Chapter - 4

Nuclear Shell Model

The final aim of any investigation in nuclear structure, is an attempt to describe the observed level structure within the purview of a nuclear model. This would require us to solve the Schrödinger Equation (SE) to obtain the wave function of the nuclear system. Mathematically it is next to impossible to analytically solve the SE, for this few body system, wherein the major hurdle stems from (i) the fact that as discussed earlier that the exact nature of the nuclear potential is not well known, (ii) computational limitations. Thus we employ nuclear models, which are based on the comparison (similarities) between the nucleus and simpler (analogous) systems which can be adequately modeled mathematically. All nuclear models, must encompass the diverse aspects viz., the microscopic as well as universal aspects, of nuclei. A microscopic model description based on the solid foundations of first principles, ensures a reliability for extrapolation to region currently inaccessible to experimental endeavours. On the other hand the universal description to the nuclear properties provided by the model ensures a kind of simplicity while providing a coherent prediction for properties across the nuclear landscape. The caveat for any nuclear model is that the model proposition limit it’s range of applicability. In this chapter we have attempted to summarize the nuclear shell model, one of the most fundamental nuclear model, which has been exhaustively used in the present thesis.

To explain the various properties of the atomic nucleus, the different models are developed. The macroscopic properties such as the Binding energy, shape etc were reasonably explained within the framework of the liquid drop model. On the other hand, the microscopic property of the nucleus such as nuclear spin, parity and magnetic moment etc. can be explained from the nuclear shell model. Like the atomic shell structure the nucleus exhibit a similar kind of shell structures. For a certain number of electrons, the atomic shell exhibits its inert nature (shell closure). A similar kind of nature has been observed in case of atomic nucleus also. It was observed that in case of certain nucleon number when the neutrons or protons = 2, 8, 20, 28, 50, 82, 126 the binding


TABLE II: Classifications of the various nuclear models of contemporary relevance.

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Energy per nucleon had a sharp increase as compared with its immediate neighbour, indicating enhanced stability for these nuclei. Several additional experimental evidence was found to support the nuclear shell structure and are discussed in the subsequent section.

4.1 Early Nuclear Models

Since, the nucleus is a few body system, which at times manifests itself as a two body system (wherein the nuclear properties are primarily explained through the interactions of a small / few nucleons only), while otherwise it presents itself as a collective ensemble, wherein the observed properties can only be explained if one were to consider the behaviour / contribution from a substantial number of nucleons. Hence, the models are naively classified as Independent Particle Models, where all nucleons are considered to be different, and Collective Models, which assumes all nucleons are similar, and examines the macroscopic properties of the nucleus such as its size, binding energies etc. Table II summarizes the classification of the prevalent nuclear models, routinely used in nuclear structure investigation.

The earliest nuclear models could be traced to the Liquid Drop and the Fermi Gas Model. The Fermi Gas model is expected to be valid for the nucleus which comprises of interacting fermions, which obey Fermi Dirac Statistics. This model assumes that since neutrons and protons are distinguishable fermions and hence are situated in two separate potential wells, wherein the binding potential is generated by all the nucleons. The nuclear potential can be considered as a rectangular well, since the nucleons inside do not experience net force, whereas the unequal ambiance for the surface nucleons results in them experiencing a net force. All the available energy states are filled by
pairs of nucleons, and the energy of the highest occupied level (state) is referred to as Fermi Energy ($E_F$). This model could predict the average binding energy, which is defined as the energy difference between the top of the well and the Fermi level to be $\sim 7 - 8$ MeV. This model however, could not reproduce the microscopic details such as the spin and parity of the observed levels. This model has proved to be a good starting point for our understanding of this intriguing few body system *viz.* the nucleus.

The Liquid Drop model was one of the earliest nuclear model, which attempts to understand the macroscopic properties of the nucleus, such as the binding energy, nuclear size, to name a few. It is a Semi Empirical model, which did posses predictive powers. This model attempts to understand the nuclear properties, based on an analogy with a drop of liquid.

