prod_{y''=y}^{y'} U(x, y'', \hat{2})\right) E^b_\pm(x, y', \hat{1}) \equiv -\sum_{y'=y}^{N-1} R_{ab} \left(S(x, y, y')\right) E^b_\pm(x, y', \hat{1}).
\]

This is the relation (3.68) in the text which was further graphically illustrated in Figure 3.12-b.

A.2.2 From loops & strings to links  (Inverse transformations)

In this part, we systematically write down all Kogut-Susskind link electric fields in terms of loop flux operators and loop electric fields. We calculate the link electric fields in three separate cases: a) \( E^a(x, y = 0; \hat{1}) \) shown in Figure 3.13-a, b) \( E^a(x, y \neq 0; \hat{1}) \) shown in Figure 3.13-b and c) \( E^a(x, y; \hat{2}) \) shown in Figure 3.13-c.

A.2.2.1 Case (a): \( E^a_\pm(x, 0; \hat{1}) \)

Consider the left electric field of a Kogut-Susskind link operator along the x axis, \( E_\pm(x, 0; \hat{1}) \). From canonical transformation A.11(figure. A.2), we have \( E^b_{[\pm]}(x, 0) = E^b_\pm(x, 0; \hat{1}) + E^b_\pm(x, 0; \hat{1}) \). Therefore,

\[
E^a_\pm(x, 0; \hat{1}) = -R_{ab}(U(x, 0, \hat{1}))E^b_\pm(x + 1, 0, \hat{1})
\]

\[= -R_{ab}(U(x, 0, \hat{1})) \left\{ E^b_{[\pm]}(x + 1, 0) - E^b_\pm(x + 1, 0; \hat{1}) \right\} \quad \text{(A.24)}
\]

Iterating this expression, we obtain

\[
E^a_\pm(x, 0; \hat{1}) = R_{ab}(T^\dagger(x, 0)) \sum_{\xi = x + 1}^{N-1} -R_{bc}(T(\xi, 0)) E^c_{[\pm]}(\xi, 0)
\]

(A.25)

Above, we have made use of the fact that \( T^\dagger(x, 0)T(\xi, 0) = U(x, 0; \hat{1})U(x + 1, 0; \hat{1}) \cdots U(\xi - 1, 0; \hat{1}) \) if \( \xi > x \). From this expression it is clear that all the \( \bar{E}_{[\pm]}(\xi, 0); \xi > x \) are parallel transported back to the point \( (x, 0) \) to give \( \bar{E}_\pm(x, 0; \hat{1}) \) so that the gauge transformations of
link and string operators are consistent with (A.25). This is a general trend which will be seen at each step of canonical transformations.

- \( \vec{E}_{[xx]} \rightarrow \vec{E}_{[y]} \rightarrow \vec{E}_{[yy]} \) 

Writing down \( \vec{E}_{[xx]}+(\bar{x},0) \) in terms of \( \vec{E}_{[xyy]}+(\bar{x},0) \) and \( \vec{E}_{[y]}+(\bar{x},1) \) using canonical transformation A.12 shown in Figure A.3-a:

\[
\vec{E}_{[xx]}^a+(\bar{x},0) = \vec{E}_{[xyy]}^a+(\bar{x},0) - \vec{E}_{[y]}^a+(\bar{x},1) = R_{ab} \left( U(\bar{x},0;\vec{2}) \right) \vec{E}_{[y]}^b(\bar{x},1;\vec{2})
\] (A.26)

We have used the fact that \( \vec{E}_{[xyy]}^a+(\bar{x},0) = 0 \) by Gauss law A.13 at \( (\bar{x},0) \). But from (A.14) and Figure A.3-b:

\[
\vec{E}_{[yy]}^a+(\bar{x},1) = \vec{E}_{[yy]}^a+(\bar{x},1) - \vec{E}_{[y]}^a+(\bar{x},1;\vec{2}) = R_{ab} \left( U(\bar{x},1;\vec{2}) \right) \vec{E}_{[y]}^b(\bar{x},1;\vec{2})
\] (A.27)

Substituting it back into eqn. A.26 for \( \vec{E}_{[xx]}^a+(\bar{x},0) \) and using \( U(\bar{x},0;\vec{2}) T^+(\bar{x},1) = T^+(\bar{x},0) \), we get

\[
\vec{E}_{[xx]}^a+(\bar{x},0) = R_{ab} \left( T^+(\bar{x},0) \right) \sum_{\bar{y}=1}^{N_{\bar{y}}-1} R_{bc} \left( T(\bar{x},\bar{y}) \right) \vec{E}_{[yy]}^c(\bar{x},\bar{y}).
\] (A.28)

Putting this into eqn. A.25 we get

\[
\vec{E}_{[y]}^c(x,0;\vec{1}) = -R_{ab} \left( T^+(x,0) \right) \sum_{\bar{x}=x+1}^{N_x-1} \sum_{\bar{y}=1}^{N_{\bar{y}}-1} R_{bc} \left( T(\bar{x},\bar{y}) \right) \vec{E}_{[yy]}^c(\bar{x},\bar{y})
\] (A.29)

- \( \vec{E}_{[yy]} \rightarrow \vec{E}_{[x]} \rightarrow \vec{E}_{[z]} \)

From canonical transformation (A.16) (Figure A.4) we have \( \vec{E}_{[yxy]}^c(\bar{x},\bar{y}) = \vec{E}_{[y]}^c(\bar{x},\bar{y}) + \vec{E}_{+}^c(\bar{x},\bar{y},\hat{1}) \) and \( \vec{E}_{[x]}^d+(\bar{x}+1,\bar{y},\hat{1}) = \vec{E}_{[x]}^d+(\bar{x}+1,\bar{y},\hat{1}) \). Therefore,

\[
\vec{E}_{[yxy]}^c(\bar{x},\bar{y}) = \vec{E}_{[yxy]}^c(\bar{x},\bar{y}) - \vec{E}_{+}^c(\bar{x},\bar{y},\hat{1}) = \vec{E}_{[yxy]}^c(\bar{x},\bar{y}) + R_{cd} \left( U(\bar{x},\bar{y},\hat{1}) \right) \vec{E}_{[x]}^d(\bar{x}+1,\bar{y},\hat{1})
\] (A.30)

Further, the canonical transformations A.18 (Figure A.5) imply:

\[
\vec{E}_{[yxy]}^c(\bar{x},\bar{y}) = \vec{E}_{[yxy]}^c(\bar{x},\bar{y}) + \vec{E}_{[x]}^c(\bar{x},\bar{y}) = -\vec{E}_{[x]}^c(\bar{x},\bar{y})
\]
Here, we have used the fact that $E_{[xyz]}^c(\bar{x}, \bar{y}) = 0$ by Gauss law at $(\bar{x}, \bar{y})$ (eqn. A.19). Also, from eqn. (A.18), $E_{[x]}^a(\bar{x}, \bar{y}) = E_{[x]}^a(\bar{x}, \bar{y})$. Therefore,

$$E_{[x]+}(\bar{x}, \bar{y}) = -R_{cd}(T_{[x]}^+(\bar{x}, \bar{y}))E_{[x]-}(\bar{x}, \bar{y}) = -R_{cd}(T_{[x]}^+(\bar{x}, \bar{y}))E_{[x]-}(\bar{x}, \bar{y})$$

(A.31)

Substituting for $\bar{E}_{[xyz]}^a, \bar{E}_{[x]+}$ in eqn. A.30 and using the relation $U(\bar{x}, \bar{y}; \hat{1})T_{[x]}^+(\bar{x} + 1, \bar{y}) = T^+(\bar{x}, \bar{y})$,

$$E_{[xy]+}(\bar{x}, \bar{y}) = R_{cd}(T_{[x]}^+(\bar{x}, \bar{y}))E_{[x]-}(\bar{x}, \bar{y}) - R_{cd}(T^+(\bar{x}, \bar{y}))E_{[x]+}(\bar{x} + 1, \bar{y})$$

(A.32)

Putting (A.32) in (A.29) and using the defining relations

$$T(\bar{x}, \bar{y})T_{[x]}^+(\bar{x}, \bar{y}) \equiv W^+(\bar{x}, \bar{y}); \quad E_{[x]}^a(\bar{x}, \bar{y}) \equiv -R_{bd}(W^+(\bar{x}, \bar{y}))E_{[x]-}(\bar{x}, \bar{y}); \quad T(\bar{x}, \bar{y})T^+(\bar{x}, \bar{y}) = 1,$$

we get a simple relation:

$$E^a_+(\bar{x}, 0; \hat{1}) = R_{ab}(T^+(\bar{x}, 0)) \sum_{\bar{x} = \bar{x} + 1}^{N_x - 1} \sum_{\bar{y} = 1}^{N_y - 1} E^b_+(\bar{x}, \bar{y}) + E^b_+(\bar{x} + 1, \bar{y})$$

(A.33)

