4 The Cashless Economy

4.1 Introduction

In this chapter we investigate the existence of a monetary equilibrium in a model which has no role for money as a medium of exchange. Rather, money enters the economy as a unit of account in terms of which prices are quoted and debts are contracted. Moreover, nominal debt is also used as a store of wealth across periods. We show that given bounds on agents’ price expectations, there exists in this model an equilibrium with a strictly positive value of money.

We label our economy as ‘cashless’ as it has no explicit modelling of the means of exchange function of money. However, the model is certainly not moneyless. Money exists as a unit of account. The fact that payoff of assets are denoted in nominal terms is an essential aspect of this model and the influence of changes in prices and price expectations on the returns of these assets is indeed the central problem studied in this chapter.

Our model follows the temporary equilibrium framework of Grandmont (1977) and Grandmont and Younes (1972). However, while Grandmont considers models with a exogenously given money stock, we extend the analy-
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sis to a situation where there is no special monetary asset whose supply is given from outside but rather agent's prior nominal commitments to other agents which determines the equilibrium price level.

4.2 The model

We consider a simple competitive pure-exchange economy. Time is discrete and there are two dates 0 and 1. There is no uncertainty. There is a single consumption good in the economy which cannot be stored across periods. There are $H$ households. Household $h$ has endowments $\omega^h_0 > 0$ and $\omega^h_1 > 0$ in the two periods respectively.

Our assumption of a nominal unit of account and nominal assets would not be meaningful if our assumption of there being only a single consumption good were to be taken literally. In that case it would only be natural to write contracts in terms of that good. The use of money prices only makes sense in a multigood world where in their absence traders would have to quote $n(n - 1)/2$ price ratios between $n$ goods.

However, our assumption of there being only a single consumption good is intended to be a simplification that allows us to focus our attention on the linkages between different periods and intertemporal terms of trade without needing to also deal with relative prices within a period. The results of this chapter would be preserved in their essentials if the single consumption good were to be replaced by a vector of goods in each period.

Spot markets in the consumption goods open on both days. Prices in
these markets are quoted in terms of a unit of account which we refer to as ‘money’. There are no future markets for commodities.

Households in the economy can make and receive commitments denoted in money terms. At the beginning of period 0, household \( h \) has a commitment, inherited from the past, to pay \( D_h \) units of money. \( D_h \) can be negative, in which case that particular household is entitled to receive money in the beginning of period 0.

In period 0 households can contract to borrow an amount \( b \) in money terms at the nominal interest rate \( i \). Repayments of these loans are made in period 1.

We take the nominal rate of interest to be exogenously determined by monetary policy as discussed in Chapter 1. Note that the absence of a means of exchange role of money does not take away the ability of the central bank to fix the nominal interest rate. It can always fix a nominal interest rate by standing ready to lend and borrow in nominal terms at a given rate of interest, as real central banks do.\(^1\)

The household’s problem in period 0 is to decide on an optimum consumption plan \((x_0, x_1)\) for the two periods. Each household’s preference ordering over consumption plans is given by a continuous, monotonic and strictly quasi-concave utility function \( u^h(x_0, x_1) \). The budget constraint faced

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\(^1\)This issue is also discussed in (Woodford, 2003, Section 1.3).
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by each household is:

\[ px_0 \leq p\omega_0 + b - D \]
\[ p_1^e x_1 + (1 + i)b \leq p_1^e \omega_1 \]
\[ x_0 \geq 0, \quad x_1 \geq 0 \]

Where \( p \) is the price of the consumption good in the spot market on day 0
and \( p_1^e \) is the price expected to prevail in the spot market on day 1.

Since our agent live only on the two date 0 and 1 they do not borrow in
period 1. The model could be extended to infinitely lived agents who could
then borrow in each period. In that case the budget constraint above would
have to be augmented with a no-Ponzi-game condition to keep borrowing
bounded.

