Chapter 2

Violation of Weak Equivalence Principle in quantum mechanics

The Equivalence Principle which is a consequence of the equality of gravitational and inertial masses [Galilei, 1638], was tested experimentally for classical test bodies to high precision. The validity of the Equivalence Principle for quantum mechanical particles has been tested through the gravity-induced interference experiments [Colella, Overhauser, and Werner, 1975; Peters, Chung, and Chu, 1999]. The possibility of violation of the weak equivalence principle in quantum mechanics (WEQ), which is the weak version of the equivalence principle adopted to the quantum mechanical framework, is discussed in a number of papers, for instance using neutrino mass oscillations in a gravitational potential [Adunas, Milla, and Ahluwalia, 2001; Gago, Nunokawa, and Funchal, 2000; Gasperini, 1988; Halprin and Leung, 1996; Krauss and Tremaine, 1988; Mureika, 1997]. The WEQ can be shown to be violated in quantum mechanics both theoretically [Greenberger, 1968, 1983; Greenberger and Overhauser, 1979] and experimentally [Colella, Overhauser, and Werner, 1975; Peters, Chung, and Chu, 1999]. One of the quantum mechanical approach of the violation of WEQ was given by Davies [Davies, 2004] for a quantum particle in a uniform gravitational field using a model quantum clock [Peres, 1980] as shown through the mass dependence of the tunnelling depth of the particle in the classically forbidden region.

In another way, Ali et al. have given the example of violation of WEQ for quantum particles represented by Gaussian wave packets projected upwards against gravity
[Ali et al., 2006]. They have shown the violation through the explicit mass dependence of the position detection probabilities around both the classical turning point and the point of initial projection and of the mean arrival time of freely falling particles at an arbitrary detector location using Gaussian wave packet.

It is now interesting to know how WEQ works for quantum particles represented by parametrized non-Gaussian wave packets. In real experiments, realization of exactly Gaussian wave packets is rather difficult. One of the aims of our study is to enable relaxation of the wave packet to be Gaussian. Therefore, the non-Gaussian nature of the wave packet can facilitate the experimental observation of the violation of WEQ, as well as enable a quantitative verification of the way the violation of WEQ depends on the departure from the Gaussian nature of the wave packet.

### 2.1 Initial wave function and it’s time evolution

Let us consider an ensemble of quantum particles in an external gravitational potential. The initial state of each particle is represented by a one-dimensional non-Gaussian wave function, given as

\[
\psi(z, t = 0) = N \left[ 1 + \alpha \sin \left( \frac{\pi}{4} \sigma_0 z \right) \right] e^{-\frac{z^2}{4\sigma_0^2}} + ik_0 z. \tag{2.1}
\]

Here, \(\alpha\) is the tuneable parameter varying from 0 to 1. \(N\) is the normalisation factor given by

\[
N = \left( \sqrt{2\pi} \sigma_0 \left[ 1 + \frac{\alpha^2}{2} (1 - e^{-\pi^2/8}) \right] \right)^{-1/2}. \tag{2.2}
\]

The salient features corresponding to the above wave packet are its asymmetry due to the presence of sine function, its infinite tail, and its reducibility to a Gaussian wave packet upon a continuous decrement of the parameter \(\alpha\) to zero. The property of an infinite tail of the above wave function is not associated with non-Gaussian forms that are generated by truncating the Gaussian distribution. For \(\alpha = 0\), we get the Gaussian wave function. Here, we have restricted ourselves to a one-dimensional representation along the vertical \(z\) direction.
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The initial group velocity is given by $u = \frac{\hbar k_0}{m}$, where $m$ is the mass of the particle and $k_0$ is the wave number. In the Schrödinger picture, the mean initial conditions for the initial wave functions $\psi_1$ and $\psi_2$ representing particles 1 and 2 respectively are

$$\langle \hat{z} \rangle_{\psi_1} = \langle \hat{z} \rangle_{\psi_2} = \langle \hat{z}(0) \rangle,$$
$$\frac{\langle \hat{p}_z \rangle_{\psi_1}}{m_1} = \frac{\langle \hat{p}_z \rangle_{\psi_2}}{m_2} = u,$$ (2.3)

where $\langle \hat{z} \rangle_\psi$ and $\langle \hat{p}_z \rangle_\psi$ represent the expectation values for position and momentum operators respectively. The propagator of a particle in the linear gravitational potential $V = mgz$ is given below

$$G(z, t | y, 0) = \sqrt{\frac{m}{2\pi i \hbar t}} e^{i\frac{m}{2\hbar t}(z-y)^2 - \frac{i m g t}{2\hbar}(z+y) - \frac{i m^2 v^2 t^3}{24 \hbar}}$$ (2.4)