$\bullet \bar{E}_1 \rightarrow \bar{E}_1$

To write $E^a_+(\bar{x}, 0; \hat{1})$ in terms of the final plaquette loop electric fields $\mathcal{E}_1^b$, we first use the canonical transformation eqn. A.20 (Figure. A.6), $E^b_+(\bar{x}, \bar{y}) = E^b_+(\bar{x}, \bar{y}) - E^b_+(\bar{x}, \bar{y} + 1)$, to write down the first term in the eqn. (A.33) in terms of $\bar{E}_1$ as follows:

$$\sum_{\bar{y} = 1}^{N_y - 1} E^b_+(\bar{x}, \bar{y}) = \sum_{\bar{y} = 1}^{N_y - 1} \left[ E^b_+(\bar{x}, \bar{y}) - E^b_+(\bar{x}, \bar{y} + 1) \right] = E^b_+(\bar{x}, 1)$$

$$= -R_{bc}(\bar{W}^+(\bar{x}, 1)) \mathcal{E}_1^c(\bar{x}, 1) = -R_{bc}(\mathcal{W}(\bar{x}, 1)) \mathcal{E}_1^c(\bar{x}, 1) = \mathcal{E}_1^b(\bar{x}, 1).$$

(A.34)

Here, we have used the fact that at the lower boundary, $\bar{W}^+(\bar{x}, 1) = \mathcal{W}(\bar{x}, 1)$. We now write down the second term in eqn. (A.33) in terms of $\bar{E}_1$. Again using canonical transformation eqn. (A.20) (Figure. A.6) as follows:

$$E^b_+(\bar{x} + 1, \bar{y}) = -R_{bc}(\bar{W}(\bar{x} + 1, \bar{y}))E^c_+(\bar{x} + 1, \bar{y}) = -R_{bc}(\mathcal{W}(\bar{x} + 1, \bar{y})) [E^c_+(\bar{x} + 1, \bar{y}) - E^c_+(\bar{x} + 1, \bar{y} + 1)]$$

$$= E^b_+(\bar{x} + 1, \bar{y}) - R_{bc}(\mathcal{W}(\bar{x} + 1, \bar{y}) \mathcal{W}^+(\bar{x} + 1, \bar{y} + 1)) E^c_+(\bar{x} + 1, \bar{y} + 1)$$

$$= E^b_+(\bar{x} + 1, \bar{y}) - R_{bc}(\mathcal{W}(\bar{x} + 1, \bar{y} + 1)) E^c_+(\bar{x} + 1, \bar{y} + 1)$$

$$= \mathcal{E}^c_1(\bar{x} + 1, \bar{y} + 1)$$

(A.35)
Putting both the terms back into eqn. A.33 for \( E^a_+(x, 0; \hat{1}) \), we get

\[
E^a_+(x, 0; \hat{1}) = R_{ab}(T^t(x, 0)) \sum_{x=x+1}^{N_x-1} \left\{ E^b_+(x, 1) + \sum_{y=1}^{N_y-1} \left[ E^b_+(x+1, y) + E^b_+(x+1, \bar{y} + 1) \right] \right\}
\]

\[
= R_{ab}(T^t(x, 0)) \left\{ E^b_+(x+1, 1) + \sum_{x=x+2}^{N_x-1} \sum_{y=1}^{N_y-1} L^b(x, y) \right\}
\]

(A.36)

Above, \( L^a(x, \bar{y}) \equiv E^a_+(x, \bar{y}) + E^a_-(x, \bar{y}) \).

A.2.2.2 Case (b): \( E^a_+(x, y \neq 0; \hat{1}) \)

The canonical transformation (A.16) and Figure A.4 state \( E^b_{[x]+}(x, y) = E^b_-(x, y; \hat{1}) \). Therefore,

\[
E^a_+(x, y, \hat{1}) = -R_{ab}(U(x, y, \hat{1})) E^b_-(x+1, y; \hat{1}) = -R_{ab}(U(x, y, \hat{1})) E^b_{[x]+}(x+1, y)
\]

Using the relations (A.31) and (A.35)

\[
E^b_{[x]+}(x+1, y) = -R_{bc}(T^t_{[x]}(x+1, y)) E^c_-(x+1, y); \quad E^c_-(x+1, y) = E^c_+(x+1, y) + E^c_-(x+1, y+1)
\]

and relation \( T^t(x, y) = U(x, y, \hat{1}) T^t_{[x]}(x+1, y) \), we get

\[
E^a_+(x, y, \hat{1}) = \left[ R_{ab}(U(x, y, \hat{1})) R_{bc}(T^t_{[x]}(x+1, y)) \right] E^c_-(x+1, y) = R_{ac} \left( T^t(x, y) \right) E^c_-(x+1, y)
\]

\[
= R_{ac} \left( T^t(x, y) \right) \left\{ E^c_+(x+1, y) + E^c_-(x+1, y+1) \right\}
\]

(A.37)

Clubbing case (a) and case (b) together,

\[
E^a_+(x, y; \hat{1}) = R_{ab}(T^t(x, y)) \left( E^b_-(x+1, y+1) + E^b_+(x+1, y) + \delta_{y,0} \sum_{x=x+2}^{N_x-1} \sum_{y=1}^{N_y-1} L^b(x, y) \right).
\]

(A.38)

We have defined \( E_\pm(x, 0) \equiv 0 \); \( E_\pm(0, y) \equiv 0 \) for notational convenience. The relations (A.36) were used in (3.72) and (3.83), (3.84) to write down the Kogut-Susskind Hamiltonian in terms of loop operators.

A.2.2.3 Case (c): \( E_+(x, y; \hat{2}) \)

The canonical transformations (A.14) (Figure A.3(b)) state \( E^c_{[y]+}(x, y) = E^c_-(x, y, \hat{2}) \). Therefore,

\[
E^a_+(x, y; \hat{2}) = -R_{ac}(U(x, y; \hat{2})) E^c_-(x, y+1, \hat{2}) = -R_{ac}(U(x, y; \hat{2})) E^c_{[y]+}(x, y+1)
\]
Using the relation $E^c_{[y]+}(x, y) = E^c_{[yy]+}(x, y) - E^c_{+}(x, y, \hat{2})$ from the canonical transformation eqn.A.14 (Figure.A.3(b)),

$$E^c_+ (x, y; \hat{2}) = -R_{ac}(U(x, y, \hat{2})) \left\{ E^c_{[yy]+}(x, y + 1) - E^c_+ (x, y + 1, \hat{2}) \right\}$$

$$= -R_{ac}(U(x, y, \hat{2})) E^c_{[yy]+}(x, y + 1) - R_{ac}(U(x, y, \hat{2}) U(x, y + 1, \hat{2})) E^c_{[yy]+}(x, y + 2) - \cdots$$

$$= -R_{ab}(T^+(x, y)) \sum_{g = y+1}^{N-1} R_{bc} (T(x, g)) E^c_{[yy]+}(x, g)$$

(A.39)

Using eqn.A.32, $E^c_{[yy]+}(x, g) = R_{cd}(T^+(x, g)) E^d_+(x, g) - R_{cd}(T^+(x, g)) E^d_+(x + 1, g)$ and the expression $W^I(x, \hat{g}) = T(x, \hat{g}) T^+(x, \hat{g})$ from eqn. A.18,

$$E^c_+ (x, y; \hat{2}) = R_{ab}(T^+(x, y)) \sum_{g = y+1}^{N-1} \left[ -R_{bc} \left( W^I(x, \hat{g}) \right) E^c_+(x, g) + E^c_+(x + 1, g) \right].$$

(A.40)

From eqn. A.35, we have $E^c_+(x, g) = E^c_+(x, g) + E^c_+(x + 1, g)$. Therefore, $E^c_+(x + 1, \hat{g}) = E^c_+(x + 1, \hat{g}) + E^c_+(x + 1, \hat{g} + 1)$ and

$$\sum_{g = y+1}^{N-1} -R_{bc} \left( W^I(x, \hat{g}) \right) E^c_+(x, g) = \sum_{g = y+1}^{N-1} -R_{bc} \left( W^I(x, \hat{g}) \right) \left[ E^c_+(x, g) + E^c_+(x, g + 1) \right]$$

$$= \sum_{g = y+1}^{N-1} R_{bc} \left( W^I(x, g - 1) \right) E^c_+(x, g) - \sum_{g = y+1}^{N-1} R_{bc} \left( W^I(x, \hat{g}) \right) E^c_+(x, g + 1)$$

$$= R_{bc} \left( W^I(x, y) \right) E^c_+(x, y + 1)$$

(A.41)

Above, we have used the relations: $W^I(x, \hat{g}) = W^I(x, \hat{g} - 1) W^I(x, \hat{g})$ and $R_{cd}(W^I(x, \hat{g})) E^d_+(x, \hat{g}) = E^c_+(x, \hat{g})$. Putting these two terms back into eqn. (A.40),

$$E^c_+ (x, y; \hat{2}) = R_{ab}(T^+(x, y)) \left\{ R_{bc} \left( W^I(x, y) \right) E^c_+(x, y + 1) + \sum_{g = y+1}^{N-1} \left[ E^c_+ (x + 1, g) + E^c_+ (x + 1, \hat{g} + 1) \right] \right\}$$

(A.42)

Therefore,

$$E^c_+ (x, y; \hat{2}) = R_{ab}(T^+(x, y)) \left( E^c_+ (x + 1, y + 1) + R_{bc}(W_{x,y}(x, y)) E^c_+(x, y + 1) + \sum_{g = y+2}^{N-1} L^I (x + 1, g) \right)$$

(A.43)

Above, $W_{x,y}(x, y) \equiv W(x, 1) W(x, 2) \cdots W(x, y)$ as defined in (3.72). The relation (A.43) was stated in (3.72) and used later in (3.83) to get the SU(N) loop Hamiltonian.