We can eliminate \( b \) from the day 0 and day 1 constraints above to obtain
the single constraint:

\[ px_0 + \frac{p_1^e}{1 + i} x_1 \leq p\omega_0 + \frac{p_1^e}{1 + i} \omega_1 - D \quad x_0 \geq 0, \quad x_1 \geq 0 \quad (4.1) \]

4.3 Expectations

The expectation \( p^e \) of the spot-price on day-1 will in general depend on all
information available to agents on day-0. Also, different agents will, in gen-
eral, form different expectations even on the basis of the same information.
While there is a considerable literature which has studied the existence of
monetary equilibrium under the assumption of rational expectations or perfect foresight, we do not feel the use of these assumptions justified in the absence of an argument showing the convergence over time of learning behaviour to rational expectations. In our model, where agents form expectations only once, on date 0, it is more natural to assume the expectation-formation mechanism to be given exogenously.

We therefore assume that \( p_1^e \) is some function \( \zeta(p) \) of current prices. While expectations may also depend on other information including the past history of prices, these factors are fixed in our model and therefore the dependence of expectations on them does not need to be explicitly modelled.

With these assumptions, the household's budget constraint now becomes:

\[
px_0 + \frac{\zeta(p)}{1+i} x_1 \leq p\omega_0 + \frac{\zeta(p)}{1+i} \omega_1 - D \quad x_0 \geq 0, \quad x_1 \geq 0
\]

For convenience we define an auxiliary function \( \eta(p) \) as:

**Definition 1.**

\[
\eta^h(p) = \frac{\zeta^h(p)}{1+i}
\]

This allows us to simplify the budget constraint to:

\[
px_0 + \eta(p)x_1 \leq p\omega_0 + \eta(p)\omega_1 - D \quad x_0 \geq 0, \quad x_1 \geq 0 \quad (4.2)
\]

### 4.3.1 Bankruptcy

In (4.2), if the day-0 price is such that \( p\omega_0 + \eta(p)\omega_1 - D < 0 \) then there are no consumption plans satisfying the budget constraint. We refer to this
outcome as 'bankruptcy'. In effect, prices are so low that the earnings of the agent from selling its endowments is insufficient even to meet its prior nominal obligations. While we have not explicitly modeled the origin of $D$, one way to understand it would be to see it as $(1 + i)b_{-1}$ where $b_{-1}$ is the amount borrowed by the agent in the period before period 0. In our model, agents never willfully plan to default on their borrowings. Therefore bankruptcy occurs only when prices are lower than those expected by the agents when formulating their plans in the previous period. Thus the occurrence of bankruptcy is a direct result of our not assuming any form of perfect foresight.

We handle bankruptcy in our model with two assumptions. First, agents who go bankrupt are assumed to consume nothing. Formally, we model this by setting their money income to zero. Second, the payments received by individuals with a negative $D^h$ remains unchanged even if some other agents go bankrupt. We may motivate this by assuming that there is a perfect debt guarantee and choosing to ignore in the current setup the informational problems associated with implementing such a guarantee.

With these assumptions about bankruptcy in hand, we define an agent’s expected money income as:

**Definition 2.** The expected money income of agent $h$ is given by the function:

$$M^h(p_0) = \max(p_0\omega^h_0 + \eta^h(p_0)\omega^h_1 - D^h, 0)$$
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Now, an household's budget constraint can be written as:

\[ px_0 + \eta(p)x_1 \leq M(p) \quad x_0 \geq 0, \quad x_1 \geq 0 \] (4.3)

4.4 Equilibrium

Let the household's demand for consumption goods on day 0 be given by the function \( d^h(p) \) (we show in this appendix that this function is well-defined). Then that household's excess demand for is given by \( \zeta^h(p) = d^h(p) - \omega^h_0 \). The aggregate excess demand is defined by,

\[ \zeta(p) = \sum_h \zeta^h(p) \]

A price \( \hat{p} > 0 \) is a temporary equilibrium for period 0 if it is the case that,

\[ \zeta(\hat{p}) = 0 \]

We investigate below the sufficient conditions for the existence of such a temporary equilibrium.

4.4.1 Continuity

We begin by assuming that expected future prices depend continuously on current prices.

Assumption 1. \( \eta(p) \) is a continuous function of \( p \).
Under this assumption we can show that,

**Lemma 1.** For all \( h \xi^h(p) \) is a continuous function of \( p \).

**Proof.** Deferred to section 4.7.

Since the aggregate excess demand function is a sum of household excess demand functions, it follows that the aggregate excess demand too is continuous.