### 4.2 Spherical Shell Model

When one investigates the properties of atoms, it is observed that elements with atomic numbers $Z = 2, 10, 18, 36...$ show enhanced stability, *ie* they have unusually high electron ionization energy, they do not react to form molecules, to name a few. The concept of atomic shell helps us understand this observation. Hence, it seemed appropriate to develop an analogous (similar) shell structure (model) for the nucleus, based on the following experimental observations.

1. The nucleus for which neutron or proton numbers or both are magic numbers ($N, Z = 2, 8, 20, 28, 50, 82, 126$), such as $^4\text{He}$, $^{16}\text{O}$, $^{40}\text{Ca}$ and $^{208}\text{Pb}$ are particularly more stable and have greater binding energy than its immediate neighbours.

2. The high natural abundance of nuclei with neutron or proton number or both equal to the nucleon magic number.

3. Sudden increase of $\bar{B}$ (binding energy per nucleon) as a function of $A$ for nuclei such as $^4\text{He}$, $^{88}\text{Sr}$, $^{140}\text{Ce}$ and $^{208}\text{Pb}$.
4. There are large number of stable isotopes or isotones of nuclei having neutron or proton or both equal to the magic number.

5. The nuclei with magic neutron number have low neutron capture cross-section.

6. The nuclei with magic neutron or proton number or both have a higher excitation energy for the first excited state.

These observations hint at the existence of shell structures in the nucleus that forms the basis of the nuclear shell model. Maria Goeppert Mayer developed this model, which explains the existence of structured shells in which nucleons are distributed within the nucleus. For this work, she shared the 1963 Nobel Prize in Physics [4–6]. Goeppert Mayer is only the second woman to receive the Nobel Prize in Physics. According to the shell model, each nucleon in the nucleus experiences a one-body mean field, due to the motion of all other nucleons. Each nucleon moves independently and Pauli’s exclusion principle, ensures that they do not collide with each other. The shell model classifies the energy levels in terms of quantum numbers \( n, l, j \) following an analogous prescription to atomic physics. For a spherically symmetric potential the wave function for nucleon whose polar co-ordinates are \((r, \theta, \phi)\) has the following analytical form

\[
\Psi_{nlm} = R_{nl}(r)Y_{l}^{m}(\theta, \phi). \quad (63)
\]

Till date the exact nature of the nuclear potential is not well known. We usually model this potential and the choices are

- 3-D spherical harmonic oscillator, which has the familiar form

\[
V(r) = -V_0 + (1/2)(m\omega^2r^2) \quad (64)
\]

Where \( \omega \) is the harmonic oscillator frequency, \( m \) is the mass of the nucleon and \( V_0 \) is the depth of the potential well.

- Spherical infinite square well, which is represented by

\[
V(r) = \begin{cases} 
-V_0 & \text{for } +\infty \leq r \leq R \\
+\infty & \text{for } r > R \\
\end{cases} \quad (65)
\]

\( R \) is the nuclear radius
• Woods Saxon potential, as a function of the distance \( r \) from the center of nucleus, is

\[
V(r) = -\frac{V_0}{1 + \exp\left(\frac{r-R_a}{a}\right)}
\]  

(66)

where \( V_0 \) (having dimension of energy) represents the potential well depth. The constant nucleon density in the nuclear interior justifies the flat central part for the potential. The potential can be solved numerically, and it tends to zero at distances well beyond the nuclear radius. At the halfway radius \( R \), we have \( R = R_0 A^{1/3} \) (\( A \) is the mass number of the nucleus and \( R_0 \sim 1.2 \) fm). It is customary to have the value of diffuseness \( a \) to be \( \sim 0.5 - 0.8 \) fm.