Using this propagator in the relation

$$\psi(z, t) = \int dy G(z, t | y, 0) \psi(y, 0),$$ (2.5)

one can obtain the Schrödinger time evolved wave function $\psi(z, t)$ at any subsequent time $t$ as

$$\psi(z, t) = N \sqrt{\frac{\sigma_0}{s_t}} e^{i\frac{m}{2\pi t} (z^2 - gt^2 z - \frac{\sigma_0^2}{4})} \cdot \left[ e^{-i\frac{m}{2\pi t} \frac{\sigma_0}{4} \left[ z - \frac{\hbar}{m} k_0 + \frac{\alpha^2}{2} \right]^2} + \frac{\alpha}{2 i} e^{-i\frac{m}{2\pi t} \frac{\sigma_0}{4} \left[ z - \frac{\hbar}{m} (k_0 + \beta) + \frac{\alpha^2}{2} \right]^2} - \frac{\alpha}{2 i} e^{-i\frac{m}{2\pi t} \frac{\sigma_0}{4} \left[ z - \frac{\hbar}{m} (k_0 - \beta) + \frac{\alpha^2}{2} \right]^2} \right],$$ (2.6)

where $s_t = \sigma_0 \left( 1 + i \frac{\hbar t}{2m \sigma_0^2} \right)$ and $\beta = \frac{\pi}{4\sigma_0}$. The probability density $|\psi(z, t)|^2$ is given by

$$\rho(z, t) = N^2 \frac{\sigma_0}{\sigma_G} e^{E_1} \left[ e^{E_2} + \alpha \left( e^{E_3} \sin A_1 + e^{E_4} \sin A_2 \right) + \frac{\alpha^2}{4} \left( 1 + e^{E_5} - 2e^{E_6} \cos A_3 \right) \right],$$ (2.7)
where

\[ E_1 = -\frac{\left(\pi \hbar t + 4 m \sigma_0 \left( z - u t + \frac{1}{2} g t^2 \right) \right)^2}{32 m^2 \sigma_G^2 \sigma_0^2}, \]

\[ E_2 = \frac{\pi \hbar t \left(\pi \hbar t + 8 m \sigma_0 \left( z - u t + \frac{1}{2} g t^2 \right) \right)}{32 m^2 \sigma_G^2 \sigma_0^2}, \]

\[ E_3 = \frac{\pi \hbar t \left(\pi \hbar t + 24 m \sigma_0 \left( z - u t + \frac{1}{2} g t^2 \right) \right)}{64 m^2 \sigma_G^2 \sigma_0^2}, \]

\[ E_4 = \frac{\pi \hbar t \left(\pi \hbar t + 8 m \sigma_0 \left( z - u t + \frac{1}{2} g t^2 \right) \right)}{64 m^2 \sigma_G^2 \sigma_0^2}, \]

\[ E_5 = \frac{\pi \hbar t \left( z - u t + \frac{1}{2} g t^2 \right)}{2 m \sigma_G^2 \sigma_0^2}, \]

\[ E_6 = \frac{E_5}{2}, \]

\[ A_1 = \frac{\pi \left( -\pi \hbar t + 8 m \sigma_0 \left( z - u t + \frac{1}{2} g t^2 \right) \right)}{32 m \sigma_G^2 \sigma_0^2}, \]

\[ A_2 = \frac{\pi \left(\pi \hbar t + 8 m \sigma_0 \left( z - u t + \frac{1}{2} g t^2 \right) \right)}{32 m \sigma_G^2 \sigma_0^2}, \]

\[ A_3 = \frac{\pi \sigma_0 \left( z - u t + \frac{1}{2} g t^2 \right)}{2 \sigma_G^2}, \]