**Proposition 4.** \( \zeta(p) \) is a continuous function of \( p \).

### 4.4.2 Boundary conditions

Since \( \zeta(p) \) is a continuous function of \( p \), we can obtain an equilibrium using the intermediate-value theorem provided we can find one price where excess demand is negative and another price where it is positive.

Naively, we would expect excess demand to go up as prices go to zero and excess demand to go down as prices go to infinity. However, the assumptions that we have made so far are not sufficient to establish this behaviour.

As is usual, we can divide the effect of a price change of demand into an income and a substitution effect. However, the existence of price expectations and nominal commitments requires some adjustments to this decomposition.

First, in addition to the usual income effect, we have an additional effect arising from changes in the real value of money commitments. This is the real balance effect. A rise in current prices raises the real income of creditors and reduces the real income of debtors.
Relative prices are given by,

\[
\frac{p_e}{p} = \frac{\zeta(p)}{\tilde{\zeta}(p)}
\]

Hence the substitution effect of a price change depends on the responsiveness of price expectations to current prices. Taking logarithms in the above equation and differentiating, we have,

\[
\frac{\dot{p}_e}{p} - \frac{\dot{p}}{p} = \frac{1}{p} \left( \frac{p\zeta'(p)}{\tilde{\zeta}(p)} - 1 \right)
\]

Thus the direction of the substitution effect depends on the elasticity of expectations, viz.

\[
e = \frac{p\zeta'(p)}{\tilde{\zeta}(p)}
\]

When \( e = 1 \) there is no substitution effect at all since expected prices move in the same proportion to current prices. When \( e > 1 \), the substitution effect of a current price increase actually leads to an increase in current consumption since the price of future consumption rises by an even greater proportion. It is only when \( e < 1 \) that the substitution effect works in the usual direction.

The presence of elastic expectations can cause our model to not have a solution since the reverse substitution effect can result in demand not becoming positive when price goes to 0 or not becoming negative when price goes to infinity. Thus any sufficient condition for existence of equilibrium in our model will require restrictions on admissible expectations functions.
The assumption we choose is that expectations are bounded both above and away from zero below. More specifically,

**Assumption 2.** For each household \( h \), there are numbers \( L^h > 0 \) and \( U^h \) such that \( L^h (1 + i) \leq \zeta(p) \leq U^h (1 + i) \) for all \( p \), which implies that \( L^h \leq \eta(p) \leq U^h \) for all \( p \).

An expectations function which satisfies this assumption cannot be elastic for all prices. One economic interpretation of this assumption is that agents have a notion of 'normal' prices and do not expect prices to go beyond these 'normal' limits. An extreme case of an expectation function which satisfies this assumption is zero-elastic expectations, i.e. an expectation which does not change at all in response to current prices. It is this boundedness of expectations which essentially guarantees the existence of monetary equilibrium in our model.

While our assumption takes care of the substitution effect, the income effect for debtors can still cause problems. As prices fall the real income of debtors fall, and in the limit they become bankrupt and cannot consume anything. Therefore an economy consisting only of debtors may once again fail to have a non-negative demand at any prices as it may happen that before prices can fall enough to make demand positive, all agents have become bankrupt. To rule out this pathological case, we assume that there is at least one creditor household.

**Assumption 3.** There is at least one household \( h \) with \( D^h < 0 \)

Taken together, the assumptions above are sufficient to establish the exis-
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tence of a monetary equilibrium in our model.

**Theorem 1.** There exists a \( p > 0 \) such that \( \zeta p = 0 \)

**Proof.** Deferred to section 4.7.

Thus we have been able to show the existence of a monetary equilibrium in a model without an explicit medium of exchange role of money. The assumption of pre-existing nominal obligations and the inelasticity of nominal expectations are both crucial for our result. The first provides a link between the past and the present, thus ruling out the imaginary monetary reforms so popular with expositors of the Quantity Theory which at one stroke multiply all nominal quantities by the same proportion. In real economies such scalar multiplication do not occur except in the rare situation of an actual monetary reform precisely because at any point of time there are outstanding contracts in the economy which are fixed in nominal terms.