Consider the links meeting at a site $(x, y)$. We now show that the local Gauss law at $(x, y)$ are redundant in terms of the new dual loop operators except when $(x, y) = (0, 0)$ where it leads to a global Gauss law. We only show explicit calculations for the case when $(x \neq 0, y \neq 0)$ and $(x = 0, y = 0)$. The cases $(x \neq 0, y = 0)$ and $(x = 0, y \neq 0)$ can be similarly shown.
• $x \neq 0, y \neq 0$. The electric fields when written in terms of the physical loop operators are given by:

$$E^a_x(x, y, \hat{1}) = R_{ab}(T^x(x, y)) \left[ \mathcal{E}^b_x(x + 1, y) + \mathcal{E}^b_x(x + 1, y + 1) \right]$$

$$E^a_x(x, y, \hat{2}) = R_{ab}(T^y(x, y)) \left[ \sum_{g=y+2}^{N_y-1} \left\{ \mathcal{E}^b_x(x + 1, g) + \mathcal{E}^b_x(x + 1, g + 1) \right\} + \right]$$

$$+ R_{ab}(U(x - 1, y; \hat{1})) T^x(x, y) \mathcal{E}^b_x(x, y + 1)$$

$$- R_{ab}(U(x - 1, y; \hat{1})) T^y(x, y) \mathcal{E}^b_x(x, y)$$

$$E^a_y(x, y, \hat{1}) = -R_{ab}(U^x(x - 1, y; \hat{1}) T^x(x - 1, y) \mathcal{E}^b_y(x, y) + -R_{ab}(U^y(x - 1, y; \hat{1}) T^y(x - 1, y) \mathcal{E}^b_y(x, y)$$

$$E^a_y(x, y, \hat{2}) = -R_{ab}(T^x(x, y)) \left\{ \mathcal{E}^b_y(x + 1, y) + \mathcal{E}^b_y(x + 1, y + 1) \right\} +$$

$$- R_{ab}(T^y(x, y)) \left\{ \mathcal{E}^b_y(x + 1, y + 1) + \sum_{g=y+2}^{N_y-1} \left[ \mathcal{E}^b_y(x + 1, g) + \mathcal{E}^b_y(x + 1, g + 1) \right] \right\} +$$

$$R_{ab}(U(x - 1, y, \hat{1}) T^x(x - 1, y)) \mathcal{E}^b_x(x, y)$$

Therefore, $G^a(x, y) = E^a_x(x, y, \hat{1}) + E^a_y(x, y, \hat{2}) + E^a_x(x, y, \hat{1}) + E^a_y(x, y, \hat{2}) = 0$.

• $x = 0, y = 0$ The electric fields $E^a_x(0, 0, \hat{1})$ and $E^a_y(0, 0, \hat{2})$ are given by equations (A.38) and (A.43) respectively.

$$E^a_x(0, 0, \hat{1}) = \mathcal{E}^a_x(1, 1) + \sum_{x=2}^{N_x-1} \sum_{y=1}^{N_y-1} L^a(x, y)$$

$$E^a_y(0, 0, \hat{2}) = \mathcal{E}^a_y(1, 1) + \sum_{y=2}^{N_y-1} \sum_{x=1}^{N_x-1} L^a(1, y)$$

Therefore,

$$G^a(0, 0) = E^a_x(0, 0, \hat{1}) + E^a_y(0, 0, \hat{2}) = \sum_{x=1}^{N_x-1} \sum_{y=1}^{N_y-1} L^a(x, y).$$

Therefore, the Gauss law at the origin is not redundant in terms of the loop variables. This leads to a residual global Gauss law in the loop picture:

$$\langle \mathcal{G}^a | \text{phys} \rangle = \sum_{x=1}^{N_x-1} \sum_{y=1}^{N_y-1} L^a(x, y) | \text{phys} \rangle = 0$$

Above, $\mathcal{G}^a = G^a(0, 0)$ and $| \text{phys} \rangle$ is any state in the physical Hilbert space $\mathcal{H}^p$. 
A.3 canonical transformations in 3 + 1 dimensions

In this section we illustrate $Z_2$ and $SU(N)$ canonical transformations involved in the construction of a loop formulation on a 3 + 1 D lattice. For simplicity, construction is done explicitly for a single cube which contains all the features of the finite lattice case.

A.3.1 $Z_2$ lattice gauge theory

Figure A.7: (a)-(e) shows the plaquette canonical transformation steps involved in the construction of a loop formulation of $Z_2$ gauge theory on a cube. Just like in the 2D case, the string variables decouple. (f) shows the final plaquette loop variables that results. The plaquette loop operator corresponding to the ‘roof’ is missing. This solves Bianchi identity constraints automatically.

In this section, we will describe the canonical transformations involved in the construction of a loop formulation of $Z_2$ gauge theory on a unit 3 dimensional cube. Without any loss of generality, we will describe the construction by iterating plaquette canonical transformations defined in 3.1.2 instead of the fundamental $Z_2$ canonical transformation. For simplicity, sites of the 3D cube are labelled as: $O \equiv (0,0,0)$; $A \equiv (1,0,0)$; $B \equiv (1,0,1)$; $C \equiv (0,0,1)$; $D \equiv (0,1,0)$; $E \equiv (1,1,0)$; $F \equiv (1,1,1)$; $G \equiv (0,1,1)$. We start with the 12 link operators on the cube as shown in figure A.7. Our notation is such that $\sigma_3(O, \hat{1})$ denotes the $\sigma_3$ variable of the link which starts at site O and is in the $\hat{1}$ direction. These 12 link conjugate pairs are then systematically transformed into plaquette operators on the 4 ‘vertical walls’ and the
‘floor’ of the cube and 7 string variables along OA, AE, OD, OC, AB, EF, DG in 5 plaquette canonical transformations defined in (3.20) and (3.21). This is illustrated in the figure A.7. Just as in 2 + 1 dimensions, these string variables decouple from the physical Hilbert space \( \mathcal{H}^p \) due to local Gauss laws at the sites A, E, D, C, B, F, G. The Gauss law at the origin O is not independent of Gauss laws at these sites and hence is automatically solved.

1. We begin by performing a plaquette canonical transformation on the plaquette OABC. The plaquette loop conjugate pairs \( \{ \mu_1(B, \hat{3}); \mu_3(B, \hat{3}) \} \) around the plaquette OABC is given by

\[
\mu_1(B, \hat{3}) = \sigma_3(O, \hat{1})\sigma_3(A, \hat{3})\sigma_3(C, \hat{1})\sigma_3(O, \hat{3}) \quad \mu_3(B, \hat{3}) = \sigma_1(C, \hat{1}) \quad (A.48)
\]

Our notation is such that the plaquette operators \( \{ \mu_1(B, \hat{3}); \mu_3(B, \hat{3}) \} \) corresponds to a plaquette in the plane perpendicular to the \( \hat{3} \) direction (xy plane) with the top right corner site B.\(^1\) The string and redefined link operators around plaquette OABC is given by:

\[
\begin{align*}
\sigma_{3[xz]}(O, \hat{3}) &= \sigma_3(O, \hat{3}) \\
\sigma_{3[xz]}(A, \hat{3}) &= \sigma_3(A, \hat{3}) \\
\sigma_{3[xz]}(O, \hat{1}) &= \sigma_3(O, \hat{1}) \\
\end{align*}
\]

\[
\begin{align*}
\sigma_{1[xz]}(O, \hat{3}) &= \sigma_1(O, \hat{3})\sigma_1(C, \hat{1}) \\
\sigma_{1[xz]}(A, \hat{3}) &= \sigma_1(A, \hat{3})\sigma_1(C, \hat{1}) \\
\sigma_{1[xz]}(O, \hat{1}) &= \sigma_1(O, \hat{1})\sigma_1(C, \hat{1}) \quad (A.49)
\end{align*}
\]

The subscript \( [xz] \) on \( \sigma_{3[xz]} \) denotes that the corresponding conjugate operator \( \sigma_{1[xz]} \) is the product of link operator \( \sigma_1 \) along the x and z direction.

2. The second step involves a plaquette canonical transformation on the plaquette AEFB. The plaquette loop conjugate operators \( \{ \mu_1(F, \hat{1}); \mu_3(F, \hat{1}) \} \) are given by:

\[
\mu_1(F, \hat{1}) = \sigma_3(A, \hat{2})\sigma_3(E, \hat{3})\sigma_3(B, \hat{2})\sigma_3(A, \hat{3}) \quad \mu_3(F, \hat{1}) = \sigma_1(B, \hat{2}) \quad (A.50)
\]

The string and redefined link operators around plaquette AEFB is given by:

\[
\begin{align*}
\sigma_{3[xyz]}(A, \hat{3}) &= \sigma_3(A, \hat{3}) \\
\sigma_{3[yz]}(A, \hat{2}) &= \sigma_3(A, \hat{2}) \\
\sigma_{3[xz]}(E, \hat{3}) &= \sigma_3(E, \hat{3}) \\
\end{align*}
\]

\[
\begin{align*}
\sigma_{1[xyz]}(A, \hat{3}) &= \sigma_1(A, \hat{3})\sigma_1(B, \hat{2}) = G(B) \\
\sigma_{1[yz]}(A, \hat{2}) &= \sigma_1(A, \hat{2})\sigma_1(B, \hat{2}) \\
\sigma_{1[xz]}(E, \hat{3}) &= \sigma_1(E, \hat{3})\sigma_1(B, \hat{2}) \quad (A.51)
\end{align*}
\]

As a result of the Gauss law at B, the string operators \( \{ \sigma_{3[xyz]}(A, \hat{3}); \sigma_{3[xyz]}(A, \hat{3}) \} \equiv \{ \sigma_1(B, \hat{3}); \sigma_3(B, \hat{3}) \} \) decouples from the physical Hilbert space. Our notation is such that

\(^1\) Top right corner is the point \((x,y,z)\) with the maximum value of \((x+y+z)\) on the plaquette.
the string operators \{\bar{\sigma}_1(B, \hat{3}); \bar{\sigma}_3(B, \hat{3})\} are denoted by the end point \(B\) of the string and the direction \(\hat{3}\) of the string.