and

\[ \sigma_G = |s_\epsilon| = \sigma_0 \left(1 + \frac{\hbar^2 t^2}{4 m^2 \sigma_0^4}\right)^{\frac{1}{2}} \quad (2.8) \]

is the spreading of the Gaussian wave packet. The probability density is explicitly mass dependent. Here, the spreading of the wave packet is given by

\[ \sigma_{NG} = \frac{\sqrt{\lambda(0) + \lambda(2) \alpha^2 + \lambda(4) \alpha^4}}{4 m \sigma_0 \left[2 e^{\frac{\pi^2}{\alpha^2}} + \alpha^2 \left(e^{\frac{\pi^2}{\alpha^2}} - 1\right)\right]}, \quad (2.9) \]

where

\[ \lambda(0) = 64 e^{\frac{\pi^2}{\alpha^2}} m^2 \sigma_0^2 \sigma_G^2, \]

\[ \lambda(2) = 8 e^{\frac{\pi^2}{\alpha^2}} \pi^2 m^2 \sigma_0^4 \left(1 - 2 e^{\frac{\pi^2}{\alpha^2}}\right) + 64 e^{\frac{\pi^2}{\alpha^2}} m^2 \sigma_0^2 \sigma_G^2 \left(e^{\frac{\pi^2}{\alpha^2}} - 1\right) + 2 e^{\frac{\pi^2}{\alpha^2}} \pi^2 \hbar^2 t^2, \]

\[ \lambda(4) = \left(e^{\frac{\pi^2}{\alpha^2}} - 1\right) \left[16 m^2 \sigma_0^2 \sigma_G^2 \left(e^{\frac{\pi^2}{\alpha^2}} - 1\right) + 4 \pi^2 m^2 \sigma_0^4 + e^{\frac{\pi^2}{\alpha^2}} \pi^2 \hbar^2 t^2\right], \quad (2.10) \]
which is also mass dependent. From Fig. 2.1, we can see that for large masses, \( \sigma_{NG} \) becomes almost constant with \( \alpha \). For a particular value of \( \alpha \), \( \sigma_{NG} \) decreases with increasing value of mass \( m \).

![Figure 2.1](image)

*Figure 2.1:* The variation of the width of the wave packet with \( \alpha \) for different values of \( m \) (in a.m.u.). We take \( \sigma_0 = 10^{-3} \) cm, \( t = 1 \) sec and \( u = 10^3 \) cm s\(^{-1} \).

Using the time evolved wave function \( \psi(z, t) \), the expressions for expectation value of position \( z \) and momentum \( p \) are calculated and are given by

\[
\langle z \rangle = \frac{\pi \alpha \sigma_0}{2 + \alpha^2 \left(1 - e^{-\frac{z^2}{\sigma_0^2}}\right)} e^{-z^2/\sigma_0^2} + ut - \frac{1}{2}gt^2 ,
\]

\[
\langle p \rangle = m(u - gt) .
\]

(2.11)

Here, \( \langle z \rangle \) is \( \alpha \)-dependent but mass independent. The value of \( z_{peak} \) is obtained from the numerical solution of the equation \( \frac{d|\psi|^2}{dz} = 0 \), for which \( |\psi|^2 \) is maximum. In Table 2.1 and Table 2.2, it is shown numerically how \( z_{peak} \) and \( \langle z \rangle \) vary with \( \alpha \) for \( 0 \leq \alpha \leq 1 \) and mass \( m \) respectively. It is clear that \( z_{peak} \neq \langle z \rangle \) and \( z_{peak} \) increases with \( \alpha \) and mass \( m \), whereas \( \langle z \rangle \) increases with \( \alpha \) only and remains constant for all masses. Here, the difference between mean and peak occurs due to the presence of the asymmetry of the wave packet. In the Gaussian limit (i.e., \( \alpha \to 0 \)), \( \langle z \rangle = z_{peak} \).