The Schrödinger equation for the single particle levels is given by.

\[
\sum_{i=1}^{A} \left( -\frac{\hbar^2}{2m} \nabla_i^2 + V(r_i) \right) \psi_i = \epsilon_i \psi_i
\]

(67)

The first term represents the kinetic energy of the individual nucleons and the second term corresponds to the potential energy. The \( \psi_i \) corresponds to the single particle wave function and \( \epsilon_i \) represents to the single particle energy of the \( i \)-th particle. After substituting the appropriate choice of potential, for example either the Wood-Saxon, or the harmonic oscillator, in the above Schrödinger equation we obtain the energy eigenvalues,

\[
E_N = -V_0 + (N + 3/2)\hbar\omega
\]

(68)

with

\[
N = 2(n-1) + l
\]

\[
\begin{align*}
N &= 2(n-1) + l \\
n &= 1, 2, 3... & \text{principal quantum number} \\
l &= 0, 1, 2... & \text{orbital angular momentum}
\end{align*}
\]

(69)

The energy eigenvalues are degenerate in magnetic quantum number \( m \). The energy levels are usually bunched together (for a given \( N \)) and are referred to as shells. Levels corresponding to different \( N \) are well separated in energy, and the gap between two major principal quantum numbers is referred to as a shell gap. The maximum number of nucleons in each energy level is \( 2(2l + 1) \), where the factor 2 corresponds to the \(+1/2, -1/2\) spin states of the nucleon. For example, the lowest level designated as \( 1s \), and would correspond to \( n, l = 1, 0 \) and would have a maximum occupancy of two nucleons, whereas the next level \( 1p (n, l = 2, 1) \) would have a maximum occupancy of
nucleons. Thus the number of nucleons in each major shell would then correspond to 2, 8, 20, 40, 70, 112, 168. This exercise resulted in the reproduction of the lowest three magic numbers, but failed in its representation (interpretation) of the higher magic numbers.

The major difficulty one could envisage while porting the atomic shell concept to nuclear shells, is that nucleus contains particles with spin up and spin down ($m_s = \pm 1/2$) and as a result, the total angular momentum $j$ has the magnitude $j = l + (1/2) & l - (1/2)$. So in a sense, spin, could be either parallel or anti-parallel to the orbital angular momentum. Once, this effect was incorporated into the nuclear potential, a seminal milestone achieved by Mayer et al [4–6], all the nuclear magic numbers could be reproduced naturally. Their formalism resulted in the re-distribution (ordering) of the orbitals and the recalculation of the shell closures that now complied with the experimental observations for magic numbers. They introduced the the strong nuclear spin orbit interaction (spin orbit potential is proportional to $\vec{l}.\vec{s}$), in the nuclear potential. Hence the effective potential in which the nucleons move is now given by the equation

$$V_{eff}(r) = V(r) - V_{ls} \frac{\partial V(r)}{\partial r} \vec{l}.\vec{s}$$

(70)

Where $V_{ls}$ is the strength of the spin-orbit interaction. The inclusion of the spin-orbit term results in a wider nuclear potential (wider well results in lowering the energy of the states) for nucleons with spin parallel to the orbital angular momentum, and a narrower potential for nucleons with spin opposite to the orbital angular momentum.

The $\vec{l}.\vec{s}$ term causes the degenerate energy levels with same $l$ to split into $j = l \pm 1/2$ levels with degeneracy $(2j + 1)$. Each orbitals contains $(2j + 1)$ number of nucleons. For example, the $2p$ ($l = 1$) level would now split into $p_{3/2}$, $(l + 1/2)$ and $p_{1/2}$, $(l - 1/2)$ levels, with and the $2p_{3/2}$ orbital contains $(2 \times 3/2 + 1) = 4$ number of nucleons. The total angular momentum of a nucleon is given by

$$\vec{j} = \vec{l} + \vec{s}$$

(71)

and hence

$$\vec{l}.\vec{s} = 1/2(j^2 - l^2 - s^2)$$

(72)
FIG. 26: Energy levels for Nuclear Shell Model. The right hand side of the figure corresponds to energy levels after incorporation of the Spin Orbit interaction term. (Adapted from Meyerhof).

