In this chapter, we show that the gauge invariant physical or loop Hilbert space $\mathcal{H}^p$ of pure SU(2) lattice gauge theory can be completely and most economically realized in terms of the Wigner coupled bound energy eigenstates $|n \, l \, m\rangle$ of hydrogen atoms [54]. One hydrogen atom is assigned to every plaquette of the lattice. The SU(2) global invariance (3.75) implies that the total angular momenta of all hydrogen atoms vanish. This Wigner coupled hydrogen atoms basis describe quantized SU(2) loop electric fluxes in terms of $(n, l, m)$ and is orthonormal as well as complete in $\mathcal{H}^p$. Following Fock [148–150], we describe $P$ hydrogen atoms on their momentum hypersphere $S^3$ (see section 5.2) so that their hidden $SU(2) \times SU(2)$ symmetries become manifest. We show that the equivalence of the gauge theory and hydrogen atom Hilbert spaces has its origin in the identification of SU(2) group manifold $S^3$ associated with each plaquette loop holonomy with the $S^3$ of the corresponding hydrogen atom (see discussion towards the end of section 5.2).

5.1 HYDROGEN ATOM & SO(4) SYMMETRY

The hydrogen atom can be elegantly solved using group theory[148–153] which exploits manifest rotational and hidden Runge Lenz symmetries generated by angular momentum
(\vec{L}) and Laplace Runge Lenz (\vec{A}) operators\(^1\) respectively. These generators commute with the hydrogen atom Hamiltonian and satisfy \(\vec{L} \cdot \vec{A} = 0\). This leads to \(\text{SU}(2) \otimes \text{SU}(2)\) symmetry algebra generated by

\[
\vec{J}_+ = \frac{1}{2}(\vec{L} + \vec{A}),
\]

on the bound states of hydrogen atom \((E < 0)\) [151, 152]. Further, the two angular momentum operators satisfy

\[
[j_+^a, j_-^b] = 0, \quad j_+^2 = j_-^2 \equiv j^2.
\]

As a consequence, the two equivalent complete set of commuting operators (CSCO) are

Uncoupled basis : \([([\vec{J}_+)^2, j_+^0]; \quad ([\vec{J}_-)^2, j_-^0] \equiv ([\vec{J})^2, j_+^0, j_-^0] : \quad (\text{CSCO} - I), \quad (5.3)\]

Coupled basis : \([([\vec{J})^2, (\vec{J}_+ + \vec{J}_-)^2, (\vec{J}_+ + \vec{J}_-)^2] \equiv ([\vec{J}_2, \vec{L}_2, \vec{z}_2] : \quad (\text{CSCO} - II) \quad (5.4)\]

Following Wybourne [151, 152], we define :

\[
j_+^a \equiv a^\dagger \left(\frac{\sigma^a}{2}\right) a; \quad j_-^a \equiv b^\dagger \left(\frac{\sigma^a}{2}\right) b. \quad (5.5)\]

In \((5.5)\) \((a_1^a, a_2^a)\) and \((b_1^b, b_2^b)\) represent \(\text{SU}(2)\) doublets of Schwinger boson (prepotential) creation operators, \(\sigma^a\) \((a=1,2,3)\) are the Pauli matrices. The condition \(\vec{J}_2 \cdot \vec{J}_2^\dagger\) implies \(N_a = N_b\) where \(N_a = a^\dagger \cdot a\) and \(N_b = b^\dagger \cdot b\) are the total number operators. The orthonormal and complete basis diagonalizing CSCO-I is given by [151, 152]:

\[
|j = j_+ = j_-, m_-, m_+\rangle = |j, m_-\rangle \otimes |j, m_+\rangle; \quad (5.6)\]

\[
|j_+ = j, m_-\rangle \equiv \frac{(a_1^a)^{j+m_-}(a_2^a)^{(j-m_-)}}{(j+m_-)! (j-m_-)!} |0\rangle, \quad |j_+ = j, m_+\rangle \equiv \frac{(c_1^b)^{j+m_+}(c_2^b)^{(j-m_+)}}{(j+m_+)! (j-m_+)!} |0\rangle. \quad (5.7)\]

Above\(^2\), \(\epsilon_{\alpha\beta} \equiv \epsilon_{\alpha\beta}\).
\[ |n l m\rangle \equiv \sum_{m_-m_+} C_{j m_-j m_+}^{|l m|} |j, m_-\rangle \otimes |j, m_+\rangle. \]  