Fig. 2.2 shows the time variation of \( z_{peak} \) and \( \langle z \rangle \). Dotted curve shows the motion of \( z_{peak} \) whereas the continuous curve shows the motion of \( \langle z \rangle \). As we take
### Table 2.1: The variation of $\langle z \rangle$ and $z_{\text{peak}}$ in cm with $\alpha$ for mass $m = 10$ a.m.u., $t = 2$ sec and $\sigma_0 = 0.1$ cm.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$z_{\text{peak}}$</th>
<th>$\langle z \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>38.59999999999815</td>
<td>38.59999999999991</td>
</tr>
<tr>
<td>0.1</td>
<td>38.61540633397262</td>
<td>38.61149836659820</td>
</tr>
<tr>
<td>0.2</td>
<td>38.62925626009014</td>
<td>38.62275565386810</td>
</tr>
<tr>
<td>0.3</td>
<td>38.64090000000000</td>
<td>38.63350000000000</td>
</tr>
<tr>
<td>0.4</td>
<td>38.65026513575232</td>
<td>38.64367969597697</td>
</tr>
<tr>
<td>0.5</td>
<td>38.65786585282045</td>
<td>38.65299987691765</td>
</tr>
<tr>
<td>0.6</td>
<td>38.66400000000000</td>
<td>38.66140000000000</td>
</tr>
<tr>
<td>0.7</td>
<td>38.66910000000000</td>
<td>38.66880000000000</td>
</tr>
<tr>
<td>0.8</td>
<td>38.67337906376510</td>
<td>38.67524619824348</td>
</tr>
<tr>
<td>0.9</td>
<td>38.67696171641449</td>
<td>38.68068943521189</td>
</tr>
<tr>
<td>1.0</td>
<td>38.68002216630916</td>
<td>38.68519766066151</td>
</tr>
</tbody>
</table>

### Table 2.2: The variation of $\langle z \rangle$ and $z_{\text{peak}}$ in cm with mass $m$ in a.m.u. for $\alpha = 0.5$, $t = 2$ sec and $\sigma_0 = 0.1$ cm.

<table>
<thead>
<tr>
<th>Mass ($m$)</th>
<th>$z_{\text{peak}}$</th>
<th>$\langle z \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>38.65786589882798</td>
<td>38.65299987691765</td>
</tr>
<tr>
<td>60</td>
<td>38.65786590313615</td>
<td>38.65299987691765</td>
</tr>
<tr>
<td>90</td>
<td>38.65786590393416</td>
<td>38.65299987691765</td>
</tr>
<tr>
<td>120</td>
<td>38.65786590421307</td>
<td>38.65299987691765</td>
</tr>
<tr>
<td>150</td>
<td>38.65786590434229</td>
<td>38.65299987691765</td>
</tr>
</tbody>
</table>

$\sigma_0 = 10^{-7}$ cm, the $\alpha$-dependence of $\langle z \rangle$ and $z_{\text{peak}}$ becomes vanishingly small. So, this $\alpha$-dependence can not be shown graphically. As a result, the curves for $z_{\text{peak}}$ and $\langle z \rangle$ coincide totally in this graph.
2.2 Quantum mechanics in terms of classical concepts

When a wave packet satisfies Ehrenfest’s theorem and classical Liouville’s equation using Wigner distribution, then one can conclude that the dynamics of $\langle z \rangle$ is like a classical point particle.

2.2.1 Ehrenfest’s theorem

According to Ehrenfest’s theorem, the expectation value of quantum mechanical operators obey classical laws of motion, the instances of which are given below

$$\frac{d\langle z \rangle}{dt} = \frac{\langle p \rangle}{m},$$

$$\frac{d\langle p \rangle}{dt} = \left\langle -\frac{\partial V}{\partial z} \right\rangle.$$  \hspace{1cm} (2.12)

From Eq. (2.11), it is possible to check that Ehrenfest’s theorem is satisfied for our non-Gaussian wave packet.
2.2.2 Wigner distribution function

If a full classical description is to emerge from quantum mechanics, we must be able to describe quantum systems in phase-space. To observe the expression of $\langle z \rangle$ in the classical limit, we use the Wigner distribution function [Ford and O’Connell, 2002; Robinett, Doncheski, and Bassett, 2005; Wigner, 1932], which is one of the important phase space distributions.