3. The third step involves a plaquette canonical transformation along the plaquette DEFG. The resulting plaquette loop conjugate pairs \(\{\mu_1(F, \hat{2}); \mu_3(F, \hat{2})\}\) are given by:

\[
\mu_1(F, \hat{2}) = \sigma_3(D, \hat{1})\sigma_3[xy](E, \hat{3})\sigma_3(G, \hat{1})\sigma_3(D, \hat{3})
\]

\[
\mu_3(F, \hat{2}) = \sigma_1(G, \hat{1})
\]  
(A.52)

The redefined link operators and string operators around DEFG are given by:

\[
\sigma_3[xx](D, \hat{1}) = \sigma_3(D, \hat{1})
\]

\[
\sigma_1[xx](D, \hat{1}) = \sigma_1(D, \hat{1})\sigma_1(G, \hat{1})
\]

\[
\sigma_3[xy](E, \hat{3}) = \sigma_3[xy](E, \hat{3})
\]

\[
\sigma_1[xy](E, \hat{3}) = \sigma_1[xy](E, \hat{3})\sigma_1(G, \hat{1}) = \mathcal{G}(F)
\]

\[
\sigma_3[xz](D, \hat{3}) = \sigma_3(D, \hat{3})
\]

\[
\sigma_1[xz](D, \hat{3}) = \sigma_1(D, \hat{3})\sigma_1(G, \hat{1})
\]

As a result of the Gauss law at \(F\), the string operators \(\{\sigma_1[xy](E, \hat{3}); \sigma_3[xy](E, \hat{3})\} \equiv \{\bar{\sigma}_1(F, \hat{3}); \bar{\sigma}_3(F, \hat{3})\}\) decouples from the physical Hilbert space.

4. The fourth step involves a plaquette canonical transformation along the plaquette ODGC. The resulting plaquette loop conjugate pairs \(\{\mu_1(G, \hat{1}); \mu_3(G, \hat{1})\}\) are given by:

\[
\mu_1(G, \hat{1}) = \sigma_3(O, \hat{2})\sigma_3[xz](D, \hat{3})\sigma_3(C, \hat{2})\sigma_{3[xy]}(O, \hat{3})
\]

\[
\mu_3(G, \hat{1}) = \sigma_1(C, \hat{2})
\]  
(A.53)

The redefined link operators and string operators around ODGC are given by:

\[
\sigma_3[xy](O, \hat{3}) = \sigma_3[xy](O, \hat{3})
\]

\[
\sigma_1[xy](O, \hat{3}) = \sigma_1[xy](O, \hat{3})\sigma_1(C, \hat{2}) = \mathcal{G}(C)
\]

\[
\sigma_3[xz](D, \hat{3}) = \sigma_3[xz](D, \hat{3})
\]

\[
\sigma_1[xz](D, \hat{3}) = \sigma_1[xz](D, \hat{3})\sigma_1(C, \hat{2}) = \mathcal{G}(G)
\]

\[
\sigma_3[yy](O, \hat{2}) = \sigma_3(O, \hat{2})
\]

\[
\sigma_1[yy](O, \hat{2}) = \sigma_1(O, \hat{2})\sigma_1(C, \hat{2})
\]  
(A.54)

As a result of the Gauss law at \(C\) and \(G\), the string operators \(\{\sigma_1[xy](O, \hat{3}); \sigma_3[xy](O, \hat{3})\} \equiv \{\bar{\sigma}_1(C, \hat{3}); \bar{\sigma}_3(C, \hat{3})\}\) and \(\{\sigma_1[xy](D, \hat{3}); \sigma_3[xy](D, \hat{3})\} \equiv \{\bar{\sigma}_1(G, \hat{3}); \bar{\sigma}_3(G, \hat{3})\}\) decouples from the physical Hilbert space. This completes the construction of the plaquettes along the ‘vertical walls’.

5. The final step involves a plaquette canonical transformation along the plaquette OAED. The resulting plaquette loop conjugate pairs \(\{\mu_1(E, \hat{3}); \mu_3(E, \hat{3})\}\) are given by:

\[
\mu_1(E, \hat{3}) = \sigma_3[xz](O, \hat{1})\sigma_3[yy](A, \hat{2})\sigma_3[xz](D, \hat{1})\sigma_3[yy](O, \hat{2})
\]

\[
\mu_3(E, \hat{3}) = \sigma_1[yy](D, \hat{1}) = \sigma_1(D, \hat{1})\sigma_1(G, \hat{1})
\]  
(A.55)
As a result of local Gauss laws at the sites, all the \( \{ \sigma_1[xxx](O, \hat{1}); \sigma_3[xxx](O, \hat{1}) \} \) and \( \{ \sigma_1[xyy](A, \hat{2}); \sigma_3[xyy](A, \hat{2}) \} \) decouple from the physical Hilbert space, due Gauss laws at O,D,G,C,E,F.

The resulting string operators around OAED are given by:

\[
\sigma_3[xxx](O, \hat{1}) = \sigma_3[xxx](O, \hat{1}) \quad \quad \quad \sigma_1[xxx](O, \hat{1}) = \sigma_1[xxx](O, \hat{1}) \sigma_1[xx](D, \hat{1}) = \mathcal{G}(O) \mathcal{G}(D) \mathcal{G}(G) \mathcal{G}(C)
\]

\[
\sigma_3[xyy](A, \hat{2}) = \sigma_3[yy](A, \hat{2}) \quad \quad \quad \sigma_1[xyy](A, \hat{2}) = \sigma_1[yy](A, \hat{2}) \sigma_1[xx](D, \hat{1}) = \mathcal{G}(E) \mathcal{G}(F)
\]

\[
\sigma_3[xyy](O, \hat{2}) = \sigma_3[yy](O, \hat{2}) \quad \quad \quad \sigma_1[xyy](O, \hat{2}) = \sigma_1[yy](O, \hat{2}) \sigma_1[xx](D, \hat{1}) = \mathcal{G}(D) \mathcal{G}(G)
\]

As a result of local Gauss laws at the sites, all the 7 string operators \( \{ \sigma_1[xxx](O, \hat{1}); \sigma_3[xxx](O, \hat{1}) \} \), \( \{ \sigma_1[xyy](A, \hat{2}); \sigma_3[xyy](A, \hat{2}) \} \), \( \{ \sigma_1(C, \hat{3}); \sigma_3(C, \hat{3}) \}, \{ \sigma_1(1, \hat{3}); \sigma_3(1, \hat{3}) \}, \{ \sigma_1(A, \hat{1}); \sigma_3(A, \hat{1}) \}, \{ \sigma_1(E, \hat{2}); \sigma_3(E, \hat{2}) \}, \{ \sigma_1(D, \hat{2}); \sigma_3(D, \hat{2}) \} \) decouple from the physical Hilbert space, leaving behind the physical plaquette loop operators \( \{ \mu_1(B, \hat{3}); \mu_3(B, \hat{3}) \}, \{ \mu_1(F, \hat{1}); \mu_3(F, \hat{1}) \}, \{ \mu_1(F, \hat{2}); \mu_3(F, \hat{2}) \}, \{ \mu_1(G, \hat{1}); \mu_3(G, \hat{1}) \}, \{ \mu_1(E, \hat{3}); \mu_3(E, \hat{3}) \} \).

### A.3.2 SU(N) lattice gauge theory

In this section, we iterate the fundamental SU(N) canonical transformation to reformulate SU(N) lattice gauge theory on a single cube in terms of plaquette loop operators. Canonical transformations transform the link operators on the unit cube to the following operators:
1. Physical plaquette loop conjugate operators on the plaquettes corresponding to the 4 ‘walls’ and the ‘floor’ of the cube

2. String operators to each site on the cube except the origin.

These string operators decouple, leaving behind the plaquette loop operators. As before, no new constraints are introduced. Just like in the $Z_2$ case, the sites of the 3D cube are labelled as: $O \equiv (0,0,0)$; $A \equiv (1,0,0)$; $B \equiv (1,0,1)$; $C \equiv (0,0,1)$; $D \equiv (0,1,0)$; $E \equiv (1,1,0)$; $F \equiv (1,1,1)$; $G \equiv (0,1,1)$. This is illustrated in figure A.8. The loop formulation of SU(N) lattice gauge theory on a single cube is achieved in 6 set of canonical transformations as follows.