The second crucial assumption for us is that of inelastic nominal expectations which give a rigidity to the link between the past and the present. In a world where prices have been stable, agents will develop a notion of the normal price level and would not give credence to the possibility of large deviations from this normal level. Alternatively and equivalently, we can say that in a world with a stable price regime agents would account for the entire recent history of prices would enter into the determination of agents' expectations of the future and therefore the price in the latest period would have a small weight in the determination of expectations.

The use of the historical stability of prices to guarantee the determinate-
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ness of present prices might seem to be begging the question. But this accords well with the experience with hyperinflations: they are hard to start but once started they are hard to stop.

If we ignore the inertia imposed by past beliefs and past contracts our model would fail to capture the stability that real monetary systems usually possess. This is precisely what happens with the overlapping generations model of money which also does not assume a medium of exchange role of money but in which the usual expectational assumption is that of perfect foresight.²

4.5 Overlapping generations models of money with perfect foresight

The overlapping generations model of general equilibrium was first presented in Samuelson (1958) as a theory of interest rate determination and further developed in Diamond (1965) and Cass and Yaari (1966) among others.³

The overlapping generations model is of considerable interest theoretically since it is possible to have non-Pareto-optimal competitive equilibrium in this model even though its assumptions seem to deviate minimally from

²We should note that it is possible to develop overlapping generations models with learning rather than perfect foresight (Evans and Honkapohja, 2001). However, perfect foresight is the expectational assumption that has predominantly been used in the literature on the use of overlapping generations models in monetary economics.

³Geanakoplos and Polemarchakis (1991) and Weil (2008) provide surveys of the overlapping generations model.
the standard Arrow-Debreu model of general equilibrium. For the Arrow-Debreu model, the so-called First Fundamental Theorem of Welfare Economics establishes that all competitive equilibria are Pareto optimal. This result also holds for the infinite-horizon Ramsey-Cass-Koopmans model. The failure for this result to hold for overlapping generations model is therefore surprising. The reason for this failure was identified by Shell (1971) as the existence of a double infinity of agents and commodities. The same double-infinity allows intrinsically valueless money to have a positive value in perfect-foresight equilibria of this model even if money serves no transactions purpose.

4.5.1 Monetary equilibria and Pareto optimality

Samuelson (1958) showed that the introduction of intrinsically valueless money could lead to a Pareto-optimal monetary equilibrium in an overlapping generations model whose non-monetary equilibrium was not Pareto optimal. The logic for this can be illustrated in a simple overlapping generations model. Let's assume that there is no uncertainty and no production. An agent born in period \( i \) lives for two periods \( i \) and \( i + 1 \). The agents are said to be "young" and "old" respectively in the two periods. In period \( i \) the number of agents born is \((1 + n)^i\), so that population grows at the rate \( n \). There is a single non-storable consumption good and each agent is endowed with \( w_y \) and \( w_o \) units of it in the first and second period of their

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4 The discussion in this section is based on Wallace (1980) which contains detailed proofs of all the propositions advanced here.

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life respectively. Additionally in the initial period (let's say period 0) there is a generation of "old" agents endowed with $\omega_o$ units of the consumption good. Each agent maximises an utility function defined over their consumption in the two periods of their lives, except for the initial old who maximise their consumption in the initial period. Assume that the utility function is the same for every generation of agents.

The only equilibrium in this model is the autarkic one. Since there is only one consumption good trade must involve offers of the good in one period in return for goods in another period. Since each generation consists of identical agents such trades cannot occur within a generation in equilibrium. Since the young and the old meet only for a single period such trade cannot happen between different generations either. Autarky is the only possibility remaining.

Whether the autarkic equilibrium thus obtained is Pareto optimal or not depends on the marginal rates of substitution between present and future goods that agents face at their endowment point. Because population is growing at the rate $n$, if each generation gives up 1 unit when young, it will receive $(1 + n)$ units of the good when old. If the MRS at the autarkic equilibrium is less than this then a benevolent planner can bring about a Pareto improvement by asking the young to give up some consumption on the margin. This depends on there being a infinite time horizon. If the time horizon were finite then the last period's young would have to give up something while getting nothing in return. For the same reason a Pareto improvement is not possible if the MRS is greater than $1 + n$ because the
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old of the first period would have to give up something without getting anything in return.