In (5.8), \( n \equiv (2j + 1) = 1, 2, \ldots; l = 0, 1, 2, \ldots, 2j \equiv (n - 1); m = -l, \ldots, +l; C_{j m_-j m_+}^{|l m|} \) are the Clebsch-Gordan coefficients. The states \(|j, m_-\rangle\) and \(|j, m_+\rangle\) are coupled together using Clebsch-Gordon coefficients to give the coupled hydrogen atom states \(|n l m\rangle\). * in Figure 5.2 and Figure 5.3 a,b represents the j-j coupling in eqn. (5.8). These \(|n l m\rangle\) states are eigenstates of \(J^2, L^2, L^z\) with eigenvalues \(j(j + 1) \equiv \frac{1}{4}(n^2 - 1)\), \(l(l + 1)\), \(m\) respectively. They are also the eigenstates of the hydrogen atom Hamiltonian with energy \([151, 152]\) \(E_n \sim -1/n^2\). The states \(|n, l, m\rangle\) are graphically represented by tadpoles as shown in Figure 5.2-b.

5.2 SU(2) LOOP STATES & HYDROGEN ATOM BOUND STATES

a. Single plaquette case

We first start with the simple single plaquette case to illustrate the correspondence between hydrogen bound states and SU(2) loop states. We identify the hydrogen atom angular momentum \(\vec{L}\), Lenz vector operators \(\vec{A}\) (combinations of \(\vec{L}\) and \(\vec{A}\) to be precise) with the SU(2) loop electric field operators \(\vec{E}_\pm\) of the gauge theory:

\[ J_\pm^a \leftrightarrow E_\pm^a. \]
Above, \( J_+^2 = \frac{1}{2}(L^a \pm A^a) \). This identification further implies:

\[
L^a \ (\equiv f_+^a + f_-^a) \leftrightarrow E^a_+ + E^-_+ \tag{5.10}
\]

where \( L^a \) are the angular momentum operators of a hydrogen atom and \( L^a = E^a_+ + E^-_+ \) are the generators of Gauss law \((3.48)\) in a single plaquette case. The SU(2) Gauss law constraint on a single plaquette and its hydrogen atom analogue is:

\[
G^a(0, 0) = L^a = E^a_+ + E^-_+ = 0 \leftrightarrow L^a \ (\equiv f_+^a + f_-^a) = 0. \tag{5.11}
\]

This immediately implies that CSCO-I \((5.3)\) \([\tilde{P}, J_+^a, J_-^a]\) and CSCO-II \((5.4)\) \([\tilde{P}, L^2, L^z]\) of hydrogen atom also characterize lattice gauge theory Hilbert space. CSCO-II is more natural for gauge theory as the SU(2) Gauss law constraints \((5.11)\) are trivially solved in the coupled \(|n, l, m\rangle\) basis diagonalizing CSCO-II by demanding \(l = m = 0\). Therefore, the spherically symmetric energy eigenstates \(|n, l = 0, m = 0\rangle\) of hydrogen atom also form a complete, orthonormal loop basis of SU(2) lattice gauge theory on a single plaquette. In order to construct these states, it is convenient to use Schwinger boson (prepotential) representation of SU(2) loop electric field and flux operators \([48, 49]\):

\[
E^a_\pm = a^\dagger \left( \frac{\sigma^a}{2} \right) a; \quad E^\sigma_\pm = b^\dagger \left( \frac{\sigma^a}{2} \right) b; \quad W_{a\beta} = \frac{1}{\sqrt{(N + 1)}} \left( \tilde{a}_a^\dagger b_\beta^\dagger - a_a b_\beta \right) \frac{1}{\sqrt{(N + 1)}}. \tag{5.12}
\]

Above \( \tilde{a}_a = \epsilon_{c\gamma} a_{c\gamma} \). Like link electric fields, loop electric fields satisfy \((\tilde{E}_-)^2 = (\tilde{E}_+)^2\) (see eqn \((2.14)\)) implying \(N_a = N_b \equiv N\). Above loop basis states \(|n, l = 0, m = 0\rangle\) can be constructed in terms of loop prepotentials as follows:

\[
|n\rangle \equiv |n, l = 0, m = 0\rangle = \frac{(k_+^n)^n}{\sqrt{n!(n + 1)!}} |0\rangle.
\]

\[
\sum_{n=1}^\infty |n\rangle \langle n| = \mathcal{I}, \quad \langle m|n\rangle = \delta_{nm}. \tag{5.13}
\]