From the non-Gaussian wave function $\psi(z, t)$ (Eq. (2.6)), we have calculated the Wigner distribution function $D_w(z, p, t)$ as

$$D_w(z, p, t) = \frac{1}{\pi \hbar} \int_{-\infty}^{\infty} \psi^*(z + y, t) \psi(z - y, t) e^{\frac{2iyp}{\hbar}} dy,$$  \hspace{1cm} (2.13)

the marginals of which yield the correct quantum probabilities for position and momentum separately. $D_w(z, p, t)$ satisfies the classical Liouville’s equation [Das and Sengupta, 2002] given by

$$\frac{\partial D_w(z, p, t)}{\partial t} + \dot{z} \frac{\partial D_w(z, p, t)}{\partial z} + \dot{p} \frac{\partial D_w(z, p, t)}{\partial p} = 0.$$  \hspace{1cm} (2.14)

Here, $H = \frac{p^2}{2m} + mgz$ is the Hamiltonian of the system and from Hamilton’s equation of motion, we get

$$\dot{z} = \frac{\partial H}{\partial p} = \frac{p}{m},$$
$$\dot{p} = -\frac{\partial H}{\partial z} = -mg.$$

The corresponding position distribution function is given by

$$\rho_C(z, t) = \int_{-\infty}^{\infty} D_w(z, p, t) dp.$$  \hspace{1cm} (2.15)

So, the expression for the average value of $z$ using Wigner distribution function in the classical limit is given by

$$\langle z \rangle = \frac{\int_{-\infty}^{\infty} z \rho_C(z, t) dz}{\int_{-\infty}^{\infty} \rho_C(z, t) dz}$$
$$= \frac{\pi \alpha \sigma_0}{2 + \alpha^2 \left(1 - e^{-\frac{\pi^2}{2}}\right)} e^{-\frac{\pi^2}{2}} + ut - \frac{1}{2} gt^2$$
$$\equiv z_0 + ut - \frac{1}{2} gt^2,$$  \hspace{1cm} (2.16)
Figure 2.3: The variation of probability $P_1(m)$ with mass (in a.m.u.) for different constant values of $\alpha$. We take $u = 10^5$ cm s$^{-1}$, $\sigma_0 = 10^{-3}$ cm and $\epsilon = \sigma_0$.

which is exactly equal to the quantum mechanical expectation value of $z$ [Chowdhury et al., 2012].

### 2.3 Position detection probabilities

Now, we consider an ensemble of quantum particles projected upwards against gravity with a given initial mean position and mean velocity. Classically, the particle moving upwards from the reference point at $z = 0$ reaches the maximum height $z = z_{\text{max}} = ut_1 - \frac{1}{2} gt_1^2$ at time $t_1 = u/g$ and returns to its initial projection point at $z = 0$ at time $t_2 = 2u/g$. Let, $P_1(m)$ and $P_2(m)$ be the probabilities of finding the particles within a very narrow detector region ($-\epsilon$ to $+\epsilon$) around $z_{\text{max}}$ and around $z = 0$ respectively, and are given by

$$P_1(m) = \int_{z_{\text{max}}-\epsilon}^{z_{\text{max}}+\epsilon} |\psi(z, t_1)|^2 dz,$$

$$P_2(m) = \int_{-\epsilon}^{+\epsilon} |\psi(z, t_2)|^2 dz. \quad (2.17)$$

From both the Fig. 2.3 and Fig. 2.4, it can be seen that both the probabilities $P_1(m)$ and $P_2(m)$ show the violation of WEQ [Chowdhury et al., 2012] through
their mass dependence for smaller masses using an initial non-Gaussian wave packet. This effect of mass dependence of the probabilities occur essentially as the spreading of the wave packet under linear gravitational potential depends on the mass of the particles. These probabilities become saturated in the limit of large mass. Again, it is seen that both the probabilities decrease for a particular value of \( m \) but the violation increases with the non-Gaussian parameter \( \alpha \), signifying a stronger violation of the weak equivalence principle.