Suppose that the autarkic equilibrium is not Pareto-optimal, so that agents are willing to accept at the margin less than \((1 + n)\) unit of the good in their old age per unit of the good sacrificed in their youth. Samuelson (1958) showed that Pareto optimality could be restored in this model by endowing the initial old with some number of intrinsically valueless tokens—"money". In an equilibrium where these tokens have a positive value the old can use them to buy consumption goods from the young, who in turn use them to buy consumption goods from the young of the next generation. The simplest such equilibria is one where the value of this money rises at the rate \(n\), the rate of population growth, so that the per-capita real value of money remains constant. Because the intertemporal terms of trade offered by holding money is the same as the terms at which society can transfer wealth steadily from the young to the old (both being equal to \((1 + n)\)) this monetary equilibrium is a Pareto optimum.

Conversely, if the initial autarkic equilibrium had been Pareto-optimal then the economy would not support a monetary equilibrium. This can be seen from the fact that when money is valued each generation consumes less when young and more when old compared to autarky. Assuming the consumption in both periods is a normal good this implies that the MRS between consuming tomorrow and consuming today is greater than \((1 + n)\). In equilibrium this MRS must equal the rate of appreciation of the value of money. But an increase in the value of money at a rate higher than the rate
of growth of population implies an increase in per-capita money holding over time. Ultimately, therefore, the real value of the money holding of the old, which equals their demand for goods, will exceed the total good endowment of the economy—a situation inconsistent with equilibrium.

So far our token money has been the only way of transferring purchasing power over time. We can extend our model to introduce other assets. Diamond (1965) initiated the study of overlapping generations models with capital. For purposes of illustration, we introduce production in the form of a storage technology for the consumption good. Let us assume that we can store the consumption good so that storing 1 unit of the consumption good in one period yields $1 + x$ units of the consumption good in the next period. Now at equilibrium the MRS between consumption today and tomorrow has to be $1 + x$ or more. In the absence of money the equilibrium with storage will be Pareto-optimal only if $x$ is greater than or equal to $n$, using the same reasoning as in the case of autarky. If $x < n$ we have inefficiency in the model without money. Once again Pareto-optimality can be restored by introducing money, with the same monetary equilibrium as above. Holding money gives a return of $n$ and storage is not used.

We can also extend our initial model to incorporate monetary policy in the form of exogenous changes in money supply. Say money supply each period grows at the rate $z - 1$. Now money can give a return upto $n/z$ without the real per-capita money holdings blowing up. This breaks the tight linkage between Pareto optimality and existence of monetary equilibria. For example, suppose $z > 1$. Then we will never be able to achieve Pareto opti-
mal equilibria with money since in such equilibria the rate of return has to be \( n \). One the other hand assume that \( z < 1 \). In that case we can have monetary equilibria even when the original non-monetary equilibrium is Pareto optimal.

### 4.5.2 Critique

In Samuelson's original analysis, 'money' was introduced as an abstract means of realizing Pareto-improving transfers along with other means such as social security. Samuelson himself did not take up the question of the extent to which the characterisation of money within the overlapping generations model captures the essential features money in modern economies. Later authors, for example Sargent (1987) or Lucas (1972), have used the model as one possible framework for studying monetary questions. However, a much stronger claim has been made in favour of overlapping generations model by Wallace (1980) who has argued that:

"... the friction in Samuelson's 1958 consumption load model, overlapping generations, gives rise to the best available model of fiat money" (emphasis in the original)

This strong claim comes up against the fact that both the existence of monetary equilibria and the relation between money and economic welfare are extremely tenous in overlapping generations model. First, as we have seen above not all overlapping generations economy have monetary equilibria. In our simple case money is valued only when the non-monetary
equilibrium in the model is not Pareto-optimal. Second, even if a particular economy has a monetary equilibrium, it would also always have an equilibrium where money is not valued. Since money is held purely as a store of value, it is not demanded if its price is zero and therefore equilibrium conditions with free disposal and a zero price of money can be satisfied. Third, the value of money in a monetary equilibrium in these models depends on the expectations that agents hold about the value that money would have in the future. This leads to a multiplicity of equilibria, including phenomena such as endogeneous cycles (Grandmont, 1985; Cass et al., 1980), sunspots (Azariadis, 1981; Cass and Shell, 1983) and hyperinflationary equilibria where money asymptotically becomes valueless (Wallace, 1980).