In \((5.13)\) \(\mathcal{I}\) is the identity operator in \(\mathcal{H}^p\). The loop creation-annihilation and number operators are defined as:

\[
k_+ \equiv a^\dagger \cdot \tilde{b}^\dagger, \quad k_- \equiv a \cdot \tilde{b}, \quad k_0 \equiv 1/2(N_a + N_b + 2).
\]

These operators form SU(1, 1) algebra which is discussed in section \(5.3\) in the context of the dynamical symmetry group SO(4,2) of hydrogen atom.
b. Finite lattice case

We now generalize the above results to a finite lattice. Some equations from the last chapter are repeated here for ease of presentation. One hydrogen atom is associated with each plaquette of the lattice. Just like in the single plaquette case, we identify $J^a_{\mp}(p)$ of the hydrogen atom associated with each plaquette $p$ with the $SU(2)$ loop electric field $E^a_{\mp}(p)$.  

\[ J^a_{\mp}(p) \leftrightarrow E^a_{\mp}(p); \quad p = 1, \cdots \mathcal{P}. \tag{5.14} \]

Above, $\mathcal{P}$ is the total number of plaquettes on the lattice. As before, this identification implies \n
\[ L^a(p) \equiv J^a_{\mp}(p) + J^a_{\mp}(p) \leftrightarrow L^a(p) \equiv E^a_{\mp}(p) + E^a_{\mp}(p); \quad p = 1, \cdots \mathcal{P}. \tag{5.15} \]

The global $SU(2)$ Gauss law constraints on a finite lattice implies that the total angular momentum of all the hydrogen atoms is 0:

\[ G^a(0,0) = \sum_p L^a(p) = 0 \leftrightarrow \sum_p L^a(p) = 0. \tag{5.16} \]

Above, the $p$ summation is over all the plaquettes. The above correspondence implies that $SU(2)$ lattice gauge theory Hilbert space on a lattice with $\mathcal{P}$ plaquettes are characterized by the following Complete Set of Commuting Operators (CSCO):

Uncoupled basis:

\[ \begin{pmatrix} \vec{J}^2_1 & \vec{J}^2_2 & \cdots & \vec{J}^2_{\mathcal{P}-1} & \vec{J}^2_{\mathcal{P}} \\ \vec{L}^2_1 & \vec{L}^2_2 & \cdots & \vec{L}^2_{\mathcal{P}-1} & \vec{L}^2_{\mathcal{P}} \\ \vec{L}^z_1 & \vec{L}^z_2 & \cdots & \vec{L}^z_{\mathcal{P}-1} & \vec{L}^z_{\mathcal{P}} \end{pmatrix}. \quad \text{(CSCO - A)} \tag{5.17} \]

Coupled basis:

\[ \begin{pmatrix} \vec{J}^2_1 & \vec{J}^2_2 & \cdots & \vec{J}^2_{\mathcal{P}-1} & \vec{J}^2_{\mathcal{P}} \\ \vec{L}^2_1 & \vec{L}^2_2 & \cdots & \vec{L}^2_{\mathcal{P}-1} & \vec{L}^2_{\mathcal{P}} \\ (\vec{L}^z_{12})^2 & (\vec{L}^z_{123})^3 & \cdots & (\vec{L}^z_{\text{total}})^2 & \vec{L}^z_{\text{total}} \end{pmatrix}. \quad \text{(CSCO - B)} \tag{5.18} \]

Above, $(\vec{L}^z_{1,2,\cdots q})^2 \equiv (\vec{L}^z_1 + \vec{L}^z_2 + \cdots + \vec{L}^z_q)^2$ with eigenvalue $l_{1,2,\cdots q}(l_{1,2,\cdots q} + 1)$ and $q = 2, 3, \cdots, \mathcal{P}$ and $(\vec{L}^z_{\text{total}})^2 \equiv (\vec{L}^z_{1,2,\cdots \mathcal{P}})^2$. For the special case of $\mathcal{P} = 4$, the uncoupled basis diagonalizing CSCO-A and the coupled basis diagonalizing CSCO-B are graphically illustrated in Figure 5.3 (a) and (b) respectively. These CSCOs were discussed in the context of SU(2) lattice gauge theory in eqns (3.76) and (3.79). CSCO-B is more suitable for SU(2) gauge theory as the SU(2) Gauss law can be trivially solved by demanding that $(\vec{L}^z_{\text{total}})^2 = \vec{L}^z_{\text{total}} = 0$. Such a gauge invariant loop basis diagonalizing CSCO-B (see equation (3.79) was already described in
section 3.2.3.3. We now interpret those results in terms of hydrogen atoms. Some of the equations are repeated for convenience. We draw tadpoles over each of the \( P \) plaquettes and then couple their emerging angular momentum fluxes \( (l_p, m_p) \) with \( p = 1, \ldots, P \) in a sequential manner as in Figure (5.3-b). It corresponds to going from decoupled tadpole basis diagonalizing CSCO-A to the coupled basis diagonalizing CSCO-B. The resulting orthonormal and complete coupled hydrogen basis describing SU(2) lattice gauge theory is given by:

\[
\begin{bmatrix}
n_1 & n_2 & \cdots & n_P \\
l_1 & l_2 & \cdots & l_P \\
l_{12} & l_{123} & \cdots & l_{12\ldots p-2}
\end{bmatrix}
\equiv \left\{ |n_1 l_1 m_1 \rangle \otimes |n_2 l_2 m_2 \rangle \otimes \cdots \otimes |n_P l_P m_P \rangle \right\}_{l_{\text{total}}=0}^{m_{\text{total}}=0}
\]

Wigner coupled states of hydrogen atoms

(5.19)

Above, we have used the fact that \( l_{12\ldots p} = 0 \) and therefore, \( l_{12\ldots p-1} = l_{12\ldots p-2} \). This follows from the Gauss law (5.16). As an example the loop states over 4 plaquettes in Figure (5.3-b) are constructed as:

\[
\begin{bmatrix}
n_1 & n_2 & n_3 & n_4 \\
l_1 & l_2 & l_3 & l_4 \\
l_{12}
\end{bmatrix}
= \sum_{m_1, m_2, m_3, m_4, m_{12}, m_{123}} C_{l_{12} m_1, l_{12} m_2}^{l_{12} m_1, l_{12} m_2} C_{l_{123} m_{123}, l_{123} m_{123}}^{l_{123} m_{123}, l_{123} m_{123}} C_{0 0}^{0 0} |n_1 l_1 m_1 \rangle |n_2 l_2 m_2 \rangle |n_3 l_3 m_3 \rangle |n_4 l_4 m_4 \rangle.
\]

(5.20)

Above, the \( |n_p l_p m_p \rangle \); \( p = 1, 2, 3, 4 \) states are coupled through Clebsch-Gordon coefficients. This is the \( ll \) coupling. The \( \bullet \) in Figure 5.3-b shows the \( ll \) coupling in eqn.(5.20). The above loop basis (3.80) will be briefly denoted by \( | [n] [l] [ll] \rangle \). As discussed in section 3.2.3.3, the symbols \( [n]; [l] \) and \( [ll] \) stand for the sets \( (n_1 \cdots n_P) \); \( P \) principle quantum numbers; \( [l_1 \cdots l_P] \); \( P \) angular momentum quantum numbers and \( (l_{12}, l_{123}, \cdots, l_{12\ldots(p-2)}) \); \( (P - 3) \) coupled angular momentum quantum numbers respectively. Thus the hydrogen
atom loop basis \((5.19)\) in \(H^p\) is labelled by \(\mathbb{N}_{SU(2)}^{d=2} = 3(P - 1)\) gauge invariant quantum numbers. As expected, this is also the dimension of quotient space \(\mathbb{N}_{SU(2)}^{d=2} = \dim \left[ \frac{\text{links}}{\text{sites}} SU(2) \right] \).

**Hydrogen atom hypersphere \(S^3\):**

As shown by Fock \([148–150]\), the \(SU(2) \otimes SU(2)\) symmetry for hydrogen bound states \((p_0^2 \equiv -2E > 0)\) becomes manifest if we transcribe the hydrogen atom dynamics on a hypersphere \(S^3: (q_0, \vec{q}; q_0^2 + \vec{q}^2 = 1)\) embedded in \(R^4: (p_0, p_1, p_2, p_3)\) through a stereographic projection:

\[
q_0 \equiv \frac{(p_0^2 - \vec{p}^2)}{(p_0^2 + \vec{p}^2)}, \quad \vec{q} \equiv \frac{2p_0\vec{p}}{(p_0^2 + \vec{p}^2)}, \quad \Omega_H(q_0, \vec{q}) \equiv q_0 \sigma_0 + i\vec{q} \cdot \vec{\sigma}, \quad q_0^2 + \vec{q}^2 = 1. \tag{5.21}
\]