### 2.4 Mean arrival time

Next, we consider quantum particles falling freely from the initial position at \( z = 0 \) under gravity with the initial state given by the Eq. (2.1) and with the initial velocity \( u = 0 \). Using the probability current approach [Dumont and Marchioro, 1993; Leavens, 1993], the mean arrival time of the particles to reach a detector location at \( z = Z \) is given by

\[
\bar{\tau} = \frac{\int_0^\infty |J(z = Z, t)| \, t \, dt}{\int_0^\infty |J(z = Z, t)| \, dt},
\]  

(2.18)
where \( J(z, t) \) is the quantum probability current density. \( J(z, t) \) can be negative, so we take the modulus sign here. From Eq. (2.18), it can be seen that the integral of the denominator converges whereas the integral of the numerator diverges formally. To converge the numerator, among the available several techniques [Damborenea, Egusquiza, and Muga, 2002; Hahne, 2003] here we use the simple technique of getting a cut-off at \( t = T \) in the upper limit of the time integral with \( T = \sqrt{2 (|Z| + 3 \sigma_T) / g} \). \( \sigma_T \) is the width of the wave packet at time \( T \). Thus, our calculations of the arrival time are valid upto the \( 3 \sigma_T \) level of spread in the wave function.

We have calculated \( J(z, t) \) with the initial non-Gaussian position distribution as

\[
J(z, t) = \frac{\hbar m}{2} \Im (\psi^* \nabla \psi) = N^2 \frac{e^{E_i}}{32 m^2 \sigma_0 \sigma_G^3} \left( \eta^{(0)} + \alpha \eta^{(1)} + \alpha^2 \eta^{(2)} \right), \tag{2.19}
\]

where

\[
\eta^{(0)} = 8 e^{E_2} \left[ \hbar^2 t \left( z - \frac{1}{2} g t^2 \right) + 4 m^2 \sigma_0^4 (u - g t) \right],
\]

\[
\eta^{(1)} = \eta^{(0)} e^{-E_2} \left( e^{E_3} \sin A_1 + e^{E_4} \sin A_2 \right) - 2 \pi \hbar^2 t \sigma_0 \left( e^{E_3} \cos A_1 + e^{E_4} \cos A_2 \right)
+ 4 \pi m \hbar \sigma_0^3 \left( e^{E_3} \sin A_1 - e^{E_4} \sin A_2 \right),
\]

\[
\eta^{(2)} = \left[ 2 \hbar^2 t \left( z - \frac{1}{2} g t^2 \right) + 8 m^2 \sigma_0^4 (u - g t) \right] (1 + e^{E_5} - 2 e^{E_6} \cos A_3)
- 2 \pi \hbar^2 t \sigma_0 e^{E_6} \sin A_3 + 2 \pi m \hbar \sigma_0^3 (e^{E_6} - 1),
\]

and where the \( E_i \)'s are defined below Eq. (2.7).

\( \bar{\tau} \) is also mass dependent through the explicit mass dependence of \( J(z, t) \). From Fig. 2.5, it is clear that for smaller masses \( \bar{\tau} \) is mass dependent. Fig. 2.6 shows that for large masses, \( \bar{\tau} \) becomes almost constant with \( \alpha \). For a particular value of \( \alpha \), \( \bar{\tau} \) decreases with increasing value of mass \( m \). This mass-dependence of the mean arrival time gradually vanishes as the mass is increased. Thus, compatibility with the equivalence principle emerges in the limit of large mass, or classical limit [Chowdhury et al., 2012]. For a particular value of mass, \( \bar{\tau} \) increases with \( \alpha \). The increment of \( \bar{\tau} \) with \( \alpha \) from the result of Gaussian case (i.e., with \( \alpha = 0 \)) signifies the stronger violation of WEQ [Chowdhury et al., 2012]. A non-Gaussian wave packet can exhibit non-classical features in a mass range where its Gaussian counterpart behaves classically.
Figure 2.5: The variation of mean arrival time with mass (in a.m.u.) for different constant values of $\alpha$. We take $\sigma_0 = 10^{-6}$ cm, $Z = -1$ cm and $u = 0$.

Figure 2.6: The variation of mean arrival time with $\alpha$ for different constant values of $m$ (in a.m.u.). We take $\sigma_0 = 10^{-6}$ cm, $Z = -1$ cm and $u = 0$. 


2.5 Bohmian interpretation

In the Bohmian model of quantum mechanics [Bohm, 1952, 1953], each individual particle is assumed to have a definite position, irrespective of any measurement. The pre-measured value of position is revealed when an actual measurement is done. If each of an ensemble of particles have the same wave function \( \psi \), which evolves with time according to the Schrödinger equation, these ontological positions are distributed according to the probability density \( \rho = |\psi|^2 \). If \( v \) is the Bohmian velocity of the particle and \( J \) is the probability current density, then the equation of motion of any individual particle is determined by the guidance equation \( v = J/\rho \). Solving the guidance equation, one gets the trajectory of the particle.