### 1. The first set of canonical transformations involves the construction of the plaquette loop operators on the XY plaquette OAEDO.

\[
\begin{align*}
U(12) & \equiv U_1 U_2 & E^a_+(12) & = E^a_-(E, \hat{2}) \\
U_{[xy]}(1) & = U_1 & E^a_+([xy])(1) & = E^a_-(A, \hat{1}) + E^a_+(A, \hat{2}) \\
U(43) & = U_4 U_3 & E^a_+(43) & = E^a_-(E, \hat{1}) \\
U_{[xy]}(4) & = U_4 & E^a_+[xy](4) & = E^a_d(D, \hat{2}) + E^a_+(D, \hat{1}) \\
U(1234) & = U_{12} U^{t}_{43} & E^a_+(1234) & = E^a_-(43) \\
U_{[xy]}(12) & = U_{12} & E^a_+([xy])(12) & = E^a_+(12) + E^a_+(43) = E^a_-(E, \hat{2}) + E^a_-(E, \hat{1})
\end{align*}
\]

### 2. Second set of canonical transformation involves construction of string operators to each site.

\[
\begin{align*}
U(16) & = U_{[xy]}(1) U_6 & E^a_+(16) & = E^a_+(6) \\
U_{[xyz]}(1) & = U_{[xy]}(1) = U_1 & E^a_+([xyz])(1) & = E^a_+([xy])(1) + E^a_+(A, \hat{3}) = G(A) \\
U((12)7) & = U_{[xy]}((12)) U_7 & E^a_+((12)7) & = E^a_+(F, \hat{3}) \\
U_{[xyz]}((12)) & = U_{[xy]}((12)) & E^a_+([xyz])(12) & = E^a_+([xy])(12) + E^a_+(E, \hat{3}) = G(E) \\
U(48) & = U_{[xy]}(4) U_8 & E^a_+(48) & = E^a_+(G, \hat{3}) \\
U_{[xyz]}(4) & = U_{[xy]}(4) & E^a_+([xyz])(4) & = E^a_+([xy])(4) + E^a_+(D, \hat{3}) = G(D)
\end{align*}
\]
3. The third set involves the construction of vertical plaquette ODGCO.

\[
\begin{align*}
U(5(12)) &= U_5 U_{12} & E^a_+(5(12)) &= E^a_+(12)
\end{align*}
\]

\[
\begin{align*}
U_{[zy]}(5) &= U_5 & E^a_+([zy])(5) &= E^a_+(C, \hat{3}) + E^a_+(C, \hat{2})
\end{align*}
\]

\[
\begin{align*}
U(485(12)) &= U(48) U^t(5(12)) & E^a_+(485(12)) &= E^a_+(O, \hat{3})
\end{align*}
\]

\[
\begin{align*}
U_{[yz]}(48) &= U(48) & E^a_+([yz])(48) &= E^a_+(48) + E^a_+(5(12))
\end{align*}
\]

4. The fourth set involves the construction of vertical plaquette ODGFEDO.

\[
\begin{align*}
U_{48(11)} &= U_{[yz]}(48) U_{11} & E^a_+(48(11)) &= E^a_-(F, \hat{1})
\end{align*}
\]

\[
\begin{align*}
U_{[xyz]}(48) &= U_{[yz]}(48) & E^a_+([xyz])(48) &= E^a_+([yz])(48) + E^a_+(G, \hat{1})
\end{align*}
\]

\[
\begin{align*}
U_{127(11)84} &= U(127) U^t(48(11)) & E^a_+(127(11)84) &= E^a_-(48(11))
\end{align*}
\]

\[
\begin{align*}
U_{[xz]}(127) &= U(127) & E^a_+([xz])(127) &= E^a_+(12(7)7) + E^a_+(48(11))
\end{align*}
\]

\[
\begin{align*}
U_{437(11)84} &= U^t(1234) U(127(11)84) & E^a_+(437(11)84) &= E^a_+(127(11)84)
\end{align*}
\]

\[
\begin{align*}
U_{[z]}(1234) &= U(1234) & E^a_+([z])(1234) &= E^a_+(1234) + E^a_+(127(11)84)
\end{align*}
\]

5. The fifth set involves the construction of vertical plaquette OAEFBAO

\[
\begin{align*}
U_{127(10)} &= U_{[xz]}(127) U_{10} & E^a_+(127(10)) &= E^a_-(F, \hat{2})
\end{align*}
\]

\[
\begin{align*}
U_{[xyz]}(127) &= U_{[xz]}(127) & E^a_+([xyz])(127) &= E^a_+([xz])(127) + E^a_-(F, \hat{2}) = \mathcal{G}(F)
\end{align*}
\]

\[
\begin{align*}
U_{127(10)16} &= U(127(10)) U(16) & E^a_+(127(10)16) &= E^a_-(127(10))
\end{align*}
\]

\[
\begin{align*}
U_{[yz]}(16) &= U(16) & E^a_+([yz])(16) &= E^a_+(16) + E^a_+(127(10))
\end{align*}
\]

6. The sixth set involves the construction of vertical plaquette OABCO.

\[
\begin{align*}
U_{58} &= U_{[yz]}(5) U_8 & E^a_+(58) &= E^a_-(G, \hat{3})
\end{align*}
\]

\[
\begin{align*}
U_{[xyz]}(5) &= U_{[yz]}(5) & E^a_+([xyz])(5) &= E^a_+([yz])(5) + E^a_+(D, \hat{3})) = \mathcal{G}(C)
\end{align*}
\]

\[
\begin{align*}
U_{1685} &= U_{[xy]}(16) U^t(58) & E^a_+(1685) &= E^a_+(58)
\end{align*}
\]

\[
\begin{align*}
U_{[xyz]}(16) &= U_{[yz]}(16) & E^a_+([xyz])(16) &= E^a_+([yz])(16) + E^a_+(58) = \mathcal{G}(B)
\end{align*}
\]

Just like before, all the string operators decouple due to the Gauss laws at A, E, D, B, F, G, C leaving behind the plaquette loop operators corresponding to the plaquettes OAEDO, ODGCO, ODGFEDO, OAEFBAO, OABCO. Note that the ‘roof’ plaquette OCGFBCO is not constructed.
In this appendix, we show that all Wilson loops are diagonal in the magnetic basis of $SU(2)$ lattice gauge theory defined in 3.56. We repeat the defining equation \(^1\) for convenience.

\[
|\omega, \hat{w}\rangle = \sum_{jm_{-}, m_{+}} \sqrt{\frac{2j+1}{2\pi^2}} D^j_{m_{-} m_{+}}(\omega, \hat{w}) |j m_{-} m_{+}\rangle = \sum_{jm_{-}, m_{+}, \mu} \sqrt{\frac{2j+1}{2\pi^2}} D^j_{m_{-} m_{+}}(\omega, \hat{w}) \langle nl\mu|jm\bar{m}\rangle |nl\mu\rangle
\]

(B.1)

Here, $(\omega, \hat{w})$ is the angle axis characterisation of a point on $S^3(SU(2))$ and $Y_{nlm}(\omega, \hat{w})$ are the hyperspherical harmonics on $S^3$. We have also used the relation \([153]\):

\[
Y_{nlm}(\omega, \hat{w}) = \sqrt{\frac{2j+1}{2\pi^2}} C_{lm}^{j} D^j_{m_{-} m_{+}}(\omega, \hat{w})
\]

The states $|\omega, \hat{w}\rangle$ forms a complete, orthonormal basis. The plaquette loop operators are diagonal in this basis.

\[
W_{\alpha\beta} |\omega, \hat{w}\rangle = z_{\alpha\beta} |\omega, \hat{w}\rangle
\]

(B.2)

where

\[
z_{\alpha\beta} = \begin{bmatrix}
\cos \frac{\omega}{2} - i \sin \frac{\omega}{2} \cos \theta & i \sin \frac{\omega}{2} \sin \theta e^{-i\phi} \\
-i \sin \frac{\omega}{2} \sin \theta e^{i\phi} & \cos \frac{\omega}{2} + i \sin \frac{\omega}{2} \cos \theta
\end{bmatrix}
\]

(B.3)

Above, we have suppressed the plaquette index of $W_{\alpha\beta}$ for simplicity. (B.2) follows from the properties of Wigner matrices $D^j_{m_{-} m_{+}}$ as shown below.

\[
W_{11} |\omega, \hat{w}\rangle = (F_N) \left[ a_1 c_1 + a_2^\dagger c_2^\dagger \right] (F_N) \sum_{j, m_{-}, m_{+}} |j| D^j_{m_{-} m_{+}}(\omega, \hat{w}) |j, m_{-}, m_{+}\rangle
\]

\[
= \sum_{j, m_{-}, m_{+}} |j| D^j_{m_{-} m_{+}}(\omega, \hat{w}) \left[ \frac{1}{\sqrt{2j+1}} \sqrt{(j + m_{-})(j + m_{+})} \frac{1}{\sqrt{2j+1}} \left| j - \frac{1}{2}, m_{-} - \frac{1}{2}, m_{+} + \frac{1}{2} \right\rangle
\]

\[
+ \frac{1}{\sqrt{2j+1}} \sqrt{(j - m_{-} + 1)(j - m_{+} + 1)} \frac{1}{\sqrt{2j+1}} \left| j + \frac{1}{2}, m_{-} - \frac{1}{2}, m_{+} - \frac{1}{2} \right\rangle
\]

(B.4)