It’s average value is given by

\[
\langle \vec{l} \cdot \vec{s} \rangle = \frac{1}{2} (\langle j^2 \rangle - \langle l^2 \rangle - \langle s^2 \rangle) 
\]  
\[ (73) \]

\[
\langle \vec{l} \cdot \vec{s} \rangle = \frac{l}{2} \hbar^2 \ \text{for} \ j = l + \frac{1}{2} 
\]  
\[ (74) \]

and

\[
\langle \vec{l} \cdot \vec{s} \rangle = -\frac{(l + 1)}{2} \hbar^2 \ \text{for} \ j = l - \frac{1}{2}. 
\]  
\[ (75) \]

The energy difference between these two states is then given by \( \frac{1}{2}(2l + 1)\hbar^2 \).

After incorporation of the spin orbit interaction term the \( l \) degeneracy is removed, and the energy levels split into \( l \pm (1/2) \) levels as shown in Fig. 26. The large spin-orbit...
effect leads to crossing over of energy levels into different shells resulting in a faithful representation of the observed magic numbers at 2, 8, 20, 28, 50, 82, 126, which are in conformity with the experimental observations.

The greatest success of the shell model is its satisfactory representation of properties of nuclei in and around the magic numbers. The level excitation energies, spin, parity, magnetic and quadrupole moments for the ground state and low-energy excited states are satisfactorily predicted / reproduced by this model.

The shell model reproduces the properties of nuclei in and around the nucleon magic numbers assuming that a single nucleon is placed in an orbit above a closed shell or that a single nucleon is removed from an orbit below a closed shell. The orbits get filled up in the sequential order $1s \rightarrow 1p \rightarrow 1d, 2s...$. The shell model can predict the ground state spin and parity of a given nucleus. In case of even-even nuclei the ground state spin is 0. As all the particles get paired up (all the magnetic sub-states are filled, $m_j = -j, (-j + 1), ... + j$) and hence have the total angular momentum as zero. The parity for a given state is given by $(-1)^l$. In even-even nuclei parity is always positive. Hence the ground state spin and parity of a given even-even nucleus is given by $J^\pi = 0^+$.  

Now let us consider the nucleus $^{24}\text{Mg}(Z = N = 12)$. The nucleons would be distributed as

$$\pi \rightarrow ((1s_{1/2})^2, (1p_{3/2}, 1p_{1/2})^6) \otimes (1d_{5/2})^4)$$

$$\nu \rightarrow ((1s_{1/2})^2, (1p_{3/2}, 1p_{1/2})^6) \otimes (1d_{5/2})^4)$$

Hence, this nucleus would have $J^\pi = 0^+$ as its ground state. The valence nucleons can be distributed within the $1d_{5/2}, 1d_{3/2}, 2s_{1/2}$ orbitals. Since these orbitals have an even $l$, the resulting states would have a positive parity. The distribution of these nucleons within the sd orbitals, would result in a maximum angular momentum of $J^\pi = 12^+$, which originates from the configuration $(\nu((1d_{5/2})^3, (1d_{3/2})^1))(\pi((1d_{5/2})^3, (1d_{3/2})^1))$. The negative parity states would however originate from the excitation of a single nucleon from sd orbital into the next major oscillator shell i.e. into the pf orbital. Hence, the energy of the lowest (first) negative-parity state encodes the information about the $sd - pf$ shell gap.
For even-odd nucleus or odd-even nucleus the ground state spin and parity is determined from the spin and parity of the unpaired nucleon. For example in case of $^{25}\text{Mg}$ nucleus the unpaired neutron would occupy the $1d_{5/2}$ orbital, which has a even $l$. Hence, it’s ground state spin would be $J^\pi = 5/2^+$. However, for odd-odd nucleus the ground state spin and parity is determined following the Nordheim rule [54]. For such a nucleus, if $j_n$ and $j_p$ correspond to the resultant angular momentum from the partially filled neutron and proton shells, and $l_p$ and $l_n$ are the orbital angular momentum quantum number for partially filled proton and neutron orbitals, then the total spin $J$ of the ground state for the odd-odd nucleus is given by the following rules:

$$J = |j_p - j_n|, \; \text{if } l_p + j_p + l_n + j_n \; \text{is even}$$

$$|j_p - j_n| < J \leq j_p + j_n, \; \text{if } l_p + j_p + l_n + j_n \; \text{is odd}$$

(76)

Example $^{32}\text{P}$ we have, $Z = 15 \& N = 17$, and the last unpaired nucleon occupies the $2s_{1/2}$ and $1d_{3/2}$ orbitals. Hence $l_p = 0 \& l_n = 2$ and $j_p = 1/2 \& j_n = 3/2$, and $l_p + j_p + l_n + j_n = 4$, and therefore the ground state spin would be $1^+$ (parity is positive as the orbitals under consideration are have even $l$). Thus this model adequately describes the level properties of nuclei in the immediate vicinity of magic numbers.

4.4 Large Basis Shell Model Calculations

With the recent advances in computational power, it is now feasible to undertake the spherical shell model calculations for nuclei with a few valance nucleons outside the magic core. The aim of such a computationally intense calculations is to arrive at a detailed understanding of the observed level structure including the excited levels. The Shell Model Hamiltonian is written as

$$H = T + V$$

$$= \sum_{i=1}^{A} \left( \frac{p_i^2}{2m} \right) + \sum_{i>k=1}^{A} V_{ik}(r_1, r_2)$$

$$= \sum_{i=1}^{A} \left[ \frac{p_i^2}{2m} + U_i(\vec{r}) \right] + \sum_{i>k=1}^{A} V_{ik}(r_1, r_2) - \sum_{i=1}^{A} U_i(\vec{r})$$

(77)

Where the shell model assumes that most of the effects of the two body interactions $V_{ik}(r_1, r_2)$ can be absorbed into an equivalent one-body central potential $U_i(\vec{r})$, such
that

\[
H \approx H_1(r_1) + H_{12}(r_1, r_2)
\]

\[
H_1(r_1) = \sum_{i=1}^{A} \left[ \frac{p_i^2}{2m} + U_i(\vec{r}) \right]
\]

\[
H_{12}(r_1, r_2) = \sum_{i>k=1}^{A} V_{ik}(r_1, r_2) - \sum_{i=1}^{A} U_i(\vec{r})
\]  

(78)

The calculations that follow assumes that \(H_{12}(r_1, r_2)\) is small enough to be treated as a perturbation. The methodology adopted for such calculations is detailed in Ref.[55], and could be summarized as:

1. Choice of \(H_1(r_1)\), the central potential, which is usually assumed as the sum of a spherical harmonic oscillator potential, a spin-orbit interaction \((\vec{l}.\vec{s})\), and a term proportional to \(l^2\).

2. Choice of inert core and the model space, (the orbits available to the valence nucleons).

3. Calculation of the one-particle eigenstates of \(H_1(r_1)\) in the desired model space.

4. Construction, of the multi-nucleon eigenstates, of \(H\), for a given number of nucleons.

5. Specification of a residual two-body interaction of \(H_{12}(r_1, r_2)\), for the desired model space.

6. Evaluation of the matrix elements of \(H_{12}(r_1, r_2)\) and calculation of the eigenvalues and eigenvectors of this matrix.