Above \(\sigma_0, \vec{\sigma}\) are the identity, Pauli matrices respectively. The mapping \((5.21)\) enables us to transform \([148–150]\) momentum space hydrogen atom Schrödinger equation into the integral equation of the 4-dimensional spherical harmonics \(Y_{n,l,m}(\Omega_H)\) representing a free particle on \(S^3\). It was later shown by Bargmann \([154]\) that \((L_1, L_2, L_3)\) and \((A_1, A_2, A_3)\) correspond to rotations in \((q_2q_3), (q_1q_3), (q_1q_2)\) and \((q_0q_1), (q_0q_2), (q_0q_3)\) planes respectively making \(SU(2) \otimes SU(2)\) symmetry of hydrogen atom manifest. We will identify hydrogen atom \(S^3\) on a plaquette \(p\) with the \(SU(2)\) group manifold \(S^3\) associated with the plaquette loop flux operator \(\mathcal{W}_{\alpha\beta}(p)\).

**Wilson loop hypersphere \(S^3\):**

We now analyze the equivalence between the Hilbert space of hydrogen atom and \(SU(2)\) lattice gauge theory in the dual magnetic description. We again start with single plaquette basis \(|j \ m_- \ m_+\rangle\) in \((5.6)\) and define states on \(SU(2)\) group manifold \(S^3\) as:

\[
|\Omega_W\rangle = \sum_{j=0}^{\infty} \sum_{m_-=0}^{\infty} |j\rangle D^j_{m_- m_+}(\Omega_W) |j, m_-, m_+\rangle. \tag{5.22}
\]

In \((5.22)\) \(|j\rangle \equiv \sqrt{\frac{(2j+1)}{2\pi}}\), \(D^j_{m_- m_+}(\Omega_W)\) are the Wigner matrices characterized by \(SU(2)\) group manifold \(S^3\):

\[
\Omega_W(w_0, \vec{w}) \equiv w_0 \sigma_0 + i\vec{w} \cdot \vec{\sigma}, \quad w_0^2 + \vec{w}^2 = 1 : S^3.
\]
5.3 Loop Dynamics & Dynamical Symmetry Group $\text{SO}(4,2)$ of Hydrogen Atom

<table>
<thead>
<tr>
<th>SU(2) lattice gauge theory</th>
<th>hydrogen atom</th>
</tr>
</thead>
<tbody>
<tr>
<td>SU(2) group manifold $S^3$</td>
<td>Momentum space $S^3$</td>
</tr>
<tr>
<td>$\Omega_W(\omega, \vec{w})$</td>
<td>$\Omega_H(\omega, \vec{w})$</td>
</tr>
<tr>
<td>$\mathcal{E}_\pm^a$</td>
<td>$J_\pm^a$</td>
</tr>
<tr>
<td>$L^a = \mathcal{E}<em>+^a + \mathcal{E}</em>-^a$</td>
<td>$L^a = J_+^a + J_-^a$</td>
</tr>
<tr>
<td>$A^a = \mathcal{E}<em>+^a - \mathcal{E}</em>-^a$</td>
<td>$A^a = J_+^a - J_-^a$</td>
</tr>
</tbody>
</table>

Table 5.1: The corresponding quantities in SU(2) lattice gauge theory and hydrogen atom

As shown in appendix B, the recursion relations of Wigner matrices show that the orthonormal and complete angular states (5.22) also diagonalize the plaquette loop operators $W_{a\beta}$.

$$W_{a\beta} |\Omega_W\rangle = (\Omega_W(\omega_0, \vec{w}))_{a\beta} |\Omega_W\rangle.$$ (5.23)

Thus the $SU(2)$ matrix $\Omega_W(\omega_0, \vec{w})$ on $S^3$ describes the $SU(2)$ magnetic fluxes on a plaquette. Under global SU(2) transformation:

$$|\Omega_W\rangle \rightarrow |A_0 \Omega_W A_0^\dagger\rangle.$$ (5.24)

The gauge generators $L^1, L^2, L^3$ in (3.47) rotate $(w_2w_3), (w_3w_1), (w_1w_2)$ planes respectively leaving $w_0$ (gauge) invariant. Defining “Lenz operators” in lattice gauge theory as $A^a = \mathcal{E}_+^a - \mathcal{E}_-^a$, we see that $(A^1, A^2, A^3)$ generate rotations in $(w_0w_1), (w_0w_2), (w_0w_3)$ planes respectively. Therefore, the actions of $(L^a, A^a)$ on $\Omega_W$ in gauge theory is exactly same as the actions of $(L^a, A^a)$ on $\Omega_H$ in hydrogen atom. Therefore, we further identify:

$$\Omega_H \sim \Omega_W \equiv \Omega.$$ (5.25)

The correspondence between quantities in SU(2) lattice gauge theory and hydrogen atom are shown in table 5.1.