\(^1\) For simplicity, the states $|\Omega(\omega, \hat{w})\rangle$ in 3.56 is denoted here as $|\omega, \hat{w}\rangle$
Above, \( F_N = \frac{1}{\sqrt{N+1}} \) and \( \langle j \rangle = \sqrt{\frac{2j+1}{2\pi}} \). Putting \( J \equiv j - \frac{1}{2} \) in the first term and \( J \equiv j + \frac{1}{2} \) in the second term and using the following \([115]\) recursion relation for Wigner D matrices

\[
\frac{\sqrt{(J - m_- + \frac{1}{2})(J - m_+ + \frac{1}{2})}}{2J + 1} D_{m_- \to m_+}^{(j - \frac{1}{2})}(\omega, \omega) + \frac{\sqrt{(J + m_- + \frac{1}{2})(J + m_+ + \frac{1}{2})}}{2J + 1} D_{m_- \to m_+}^{(j + \frac{1}{2})}(\omega, \omega) = \left( \cos \frac{\omega}{2} - i \sin \frac{\omega}{2} \cos \theta \right) D_{m_- \to m_+}^{(j - \frac{1}{2})}(\omega, \omega)
\]

Therefore,

\[
\mathcal{W}_{11} |\omega, \hat{\omega}\rangle = \left( \cos \frac{\omega}{2} - i \sin \frac{\omega}{2} \cos \theta \right) |\omega, \hat{\omega}\rangle = z_{11} |\omega, \hat{\omega}\rangle \quad (B.5)
\]

similarly, we get

\[
\begin{align*}
\mathcal{W}_{12} |\omega, \hat{\omega}\rangle &= i \sin \frac{\omega}{2} \sin \theta e^{-i\phi} |\omega, \hat{\omega}\rangle = z_{12} |\omega, \hat{\omega}\rangle \\
\mathcal{W}_{21} |\omega, \hat{\omega}\rangle &= -i \sin \frac{\omega}{2} \sin \theta e^{i\phi} |\omega, \hat{\omega}\rangle = z_{21} |\omega, \hat{\omega}\rangle \\
\mathcal{W}_{22} |\omega, \hat{\omega}\rangle &= \left( \cos \frac{\omega}{2} + i \sin \frac{\omega}{2} \cos \theta \right) |\omega, \hat{\omega}\rangle = z_{22} |\omega, \hat{\omega}\rangle 
\end{align*}
\]

(B.6)

Even though \(|\omega, \hat{\omega}\rangle\) forms a complete, orthonormal basis, they are not invariant under global gauge transformations. Under a global gauge transformation \(\Lambda\), \(|\omega, \hat{\omega}\rangle\) transforms as follows

\[
|\omega, \hat{\omega}\rangle^\Lambda = \sum_{n,l,m} Y_{nlm}(\omega, \hat{\omega}) D_{\hat{m} \to \bar{m}}^l(\Lambda) |nl\bar{m}\rangle = \sum_{n,l,m} \chi_l^j(\omega) Y_{nlm}(\hat{\omega}) D_{\hat{m} \to \bar{m}}^l(\Lambda) |n\bar{m}\rangle = \sum_{n,l,m} \chi_l^j(\omega) Y_{nlm}(\hat{\omega}^\Lambda) |nl\bar{m}\rangle \quad j = \frac{(n - 1)}{2}.
\]

(B.7)

Here, \(\chi_l^j(\omega)\) are generalized \(SU(2)\) characters \([115]\), \(Y_{nlm}(\omega, \hat{\omega})\) are the spherical harmonics on \(S^2\) and \(\hat{\omega}^\Lambda\) denotes rotated \(\hat{\omega}\). Thus, under a global gauge transformation \(\Lambda\), \(\omega\) is an invariant angle and \(\hat{\omega}\) transforms like a vector:

\[
\omega \to \omega' \quad \hat{\omega} \to R_{ab}(\Lambda) \hat{\omega}^b
\]

A completely gauge invariant angular basis on a single plaquette can be constructed by integrating out gauge transformations as follows:

\[
|\omega\rangle \equiv \int |\omega, \hat{\omega}^\Lambda\rangle \, d\mu(\Lambda) = \sum_{n,l,m} \int d\mu(\Lambda) Y_{nlm}(\omega, \hat{\omega}) D_{\hat{m} \to \bar{m}}^l(\Lambda) |n\bar{m}\rangle = \sum_n Y_{n00}(\omega, \hat{\omega}) |n00\rangle = \sum_j \chi_l^j(\omega) |j\rangle
\]

(B.8)
Here we have used the identity [115] \[ \int d\mu(\Lambda) D_{\rho \bar{m}}^{j}(\Lambda) = \delta_{\rho \bar{m}} \delta_{\bar{m} 0} \] where

\[
\int d\mu(\Lambda) = \frac{1}{4\pi^2} \sin^{2\frac{\omega}{2}} d\omega \int \sin \theta d\theta \int d\phi
\]

is the Haar measure of SU(2) where the SU(2) group element \( \Lambda \) is characterized by \( \omega, \theta, \phi \). Also, \( j = (n - 1)/2 \) and \( \chi_{j}(\omega) = \frac{\sin(2j + 1)\omega}{\sin(\frac{\omega}{2})} \) are the SU(2) characters. Therefore, for a single plaquette lattice the gauge invariant states are characterized by the gauge invariant angle \( \omega \) as expected.

For a finite lattice, a complete, orthonormal basis is given by the direct product of \( |\hat{\omega}, \omega\rangle \) corresponding to each plaquette. All Wilson loops are diagonal in this basis. A completely gauge invariant magnetic/angular basis can then be constructed by integrating out the gauge transformations:

\[
|\Omega\rangle = |\omega_p, [\Theta_j]\rangle = \int d\mu(\Lambda) \prod_p |\omega_p, \hat{\omega}_p\rangle^{\Lambda} = \int d\mu(\Lambda) \prod_p |\omega_p, \hat{\omega}_p^{\Lambda}\rangle
\]

where \([\Theta_j]\) denotes the set of \( 2p - 3 \) independent relative angles between the axes \( \hat{\omega}_p \), where \( p \) is the no of plaquettes in the lattice. These angles are invariant with respect to gauge rotations.

- **2 plaquette lattice** On a 2 plaquette lattice the gauge invariant basis states are:

\[
\int |\omega_1, \hat{\omega}_1^{\Lambda}\rangle |\omega_2, \hat{\omega}_2^{\Lambda}\rangle d\mu(\Lambda) = \sum_{n_1 l_1 m_1 n_2 l_2 m_2} \int d\mu(\Lambda) \begin{pmatrix} \chi_{l_1}^{\bar{m}_1}(\omega_1) Y_{1 m_1 l_1}(\hat{\omega}_1) & D_{l_1 m_1}^{l_2 m_2}(\Lambda) & n_1 l_1 \bar{m}_1 \\ \chi_{l_2}^{\bar{m}_2}(\omega_2) Y_{1 m_2 l_2}(\hat{\omega}_2) & D_{m_2 m_1}^{l_1 l_2}(\Lambda) & n_2 l_2 \bar{m}_2 \end{pmatrix}
\]

\[
= \sum_{n_1 l_1 m_1} (-1)^{l_1 - m_1} (-1)^{l_1 - m_1} \frac{16\pi^2}{2l_1 + 1} \chi_{l_1}^{\bar{m}_1}(\omega_1) Y_{1 m_1 l_1}(\hat{\omega}_1) |n_1 l_1 \bar{m}_1\rangle
\]

\[
= 16\pi^2 \sum_{n_1 l_1 m_1} \underbrace{C_{l_1 l_2}^{00} \Delta_{l_1 l_2}(\hat{\omega}_1) Y_{1 m_1 l_1}(\hat{\omega}_1) \chi_{l_1}^{\bar{m}_1}(\omega_1) \chi_{l_2}^{\bar{m}_2}(\omega_2) |n_1 l_1 n_2 l_2 0\rangle}_{\text{Bipolar scalar harmonics}}
\]

\[
= 4\pi \sum_{n_1 l_1 m_1} (2l_1 + 1) P_l(\cos\Theta_{12}) \chi_{l_1}^{\bar{m}_1}(\omega_1) \chi_{l_2}^{\bar{m}_2}(\omega_2) |n_1 l_1 n_2 l_2 0\rangle \equiv |\omega_1 \omega_2 \Theta_{12}\rangle
\]

Here, we have used the relations:

\[
\int d\mu(\Lambda) D_{l_1 m_1}^{l_2 m_2}(\Lambda) = (-1)^{l_1 - m_1} \frac{16\pi^2}{2l_1 + 1} \delta_{l_1 l_2} \delta_{-m_1 m_2} \delta_{-\bar{m}_1 \bar{m}_2};
\]

\[
\frac{(-1)^{l_1 - m_1}}{\sqrt{2l_1 + 1}} = C_{l_1 m_1}^{00}, \quad \sum_{n_1} C_{l_1 n_1}^{00} \Delta_{l_1 n_1}(\hat{\omega}_1) Y_{1 m_1 l_1}(\hat{\omega}_1) = \frac{2l_1 + 1}{4\pi} P_l(\cos\Theta_{12})
\]
where $\Theta_{12}$ is the angle between $\hat{w}_1$ and $\hat{w}_2$.