For a set of calculations, we specify \(H_1\) and \(H_{12}\), and define the valence orbits that are to be included in the model space for computation. This in turn decides the inert core. Given this information there are a number of computer codes, such as OXBASH [56], NuShellX@MSU [57], Antoine [58] with which it is possible to construct the complete set of basis states, calculate the matrix elements of \(H_{12}\), and then diagonalize \(H_{12}\) in this basis. With the matrix diagonalized, we solve for the eigenvectors and values,

\[
H|\Psi\rangle = E|\Psi\rangle.
\]  

(79)
The eigenvectors can then be used to compute the matrix elements of other operators which yield predictions of strengths for observables, i.e. the electric, $B(EL)$, and magnetic, $B(ML)$, multipole ($L$) transition probabilities and the singular or multiple nucleon transfer spectroscopic factors.

In reality, the exact eigenvectors and eigenvalues of the nucleus can, in principle, be obtained by solving the Schrödinger equation of the exact nuclear (assuming the particles are free to move in all the orbits)

$$H|\Psi_i> = E|\Psi_i>.$$  \hfill (80)

However, such calculations are computationally intensive to the extent that they are impossible. The shell model circumvents this problem by assuming that there is an effective Hamiltonian $H$ appropriate for the truncated model space $|\phi_i>$, a practical subset of the complete nuclear Hilbert space $|\Psi>$, such that

$$H|\phi_i> = E|\phi_i>.$$  \hfill (81)

Hence, all practical shell model calculations which utilize the effective Hamiltonian, $H$ it is imperative to use analogous effective operators $\hat{O}$ on the wave functions of the states, to calculate the value of the any observable i.e $<\Psi_f|O|\Psi_i> \rightarrow <\phi_f|\hat{O}|\phi_i>$. 

Electromagnetic interaction, which is a weak interaction, is well understood with the help of perturbation theory. The transition rates are calculated from the perturbation theory using the Fermi’s golden rule, where the transition probability is given by:

$$W = \frac{2\pi}{\hbar} |\langle \Psi_f| |H_{em}| |\Psi_i\rangle|^2 \rho(E_f)$$  \hfill (82)

Where initial and final states of the wave functions are $\langle \Psi_i|$ and $\langle \Psi_f|$ respectively, $\rho(E_f)$ is the density of final states, the number of states per unit energy interval and $H_{em}$ is the electro-magnetic perturbation. Hence, the transition probability [25, 46] from a state $j_i$ to final state $j_f$ connected by a $\gamma$-ray of energy $E_\gamma$, of multipole order $l$ is given by:

$$W(\lambda l) = \frac{8\pi(l + 1)}{\hbar l((2l + 1)!!)^2} \left(\frac{E_\gamma}{\hbar c}\right)^{(2l+1)} B(\lambda l : J_i \rightarrow J_f)$$  \hfill (83)
Where \( B(\lambda l : J_i \rightarrow J_f) \) is the reduced transition probability, and for an electric and magnetic transition, it can be expressed as

\[
B(El) = \frac{1}{2J_i + 1} |\langle J_f | \hat{Q}^l | J_i \rangle|^2
\]

and

\[
B(Ml) = \frac{1}{2J_i + 1} |\langle J_f | \hat{M}^l | J_i \rangle|^2.
\]

Where \( \hat{Q}^l \) and \( \hat{M}^l \) are the electric and magnetic multipole operators respectively. For transition \(|f⟩ \rightarrow |i⟩\) the reduced transition probability is denoted by \( B(\lambda l \downarrow) \) and for the transition \(|i⟩ \rightarrow |f⟩\) the reduced transition probability is denoted by \( B(\lambda l \uparrow) \) are related as

\[
B(\lambda l \uparrow) = \left( \frac{2J_f + 1}{2J_i + 1} \right) B(\lambda l \downarrow).
\]