The one-to-one correspondence between the lack of Pareto optimality in non-monetary equilibria and the existence of monetary equilibria in our example above might seem to capture the welfare-improving nature of money. However, this correspondence is not a general feature of overlapping generations models. Cass et al. (1980) have shown that it breaks down once heterogeneity between agents is allowed. With heterogeneity it is possible to construct economies in which simultaneously have monetary and non-monetary equilibria which are Pareto equilibria as well as economies in which there are monetary and non-monetary equilibria but no equilibria is Pareto optimal.

In addition to these difficulties intrinsic to the model, the overlapping generations framework has also been challenged for its lack of correspondence to the features of real-world monetary institutions. For example, To-
bin (1980) criticises the overlapping generations model on the grounds that monetary equilibria exist in these models only for a subset of economies whereas real money has existed and has been found useful in a diverse set of social conditions. He is also critical of the dependence of the acceptance of money in the present generation on the belief that money will be accepted by the next generation which in turn is posited on the belief that the next generation will believe that money will be accepted by the generation. Sheinkman (1980) and McCallum (1982) point to the absence of a means of exchange role in overlapping generations models and see this absence as the source of the teneousness of monetary equilibria that Wallace (1980) attributes to the essential characteristics of money.

Comparison to the temporary equilibrium model developed earlier in this chapter shows that many of these problems associated with the overlapping generations model are due to assumption of perfect foresight and its shortcomings should not be generalised to all models which ignore the means of exchange role of money.

4.6 Conclusion

The price equilibrium established in this chapter has two interesting properties. One, we showed the existence of equilibrium for an exogenously given nominal interest rate. In general, the equilibrium prices and allocations will be different for different nominal interest rates. Thus, interest rates can be used as an instrument of redistributive policy in this economy. Secondly, the
classical dichotomy between monetary and real variables does not exist in this model since multiplying prices and nominal commitments by some factor does not necessarily multiply price expectations by the same factor and thus leads to a change in budget and equilibrium sets.

4.7 Proofs

4.7.1 The budget correspondence

Definition 3. For agent $h$, the budget correspondence is a set-valued function of $p$, given, for $p \geq 0$ by:

$$B^h(p) = \{(x_0, x_1) \in \mathbb{R}^2_+ | px_0 + \eta(p)x_1 \leq M^h(p)\}$$

We now establish some properties of this correspondence. In what follows, we omit the subscript $h$ for the household when this is not likely to cause any confusion.

Lemma 2. $B(p)$ is compact for all $p \geq 0$

Proof. Since $x_1 \geq 0$,

$$x_0 \leq M(p)/p$$

Similarly, since $x_0 \geq 0$,

$$x_1 \leq M(p)/\eta(p)$$

Thus $B(p)$ is bounded.
We now show that $B(p)$ is also closed by showing that every convergent sequence in $B(p)$ converges to a point of $B(p)$. Let $(x_0^n, x_1^n)$ be a sequence in $B(p)$ converging to $(x_0, y_0)$. Since $(x_0^n, x_1^n) \in B(p)$,

$$px_0^n + \eta(p)x_1^n \leq M(p)$$

Taking limits we have,

$$px_0 + \eta(p)x_1 \leq M(p)$$

Hence $(x_0, y_0)$ also belongs to $B(p)$.

Being a closed and bounded subset of an Euclidean space, $B(p)$ is compact.

Lemma 3. If $p^n$ is a sequence of prices that converge to $p$ and $(x_0^n, x_1^n) \in B(p^n)$ is a sequence of consumption plans that converge to $(x_0, x_1)$ then $(x_0, x_1) \in B(p)$.