- **N plaquette** The above process can be generalized to N plaquettes which leads to the following gauge invariant basis on an N plaquette lattice.

$$|\omega_{P_1}, [\Theta_{ij}]\rangle = k \sum_{[n], [l], [l]} Y_{[n][l][ll]} (\omega_1, \cdots \omega_P, [\Theta_{ij}]) |[n], [l], [ll]\rangle$$ (B.11)

Here, $Y_{[n][l][ll]}$ is the scalar multipolar spherical harmonics [115]. In general, the procedure of integrating out the gauge transformation projects out the gauge invariant subspace and is equivalent to finding the subspace with 0 total angular momentum.
In this appendix, we discuss some calculational methods in the new loop formulation. Since, the degrees of freedom now lie on the plaquettes instead of the links and the magnetic part of the Hamiltonian involve a single plaquette holonomy, calculations are much more simpler in the present loop formulation.

**Variational method**

In this section, we study the ground state of SU(2) loop Hamiltonian using a ‘single’ plaquette variational ansatz. We then compare the results with those obtained from the variational analysis of the standard Kogut-Susskind formulation \[70–72, 74\]. Note that after canonical transformations each plaquette loop is a fundamental degree of freedom. Therefore, gauge invariant computations in the dual spin model become much simpler. For the ground state of SU(2) gauge theory, the magnetic fluctuations in a spatial region are almost independent of fluctuations in another spatial region which is sufficiently far away \[17, 62, 63\]. So, the largest contributions to the vacuum state comes from states with little magnetic correlations. Therefore, we use the following separable state without any spin-spin correlations as our variational ansatz:

\[
|\psi_0\rangle = e^{S/2}|0\rangle; \quad S = \alpha \sum_p \text{Tr}W(p).
\]

\[
= \prod_p |\psi_0\rangle_p. \tag{C.1}
\]

Above, \(|0\rangle\) is the strong coupling vacuum state defined by \(E_{\pm}^a(m,n)|0\rangle = 0\) and \(\alpha\) is the variational parameter. This state (C.1) satisfies Wilson’s area law criterion. We consider a Wilson loop \(\text{Tr}(W_C)\) along a large space loop \(C\) on the lattice and compute its ground state expectation value: \(\frac{\langle \psi_0 | \text{Tr}W_C | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle}\). In the dual spin model any Wilson loop \(W_C\) can be written in terms of the \(P\) fundamental loops \(W_{a\beta}\) as shown in Figure C.1:

\[
W_C = W(p_1)W(p_2)W(p_3)\cdots W(p_n). \tag{C.2}
\]
Figure C.1: A Wilson loop $W_C$ can be written as the product of fundamental plaquette loop operators $W(p_i)$. $W_C = W(p_1)W(p_2)W(p_3)\cdots W(p_n)$. The tails of the fundamental plaquette loop operators connecting them to the origin (see Figure 3.11-a) are not shown for clarity.

Here $p_1$ is the plaquette operator in the bottom right corner of $C$ and $p_n$ is the plaquette operator at the left top corner of $C$. We now show that this state satisfies Wilson’s area law. The expectation value of $TrW_C$ in $|\psi_0\rangle$ is given by

$$
\langle TrW_C \rangle \equiv \frac{\langle \psi_0|TrW_C|\psi_0\rangle}{\langle \psi_0|\psi_0\rangle} = \frac{1}{\langle \psi_0|\psi_0\rangle} \prod_{p \in p_1} \int d\mu(\omega_p, \hat{\omega}_p) \langle 0|e^{STrz(C)}|\omega_p, \hat{\omega}_p\rangle \langle \omega_p, \hat{\omega}_p|0\rangle
$$

$$
= \prod_{p} \int d\mu(\omega_p, \hat{\omega}_p) e^{2\alpha \cos \omega_p/2} \cos (\alpha(C)/2)
$$

(C.3)

In (C.3), $\int d\mu(\omega_p, \hat{\omega}_p) \equiv \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} d\omega \sin \theta d\theta d\phi$. We have also used the completeness relation of the $|\omega, \hat{\omega}\rangle$ basis. $z(C)$ is the eigenvalue of $W_C$ corresponding to the eigenstate $\prod_p |\omega_p, \hat{\omega}_p\rangle$. Since $W_C = \prod_p W(p_i)$, $z(C) = \prod_p z(p_i)$ and $Trz(C) = 2\cos (\alpha(C)/2)$. Here, $\alpha(C)$ is the gauge invariant angle characterizing the SU(2) matrix $z(C)$ in its angle axis representation. Using the expression for the product of 2 SU(2) matrices \(^1\) repeatedly, it is easy to show that $\cos(\alpha(C)/2) = \prod \cos(\alpha_p/2) + \text{terms which vanish on } \theta \text{ integration}^2$

Therefore,

$$
\langle TrW_C \rangle = 2 \left( \frac{I_0(2\alpha)}{I_0(2\alpha)} \right)^{n_c} = 2e^{-n_c \ln \left( \frac{I_0(2\alpha)}{I_0(2\alpha)} \right)}
$$

(C.5)

\(^1\) Product of 2 SU(2) matrices characterized by $(\omega_1, \hat{\omega}_1)$ and $(\omega_2, \hat{\omega}_2)$ gives an SU(2) matrix characterized by $(\omega, \hat{\omega})$ with

$$
\begin{align*}
\cos \frac{\omega}{2} &= \cos \frac{\omega_1}{2} \cos \frac{\omega_2}{2} - (\hat{\omega}_1 \cdot \hat{\omega}_2) \sin \frac{\omega_1}{2} \sin \frac{\omega_2}{2}; \\
\hat{\omega} \sin \frac{\omega}{2} &= \hat{\omega}_1 \sin \frac{\omega_1}{2} \cos \frac{\omega_2}{2} + \hat{\omega}_2 \sin \frac{\omega_2}{2} \cos \frac{\omega_1}{2} - [\hat{\omega}_1 \times \hat{\omega}_2] \sin \frac{\omega_1}{2} \sin \frac{\omega_2}{2}.
\end{align*}
$$

(C.4)

\(^2\) The integrand under $\theta$ integration contains either $\sin 2\theta$ or a $\cos \theta$, both vanish on $\theta$ integration from 0 to $\pi$. 


In (C.5), $n_c$ is the number of plaquettes in the loop $C$ and $I_l(2\alpha)$ is the $l$-th order modified Bessel function of the first kind. We have used the relation

$$I_l(2\alpha) = \frac{1}{\pi} \int_0^\pi e^{2\alpha \cos \omega} \cos l\omega \, d\omega. \tag{C.6}$$

and the recurrence relation [166]

$$I_{l-1}(2\alpha) - I_{l+1}(2\alpha) = \frac{2l}{2\alpha} I_l(2\alpha) \tag{C.7}$$

to arrive at (C.5). The string tension is given by $\sigma_T(\alpha) = \ln \left( \frac{I_l(2\alpha)}{I_l(2\alpha)} \right)$.

The local effective SU(2) spin model Hamiltonian is

$$H_{\text{spin}} = \sum_{p=1}^P \left\{ 4g^2 \bar{\mathcal{E}}^2(p) + \frac{1}{8\pi^2} [2 - (\text{Tr} \mathcal{W}(p))] \right\} + g^2 \sum_{(p,p')} \left\{ \bar{\mathcal{E}}_-(p) \cdot \bar{\mathcal{E}}_+(p') \right\}. \tag{C.8}$$

We now calculate $\alpha$ by minimizing

$$\langle H_{\text{spin}} \rangle = \frac{\langle \psi_0 | H_{\text{spin}} | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle}. \tag{C.9}$$

In order to calculate $\langle H_{\text{spin}} \rangle$, we first find the expectation value of $\mathcal{E}_-(p) \cdot \mathcal{E}_+(p')$ and $\mathcal{E}(p) \cdot \mathcal{E}(p') \equiv \mathcal{E}_+(p) \cdot \mathcal{E}_+(p) \equiv \mathcal{E}_-(p) \cdot \mathcal{E}_-(p)$ in (C.8). First, lets calculate $\langle \psi_0 | \mathcal{E}_a^\alpha (p) \mathcal{E}_b^\beta (P) | \psi_0 \rangle$. Here, $P$ is any plaquette.

$$\langle \psi_0 | \mathcal{E}_a^\alpha (p) \mathcal{E}_b^\beta (P) | \psi_0 \rangle = \left\langle 0 \left| e^{S/2} \mathcal{E}_a^\alpha (p) e^{-S/2} \right| e^{S/2} \mathcal{E}_b^\beta (P) e^{-S/2} \right| 0 \right\rangle = -\frac{1}{4} \langle \psi_0 | [\mathcal{E}_a^\alpha (p), S] [\mathcal{E}_b^\beta (P), S] | \psi_0 \rangle \tag{C.10}$$

In (C.9), we have used the fact that $\mathcal{E}_\pm |0\rangle = 0$. Evaluating $\langle \psi_0 | \mathcal{E}_a^\alpha (p) \mathcal{E}_b^\beta (P) | \psi_0 \rangle$ in a different way,

$$\langle \psi_0 | \mathcal{E}_a^\alpha (p) \mathcal{E}_b^\beta (P) | \psi_0 \rangle = \left\langle 0 \left| e^{S/2} \mathcal{E}_a^\alpha (p) e^{-S/2} \left( e^{-S/2} \mathcal{E}_b^\beta (P) e^{S/2} \right) \right| 0 \right\rangle = \frac{1}{2} \langle \psi_0 | [\mathcal{E}_a^\alpha (p), [\mathcal{E}_b^\beta (P), S]] | \psi_0 \rangle + \frac{1}{4} \langle \psi_0 | [\mathcal{E}_a^\alpha (p), S] [\mathcal{E}_b^\beta (P), S] | \psi_0 \rangle$$

The equations (C.9) and (C.10) implies:

$$\langle \psi_0 | \mathcal{E}_-(p) \cdot \mathcal{E}_+(P) | \psi_0 \rangle = \frac{1}{4} \langle \psi_0 | \mathcal{E}_a^\alpha (p), [\mathcal{E}_b^\beta (P), S] | \psi_0 \rangle \tag{C.11}$$
The expression in (C.11) vanishes when \( P \neq p \). In particular,

\[
\langle \psi_0 | \mathcal{E}_-(p) \cdot \mathcal{E}_+(p') | \psi_0 \rangle = 0,
\]

(\text{C.12})

\[
\langle \psi_0 | \mathcal{E}_-(p) \cdot \mathcal{E}_-(p) | \psi_0 \rangle = \frac{3\alpha}{16} \langle \psi_0 | \text{Tr} \mathcal{W}(p) | \psi_0 \rangle.
\]

Above \( p, p' \) are nearest neighbours. Putting \( n_c = 1 \) in equation (C.5), \( \langle \text{Tr} \mathcal{W}(p) \rangle = \frac{2\hbar(2\alpha)}{I_i(2\alpha)} \).