The transition probability \( W(\lambda l) \), is expressed as the number of decays per unit time. If \( B(El) \) and \( B(Ml) \) are expressed in units of \( e^2 fm^{2l} \) and \( \mu_N^2 fm^{2l-2} \) respectively, then \( W(\lambda l) \) and \( B(\lambda l) \) are related as

\[
W(\lambda l) = \alpha \hbar c \frac{8\pi(l + 1)}{\hbar l((2l + 1)!!)^2} \left( \frac{E_\gamma}{\hbar c} \right)^{(2l + 1)} B(\lambda l) \text{ in } e^2 fm^{2l}
\]

and

\[
W(\lambda l) = \alpha \hbar c \left( \frac{\hbar c}{2m_p c^2} \right)^2 \frac{8\pi(l + 1)}{\hbar l((2l + 1)!!)^2} \left( \frac{E_\gamma}{\hbar c} \right)^{(2l + 1)} B(\lambda l) \text{ in } \mu_N^2 fm^{2l-2}.
\]

Where \( e^2 = \alpha \hbar c = 1.44 \text{ MeV fm}, \alpha = 1/137 \) is the fine structure constant and \( m_p \) is the mass of proton and \( \mu_N = e\hbar/2m_p c^2 \) is the nuclear magneton. After substituting all the values in above equations the transition probability for the three lowest multipole order is given by [59]

\[
\begin{align*}
W(E1) &= 1.59 \times 10^{15} \ E_\gamma^3 B(E1) \\
W(E2) &= 1.23 \times 10^9 \ E_\gamma^5 B(E2) \\
W(E3) &= 5.71 \times 10^2 \ E_\gamma^7 B(E3) \\
W(M1) &= 1.76 \times 10^{13} \ E_\gamma^3 B(M1) \\
W(M2) &= 1.35 \times 10^7 \ E_\gamma^5 B(M2) \\
W(M3) &= 6.31 \times 10^6 \ E_\gamma^7 B(M3).
\end{align*}
\]
The reduced transition probability can be calculated from the experimentally obtained lifetime ($\tau$), branching ratio ($BR$), mixing ratio ($\delta$), conversion coefficient ($\alpha$) and the gamma energy ($E_\gamma$). The reduced transition probability for electric and magnetic transition can be expressed in units of \(e^2 f m^2\) and \(\mu_N f m^{2l-2}\) respectively and \(E_\gamma\) is in MeV.

\[
B(E1) = \frac{0.693}{E_\gamma^3} \frac{BR}{\tau(1+\alpha)} \frac{\delta^2}{1+\delta^2}, (\tau \text{ is in fs})
\]

\[
B(E2) = \frac{816}{E_\gamma^5} \frac{BR}{\tau(1+\alpha)} \frac{\delta^2}{1+\delta^2}, (\tau \text{ is in ps})
\]

\[
B(E3) = \frac{1760}{E_\gamma^7} \frac{BR}{\tau(1+\alpha)} \frac{\delta^2}{1+\delta^2}, (\tau \text{ is in \mu s})
\]

\[
B(M1) = \frac{56.8}{E_\gamma^3} \frac{BR}{\tau(1+\alpha)} \frac{1}{1+\delta^2}, (\tau \text{ is in fs})
\]

\[
B(M2) = \frac{74.1}{E_\gamma^5} \frac{BR}{\tau(1+\alpha)} \frac{1}{1+\delta^2}, (\tau \text{ is in ns})
\]

\[
B(M3) = \frac{0.158}{E_\gamma^7} \frac{BR}{\tau(1+\alpha)} \frac{1}{1+\delta^2}, (\tau \text{ is in s})
\]

As mentioned earlier, the shell model wave functions are then used to calculate the overlap of the initial and final states, where operated by either electric and/or magnetic operator [60]. The \(M1\) operator is

\[
(M1)^{\text{op}} = \sqrt{\frac{3}{4\pi}} \sum_{i,\tau_z} \left[ g_{\tau_z} \bar{s}_{i,\tau_z} + g_{\tau_z} \bar{l}_{i,\tau_z} + g_{\tau_z} \sqrt{8\pi} \left[ Y^2(\hat{r}_{i,\tau_z}) \otimes \bar{s}_{i,\tau_z} \right] \right] \mu_N \tag{91}
\]