The violation of WEQ arises as a consequence of the spread of wave packets, the magnitude of which itself depends on the mass. In order to illustrate this effect we present an analysis in terms of Bohmian trajectories in the free fall case. Within the context of Bohmian interpretation of quantum mechanics [Bohm, 1952, 1953], each individual particle is assumed to have a definite position, irrespective of any measurement. The equation of motion of the particle is,

\[
m \dddot{x} = -\nabla (V + Q) \big|_{\vec{x}=\vec{x}(t)}, \tag{2.20}
\]

where \( Q = -\frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} \) is called ‘quantum potential energy’ and it is explicitly mass-dependent and \( \vec{x}(t) \) determines the path of the particle. In the gravitational potential \( V = mgz \), the quantum version of Newton’s second law is given by

\[
\dddot{z} = g - \frac{1}{m} \frac{\partial Q}{\partial z} \bigg|_{z=z(t)}. \tag{2.21}
\]

According to Holland [Holland, 1993], due to the intervention of the mass-dependent quantum force term, WEQ is violated. The arrival-time problem is unambiguously solved in the Bohmian mechanics, where for an arbitrary scattering potential \( V(\vec{x}) \), the arrival-time distribution of particles [Leavens, 1993, 1996, 1998, 2002] that actually reach \( \vec{x} = \vec{X} \), is given by the modulus of the probability current density, i.e., \( | J(\vec{X}, T) | \).

The effect of nonlocal nature and the mass-dependence of the quantum potential \( Q \) on the violation of WEQ is manifested in a similar way like the violation observed
due to the spread of the wave packet. We have computed a set of Bohmian trajectories corresponding to the free fall of our non-Gaussian wave packet with the tunable parameter $\alpha$. Fig. 2.7 shows a selection of Bohmian trajectories exhibiting their mass and $\alpha$ dependent spread. One can see that for small mass, the trajectories with initial positions on the left and right of the center of the wave packet (mean $\langle z \rangle$) spread out with time evolution, indicating violation of WEQ. The magnitude of spread increases with $\alpha$, signifying stronger violation of WEQ [Chowdhury et al., 2012] with increased departure from Gaussianity. The spread of the trajectories decreases as mass is increased, leading to the emergence of WEQ in the classical limit.
2.6 Summary

To summarize, in this chapter we have studied the violation of the gravitational weak equivalence principle in quantum mechanics using a non-Gaussian wave packet. The wave packet is constructed in such a way that its departure from Gaussianity is represented by a continuous and tunable parameter $\alpha$. The asymmetry of the wave packet entails a difference between its mean and peak, and causes the peak to evolve differently from a classical point particle, whereas the mean evolves like a classical point particle modulo a constant depending upon the parameter $\alpha$. Such a result is consistent with the Ehrenfest theorem, as we have shown, and re-confirmed using the Wigner distribution. The correspondence with the results following from a Gaussian wave packet is achieved in the limit of $\alpha \to 0$.

First, we have shown the violation of the weak equivalence principle through the dependence on mass of the position detection probabilities of an ensemble of particles projected upwards against gravity. We next compute the mean arrival time of freely falling wave packets through the probability current distribution corresponding to the non-Gaussian wave packet. The mass dependence of the arrival time again causes the violation of WEQ. In both the cases, the magnitude of violation increases with the increase of the non-Gaussian parameter $\alpha$, signifying stronger violation of WEQ with larger deviation from Gaussianity. It is observed that in the limit of large mass the classical value for the mean arrival time is approached, thereby indicating the emergence of the WEQ in the classical limit. In this context, it is worthwhile mentioning that though our work follows as a natural consequence of quantum mechanics or quantum theory in the non-relativistic limit, it does not imply a violation of general covariance in the relativistic domain. An interesting connection between the non-relativistic limit of quantum theory and the principle of equivalence has recently been discussed [Padmanabhan and Padmanabhan, 2011]. Finally, a computation of Bohmian trajectories further establishes the stronger violation of the WEQ by a non-Gaussian wave packet.