5.3 LOOP DYNAMICS & DYNAMICAL SYMMETRY GROUP $\text{SO}(4,2)$ OF HYDROGEN ATOM

It is well known in hydrogen atom literature [151, 152] that the operators which takes $|n l m\rangle$ to $|n' l' m'\rangle$ form an SO(4,2) algebra on closure. This is called the dynamical symmetry group of hydrogen atom. In this section, we will see how this dynamical symmetry group naturally emerges in SU(2) lattice gauge theory. We will also study its role in describing SU(2) loop dynamics through the loop Hamiltonian (5.34).
Like single plaquette case, the remaining quadratic operators:

\[
\begin{array}{|c|c|c|}
\hline
L_{ab} &= \epsilon_{abc} (\mathcal{E}^c_- + \mathcal{E}^c_+) \\
L_{a4} &= (\mathcal{E}^a_- - \mathcal{E}^a_+) \\
L_{45} &= -\frac{i}{2} (k_+ - k_-) \\
L_{a5} &= \frac{1}{2} \text{Tr} \sigma_a \left( \mathcal{W}^{(+) - \mathcal{W}^{(-)} } \right) \\
L_{a6} &= \frac{i}{2} \text{Tr} \sigma_a \left( \mathcal{W}^{(+)} + \mathcal{W}^{(-)} \right) \\
L_{56} &= k_0 \\
\hline
\end{array}
\]

Table 5.2: All possible 15 SU(2) tensor operators on a plaquette which are U(1) gauge invariant. They form SO(4,2) algebra. We have defined \( \mathcal{W}_{\beta}^{(\pm)} \equiv -\tilde{a}^\dagger_{\alpha} b^\beta_{\alpha} \) and \( \mathcal{W}_{\alpha}^{(-)} \equiv a^\dagger_{\alpha} b_{\beta} \).

We now discuss the structure of a general gauge invariant operator in \( \mathcal{H}^P \). We first consider the single plaquette case. Any gauge invariant operator on a single plaquette can be created out of the loop creation, annihilation \((k_+,k_-)\) and number operators \((k_0)\). They

1. are invariant under U(1) gauge transformations \((2.31)\):

\[
a_a \rightarrow e^{i\theta} a_a,
\]

2. form \(SU(1,1) \subset SO(4,2)\) algebra

\[
[k_-,k_+] = 2k_0, \quad [k_0,k_\pm] = \pm k_\pm \quad \text{and}
\]

3. generate transitions \(|n\rangle \rightarrow |\tilde{n}\rangle\) within the single plaquette hydrogen atom basis in \( \mathcal{H}^P \).

Therefore, the dynamical symmetry group of SU(2) lattice gauge theory on a single plaquette is given by SU(1,1).

We now generalize the above three results to the entire lattice by constructing an algebra of U(1) invariant and SU(2) gauge covariant operators. We will then see the action of these operators on \(|n,l,m\rangle\) and show that they generate transitions \(|n,l,m\rangle \rightarrow |\tilde{n},\tilde{l},\tilde{m}\rangle\). We then show that this algebra is an SO(4,2) algebra as is expected from the corresponding result in hydrogen atom.

We note that all \(4P\) loop prepotential operators \((a^\dagger_{\alpha}(p), a_{\beta}(p))\) and \((b^\dagger_{a}(p), b_{\beta}(p))\) of the theory transform as matter doublets under the global SU(2) gauge transformations. Therefore, the basic SU(2) tensor operators which are also invariant under U(1) gauge transformations of prepotentials can be classified\(^3\) into the following four classes:

\[
\left[ a^\dagger_{\alpha}(p) b^\dagger_{\beta}(p); \quad a_{\alpha}(p) b_{\beta}(p); \quad a^\dagger_{\alpha}(p) a_{\beta}(p); \quad b^\dagger_{a}(p) b_{\beta}(p) \right] \quad p = 1,2,\cdots,P. \tag{5.25}
\]

These are 16 SU(2) gauge covariant and U(1) gauge invariant operators on every plaquette of the lattice. The magnitude of the left and the right electric fields on every plaquette being equal, the number operators on each plaquette satisfy \(a^\dagger(p) \cdot a(p) = b^\dagger(p) \cdot b(p) = \hat{N}(p)\).