Now, the \(M1\) transition matrix elements, which is an observable is given by

\[
M(M1) = \langle \Psi_f \| (M1)^{\text{op}} \| \Psi_i \rangle \tag{92}
\]

For an electric transition, we have

\[
(E2)^{\text{op}} = \sum_{i,\tau_z} e_{\tau_z} \bar{r}_{i,\tau_z}^2 Y^2(\hat{r}_{i,\tau_z}) e \tag{93}
\]

Now, the \(E2\) transition matrix elements, which is an observable is given by

\[
M_p = \langle \Psi_f \| (E2)^{\text{op}} \| \Psi_i \rangle \tag{94}
\]

The transition matrix elements, in turn are related to the corresponding transition probabilities [60] as

\[
B(M1) = \frac{[M(M1)]^2}{(2J_i + 1)}
\]

\[
B(E2) = \frac{M_p^2}{(2J_i + 1)} \tag{95}
\]
As mentioned earlier, the use of a restricted model space, demands, that we use effective values for the charges, g-factors etc. These would be model dependent hence, are not universal, and may have to be obtained from a consistent fit to the experimental observables, for a given model space.

In the present thesis the shell model calculations have been performed using the code NuShellX@MSU [57]. For example, if we wish to predict the level structure of the positive parity states in $^{24}$Mg, then from a purely practitioners hands-on-approach to a shell model calculation, we would

1. Identify the inert core, which is this case would be $^{16}$O. Hence we have total 8 valence nucleons.

2. Identify the valence orbits in which the nucleons outside the core are free to move i.e choose the model space. The 8 valence nucleons would be allowed to occupy the $2s_{1/2}$, $1d_{3/2}$, $1d_{5/2}$ orbitals, which have $l = 0$, 2, and hence would generate positive parity sequences only. The negative parity sequences, would require the excitation of a single nucleon from the $sd$ orbitals into the $fp$ orbitals, and the model space would have to be accordingly expanded to include these orbits, which is schematically represented in Fig. 26.

3. Identify the two-body matrix elements for this model space. Usually we have a one-to-one correspondence between the model space and the two-body matrix elements. Hence the choice of the two is kind of inter-leaved.

4. Run the shell model code with the above inputs to obtain the energy eigenvalues and use the wave functions (within the same model space and interaction) to calculate the overlaps, when operated by an effective operator to deduce an observable.

For the above example, we shall use the $sd$ model space, and the recently developed interaction codenamed $USDA$ [61], in the code NuShellX@MSU [57]. Since, unrestricted shell model calculations are feasible for this model space, no truncation scheme was adopted. The calculations have predicted the ground state energy as -87.13 MeV, which is in excellent agreement with the experimental value of -87.10 MeV. The
FIG. 27: Comparison of experimentally observed and shell model calculated energy levels in \(^{24}\text{Mg}\) nucleus by using \textit{USDA} and \textit{sdpfmw} interactions (only for positive parity states).

Calculated excitation energies for the low-lying positive parity states (up to \(J^\pi = 6^+\)) are compared with the reported [23] value in Fig. 27. The same exercise was extended to a larger model space which now encompasses the \(sd - pf\) shells. The corresponding interaction chosen, was codenamed \textit{sdpfmw} [9]. Unrestricted calculations within such a large model space was not computationally feasible. Hence, the model space was internally truncated by considering no nucleon excitation into the \(fp\) orbits \(i.e\), we used the \(0\hbar\omega\) restriction (0 refers to the number of nucleons excited across one major shell).

The predicted ground state energy is -87.08 MeV, which is in excellent agreement with the experimental observation. This is indicative of an appropriate truncation scheme used. The predicted excitation energies are also compared in Fig. 27 with the reported values.

The detailed shell model calculations for both the positive and negative parity states for isotopes of \(\text{Mg}\) and \(\text{Si}\) are detailed in the subsequent chapters. The model calculations helps us elucidate the underlying microscopic configurations for the established levels.