Using the above relations, the expectation value of the effective Hamiltonian \( H_{\text{spin}} \) is

\[
\frac{\langle \psi_0 | H_{\text{spin}} | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle} = 2 \mathcal{P} \left\{ \left( \frac{3\alpha}{4} g^2 - \frac{1}{g^2} \right) \frac{l_2(2\alpha)}{l_1(2\alpha)} + \frac{1}{g^2} \right\}.
\]

(C.13)

Above, \( \mathcal{P} \) is the number of plaquettes in the lattice. \( \frac{l_i(2\alpha)}{l_i(2\alpha)} \) is a monotonously increasing bounded function of \( \alpha \). It takes values between +1 and −1 with +1 at \( \alpha \to \infty \) and −1 at \( \alpha \to -\infty \). In the weak coupling limit, \( g^2 \to 0, \frac{l_i(2\alpha)}{l_i(2\alpha)} \) should be maximum for the expectation value of \( H_{\text{spin}} \) to be minimum and therefore, \( \alpha \to \infty \). But, using the asymptotic form of the modified Bessel function of the first kind \( l_i(2\alpha) \),

\[
l_i(2\alpha) \xrightarrow{\alpha \to \infty} \frac{e^{2\alpha}}{\sqrt{2\pi}(2\alpha)} \left( 1 + \frac{(1 - 2l)(1 + 2l)}{16\alpha} + \ldots \right)
\]

In the weak coupling limit, \( \frac{l_i(2\alpha)}{l_i(2\alpha)} \approx 1 - \frac{3}{4\alpha} \). Hence,

\[
\frac{\langle \psi_0 | H_{\text{spin}} | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle} = \sum_p 2 \left\{ \left( \frac{3\alpha}{4} g^2 - \frac{1}{g^2} \right) \left( 1 - \frac{3}{4\alpha} \right) + \frac{1}{g^2} \right\}
\]

(C.14)

Minimizing the expectation value in the weak coupling limit, \( \alpha = \frac{1}{g^2} \). The string tension is given by \( \sigma_T(\frac{1}{g^2}) = \ln \left( \frac{l_1(\frac{1}{g^2})}{l_2(\frac{1}{g^2})} \right) \). This is exactly the result obtained in [70–72] using variational calculation with the fully disordered ground state and Kogut-Susskind Hamiltonian (3.59) which is dual to the full non-local spin Hamiltonian. The general non-local Hamiltonian \( H \) differs from the above effective local spin Hamiltonian \( H_{\text{spin}} \) by terms of the form \( R_{ab}(W) \mathcal{E}_a^b(p) \mathcal{E}_+^b(\vec{p}) \), where \( p \) and \( \vec{p} \) are any 2 plaquettes on the lattice which are at least 2 lattice spacing away from each other. Above, \( W \) is in general the product of many plaquette loop operators. The expectation value of the full Hamiltonian in the variational ground state |\( \psi_0 \rangle \) reduces to \( \langle \psi_0 | H_{\text{spin}} | \psi_0 \rangle \) as the expectation value of the non-local terms in |\( \psi_0 \rangle \) vanishes. So, the simplified Hamiltonian with nearest neighbour interactions gives the same variational ground state to the lowest order as the full Hamiltonian.
A tensor networks ansatz

The present loop formulation is tailor-made for tensor network [155–158] and matrix product state (MPS) [159] ansatzes to explore the interesting and physically relevant part of $\mathcal{H}^p$ for low energy states. This is due to the following two reasons:

- The absence of local non-abelian Gauss laws at every lattice site.
- The presence of (spin type) local hydrogen atom orthonormal basis at every plaquette.

We first briefly discuss matrix product state approach in a simple example of spin chain with spin $s = 1$ before directly generalizing it to pure SU$(2)$ lattice gauge theory on a one dimensional chain of plaquettes. In the case of spin chain with $s_x = -1, 0, +1$ at every lattice site $x = 0, 1, \cdots, N$, any state can be written as:

$$|\Psi\rangle = \sum_{s_1, s_2, \cdots, s_N = 0, \pm 1} \Psi(s_1, s_2, \cdots s_N) |s_1, s_2, \cdots s_N\rangle. \quad (C.15)$$

The matrix product state method consists of replacing the wave functional by

$$\Psi(s_1, s_2, \cdots s_N) = \text{Tr}\left(T^{(s_1)}_1 T^{(s_2)}_2 \cdots T^{(s_N)}_N\right). \quad (C.16)$$

In (C.16) $T^s$ are $D \times D$ matrices where $D$ is the bond length. The matrix elements of $T^s$ are fixed by minimizing the spin Hamiltonian. In the hydrogen atoms loop basis we have a similar structure where the three dimensional spin states are replaced by infinite dimensional quantum states of hydrogen atoms: $|s\rangle \rightarrow |n l m\rangle$. The most general state in the hydrogen atom loop basis can be written as:

$$|\Psi\rangle = \sum_{\{n\} \{l\} \{m\}} \Psi\left(\begin{array}{ccc} n_1 & n_2 & \cdots & n_p \\ l_1 & l_2 & \cdots & l_p \\ m_1 & m_2 & \cdots & m_p \end{array}\right) \left(\begin{array}{cccc} n_1 & n_2 & \cdots & n_p \\ l_1 & l_2 & \cdots & l_p \\ m_1 & m_2 & \cdots & m_p \end{array}\right) \equiv \text{Tr}\left(T^{(n_1 l_1 m_1)}_1 T^{(n_2 l_2 m_2)}_2 \cdots T^{(n_p l_p m_p)}_{P}\right). \quad (C.17)$$

We now consider SU$(2)$ lattice gauge theory on a chain of $P$ plaquettes as shown in Figure 3.14. A simple tensor network ansatz, like (C.16) for spins, for the ground state wave function in (C.17) is

$$\Psi_0\left(\begin{array}{ccc} n_1 & n_2 & \cdots & n_p \\ l_1 & l_2 & \cdots & l_p \\ m_1 & m_2 & \cdots & m_p \end{array}\right) \equiv \text{Tr}\left[T^{(n_1 l_1 m_1)}_1 T^{(n_2 l_2 m_2)}_2 \cdots T^{(n_p l_p m_p)}_P\right]. \quad (C.18)$$

In (C.18) $T^{(n_x l_x m_x)}_x$, $x = 1, 2, \cdots, P$ are $P$ matrices of dimension $D \times D$ where $D$ is the bond length describing correlations between hydrogen atoms. Assuming a bound on the principal quantum number (e.g., $n = 1, 2$) and minimizing the energy of the spin model Hamiltonian
within spherically symmetric s-sector should give a good idea of ground state at least in the strong coupling region. The method can then be extrapolated systematically towards weak coupling by extending the range of hydrogen atom principal quantum number on each plaquette. The global SU(2) Gauss law can also be explicitly implemented through the following ansatz:

$$|\Psi\rangle = \sum_{\{n\} \{l\} \{ll\}} \Psi \begin{bmatrix} n_1 & n_2 & \cdots & n_p \\ l_1 & l_2 & \cdots & l_p \\ l_{12} & l_{13} & \cdots & l_{12-p-2} \\ l_{11} & l_{12} & \cdots & l_{12-p-2} \end{bmatrix} \begin{bmatrix} l_{11} & l_{12} & \cdots & l_{12-p-2} \\ n_1 & n_2 & \cdots & n_p \\ l_1 & l_2 & \cdots & l_p \\ l_{12} & l_{13} & \cdots & l_{12-p-2} \end{bmatrix}. \quad (C.19)$$

We can now make an explicitly gauge invariant MPS ansatz for the ground state:

$$\Psi_0 \equiv \text{Tr} \left[ T^{n_1}_{l_{11}, (1)} T^{n_2}_{l_{12}, (2)} T^{n_3}_{l_{13}, (3)} \cdots T^{n_p}_{l_{12-p}, (P)} \right]. \quad (C.20)$$

This ansatz is illustrated in Figure 3.14-b. Much more work is required to implement these ideas on a computer. This will be done in the future.