\(^3\) Like single plaquette case, the remaining quadratic operators: \([a^\dagger_{\alpha}(p) a^\dagger_{\beta}(p), b^\dagger_{a}(p) b^\dagger_{\beta}(p), a^\dagger_{\alpha}(p) b_{\beta}(p)]\) and their hermitian conjugates are not invariant under U(1) gauge transformations and therefore ignored.
Thus their number reduces to 15. These 15 operators on every plaquette, arranged as in Table 5.2, form SO(4,2) algebra\[^{151}\]:

\[
[L_{AB}, L_{CD}] = -i \left( g_{AC} L_{BD} + g_{AD} L_{CB} + g_{BC} L_{DA} + g_{BD} L_{AC} \right).
\] (5.26)

Above, $A, B = 1, \cdots, 6$ and $g_{AB}$ is the metric $(- - - + +)$. The algebra (5.26) can be explicitly checked using the prepotential representations of $E_\pm^\ell$ and $W_\pm^\ell$. Note that the fundamental loop quantization relations (3.46) are also contained in (5.26).

It is expected that since any gauge invariant operators can be constructed out of the above SO(4,2) generators, it forms the dynamical symmetry group of SU(2) lattice gauge theory. This can be seen as follows. Let $|\psi\rangle$ be a physical state and $\hat{O}$ be any gauge invariant operator. Then the state $|\psi'\rangle \equiv \hat{O} |\psi\rangle$ is also a physical state. As $|\psi\rangle, |\psi'\rangle \in \mathcal{H}'$, both can be expanded in the “hydrogen atom loop basis”. We, therefore, conclude that any gauge invariant operator $\hat{O}$ will generate a transition:

\[
|n l m\rangle \xrightarrow{\hat{O}} \sum_{\bar{n}, \bar{l}, \bar{m}} O_n^{\bar{n} \bar{l} \bar{m}} |\bar{n} \bar{l} \bar{m}\rangle.
\]

Above, $O_n^{\bar{n} \bar{l} \bar{m}}$ are some coefficients depending on the operator $\hat{O}$.

We now show that any transition from $|n, l, m\rangle \rightarrow |\bar{n}, \bar{l}, \bar{m}\rangle$ can be achieved by the SO(4,2) generators\[^{151}\]. The operator $L_{56}$ acts as a number operator which counts the $n$ value,

\[
L_{56}|nlm\rangle = n|nlm\rangle.
\] (5.27)

From the SO(4,2) algebra (table 5.2), $[L_{45}, L_{46}] = iL_{56}$. Therefore, $k_\pm = L_{45} \mp iL_{46}$ acts as ladder operators on $n$ since $[L_{56}, k_\pm] = \pm k_\pm$. Similarly, we define $L_\pm = L^{(1)} \pm L^{(2)}$ and $A_\pm = A^{(1)} \pm A^{(2)}$.

\[
k_+ |n l m\rangle = \sqrt{(n+1)(n+2)} |n+1 l m\rangle,
\]
\[
k_- |n l m\rangle = \sqrt{(n+1)(n-1)} |n-1 l m\rangle,
\]
\[
L^{(3)} |n l m\rangle = m |n l m\rangle,
\]
\[
L_\pm |n l m\rangle = \mp \frac{1}{\sqrt{2}} \sqrt{(l \mp m)(l \pm m + 1)} |n l m \pm 1\rangle,
\] (5.28)

\[
A^{(3)} |n l m\rangle = c_1 |n l + 1 m\rangle - c_2 |n l - 1 m\rangle,
\] (5.29)

\[
A_\pm |n l m\rangle = d_1 |n l m \pm 1\rangle + d_2 |n l + 1 m \pm 1\rangle + d_3 |n l - 1 m \pm 1\rangle.
\] (5.30)
The finite plaquette Hamiltonian can also be written in terms of SU($2$) algebra, the Hamiltonian can be written in terms of these SU($2$) operators. This is also clear from eqns. (5.2) and (5.3). for eg: The Hamiltonian for the single plaquette case is given by

$$H = 4g^2 \mathcal{E}^2 - \frac{K}{g^2} \text{Tr} \mathcal{W} = g^2 (L_{56})^2 - \frac{K}{g^2} \frac{1}{\sqrt{L_{56}}} L_{46} \frac{1}{\sqrt{L_{56}}}.$$  

(5.34)
5.3 Loop Dynamics & Dynamical Symmetry Group SO(4,2) of Hydrogen Atom

Figure 5.4: The SU(2) ground state picture in the hydrogen atom basis (3.80).

can also be appropriately generalized to higher SU(N) group by replacing SU(2) prepotential operators by SU(N) irreducible prepotential operators discussed in [102, 103].

5.3.0.1 A variational ansatz

The above correspondence between hydrogen atom and SU(2) gauge theory inspires the following simple variational ansatzes for the ground state $|\Psi_0\rangle$ and the first excited state $|\Psi_1\rangle$ of SU(2) lattice gauge theory:

$$
|\Psi_0\rangle = e^\Gamma |0\rangle , \quad \langle \Psi_0 | \Psi_0 \rangle = 1 ,
|\Psi_1\rangle = \Sigma^+ |\Psi_0\rangle , \quad \langle \Psi_0 | \Psi_1 \rangle = 0 .
$$

In (5.35) $\Gamma$ and $\Sigma$ are the SU(2) $\otimes$ U(1) gauge invariant operators constructed out of SO(4,2) generators in the Table 1. It is convenient to write $\Gamma = \Gamma^+ - \Gamma^-$ where $\Gamma^- \equiv (\Gamma^+)^\dagger$ and $\Gamma^+, \Sigma^+$ have the structures:

$$
\Gamma^+ \equiv G_1 \sum_{p=1}^P k_+(p) + \sum_{p_1,p_2=1}^P G_2(|p_1 - p_2|) \vec{k}_+(p_1) \cdot \vec{k}_+(p_2) + \cdots ,
\Sigma^+ \equiv F_1 \sum_{p=1}^P k_+(p) + \sum_{p_1,p_2=1}^P F_2(|p_1 - p_2|) \vec{k}_+(p_1) \cdot \vec{k}_+(p_2) + \cdots .
$$

In the first term above $k_+(p)$ is the gauge invariant SU(1,1) $\in$ SO(4,2) plaquette loop creation operator. In the second term, we have defined SU(2) adjoint loop flux creation operator $\vec{k}_+(p)$ on every plaquette $p$ using SO(4,2) generators in Table 1:

$$
k_+(p) \equiv L_{a5}(p) - iL_{a6}(p) = Tr(\sigma^a W^{(+)}) , \quad a = 1, 2, 3 .
$$

Note that the expansion (5.36) is in terms of number of fundamental loops and not in terms of coupling constant. In fact, $g^2$ dependence of the structure functions $G_1, G_2, \cdots$ and $F_1, F_2, \cdots$ have been completely suppressed. The physical interpretations of (5.35) and (5.36) are extremely simple. The operator $e^\Gamma$ acting on the strong coupling vacuum in (5.35)
creates loops of all shapes and sizes in terms of the fundamental loop operators to produce the ground state $|\Psi_0\rangle$. The first term $k_+(p)$ in (5.36) creates hydrogen atom s-states on plaquette $p$ or simple one plaquette loops. These are shown as small circles (tadpoles without legs) in Figure 5.4. The second term describes doublets of hydrogen atoms with vanishing total angular momentum. These are shown as two tadpoles joined together in Figure 5.4.

The three hydrogen atom or three tadpole states over three plaquettes $(p_1, p_2, p_3)$ can be created by including a term of the form $(\vec{k}_+(p_1) \times \vec{k}_+(p_2)) \cdot \vec{k}_+(p_3)$ in $\Gamma^+$ and so on and so forth. As shown in Figure 5.4, the ground state is a soup of all such coupled tadpoles or coupled hydrogen atom clusters, each with vanishing angular momentum. The first excited state in (5.36) is obtained by exciting loops in this ground state by a creation operator $\Sigma^+$. The sizes of the “hydrogen atom clusters” and their importance depend on the structure functions $G$ and $F$ which in turn are fixed by the loop Schrödinger equation with Hamiltonian (3.86). These qualitative features can be made more precise by putting the ansatz (5.35) in (3.86). The resulting Schrödinger equation can be analyzed for the structure constants $G_1, G_2, \cdots$ and $F_1, F_2, \cdots$ in the complete, orthonormal hydrogen atom loop basis (3.80) using its dynamical symmetry group $SO(4,2)$ algebra in (5.26).

---

4 A reasonable assumption is $G_1 >> G_2 >> \cdots$, $F_1 >> F_2 >> \cdots$ in (5.36). In this case the matrix elements of $\delta H$ in (3.86) are small in the states in (5.35): $\langle \Psi_0 | \delta H | \Psi_0 \rangle \approx 0$ and $\langle \Psi_1 | \delta H | \Psi_1 \rangle \approx 0$ as $[L^2(p), k_+(p)] = 0$ and $L^2 |0\rangle = 0$. 