Proof. From the definition of $B(p^n)$ we have,

$$p^n x_0^n + \eta(p^n)x_1^n \leq M(p^n)$$

Taking limits and making use of the continuity of $M(\cdot)$ and $\eta(\cdot)$, we have,

$$px_0 + \eta(p)x_1 \leq M(p)$$
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From which it follows that

$$(x_0, x_1) \in B(p)$$

Lemma 4. If $p^n$ is a sequence of prices that converge to $p$ and, and $(x_0, y_0) \in B(p)$ then there is a sequence $(x_0^n, y_0^n)$ such that $\lim(x_0^n, x_1^n) = (x_0, x_1)$ and there is a number $N$ such that for $n > N$, $(x_0^n, x_1^n) \in B(p_n)$.

Proof. We denote the expected cost of a particular consumption plan $(x, y)$ at current price $q > 0$ by the function:

$$f(q, x, y) = qx + \eta(q)y$$

The function $f(\cdot)$ has the following properties,

1. $f(q, \lambda x, \lambda y) = \lambda f(q, x, y)$.

2. $f(q, x, y)$ is continuous: since $\eta(\cdot)$ is.

3. $f(q, x, y) \geq 0$.

4. $f(q, x, y) = 0$ if and only if $x = y = 0$.

Using $f(\cdot)$ we can write the consumer's budget set as,

$$B(q) = \{(x, y) \in \mathbb{R}_+^2 | f(q, x, y) - M(q) \leq 0\}$$

Now we construct the sequence required by the lemma.
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If \( x_0 = y_0 = 0 \) then we let \( N = 1 \) and \((x_0^n, y_0^n) = (0, 0)\) for all \( n \). Since \((0, 0) \in B(p^n)\) for all \( n \) and \( \lim(0, 0) = (0, 0) = (x_0, y_0)\), so this sequence satisfies all the requirements.

Otherwise, let,

\[
\lambda_n = \frac{f(p, x_0, y_0) - M(p) + M(p^n)}{f(p^n, x_0, y_0)}
\]

and,

\[
(x_0^n, x_1^n) = (\lambda_n x_0, \lambda_n y_0)
\]

Since \( M(\cdot) \) is continuous, \( p^n \to p \) and \( f(p, x, y) > 0 \), we can find a \( N \) such that for \( n > N \),

\[
M(p) - M(p^n) < f(p, x, y)
\]

From (4.4) this implies that for \( n > N \), \( \lambda_n > 0 \).

We claim that \( N \) and \((x_0^n, x_1^n)\) satisfy the requirements of the lemma.

For \( n > N \),

\[
f(p^n, x_0^n, x_1^n) - M(p^n) = f(p^n, \lambda_n x_0, \lambda_n x_1) - M(p^n)
\]

\[
= \lambda_n f(p^n, x_0, x_1) - M(p^n)
\]

\[
= f(p, x_0, x_1) - M(p) + M(p_n) - M(p^n)
\]

\[
= f(p, x_0, x_1) - M(p)
\]

\[
\leq 0 \quad \text{since} \ (x_0, x_1) \in B(p)
\]

Thus \((x_0^n, x_1^n) \in B(p^n)\) for \( n > N \).

Also, \( \lim(x_0^n, x_1^n) = (x_0, x_1) \) since \( \lim \lambda_n = 1 \) by the continuity of \( f(\cdot) \) and
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\( M(\cdot). \)

4.7.2 Individual optimisation

Definition 4. If the solution to the household’s maximisation problem

\[
\max_{(x_0, x_1) \in B^h(p)} u^h(x_0, x_1)
\]

is \((x_0, x_1)\) then we define \(d^h(p) = (x_0, x_1)\) and \(d_0^h(p) = x_0\) and \(d_1^h(p) = x_1\).

Since \(B(p)\) is compact and \(u_h(\cdot)\) is continuous, this maximisation problem has a solution. Since \(u_h(\cdot, \cdot)\) is strictly quasi-concave, this solution is unique. Hence the functions \(d^h(\cdot), d_0^h(\cdot)\) and \(d_1^h(\cdot)\) are well defined.

Lemma 5. If \(p_n\) is a sequence such that \(\lim p_n = p\) and the optimal consumption bundles \((x_0^n, x_1^n) = d(p_n)\) converge to \((x_0, x_1)\) then \(d(p) = (x_0, x_1)\).

Proof. By lemma 3, \((x_0, x_1)\) belongs to \(B(p)\).

Let \((z_0, z_1)\) be an arbitrary bundle in \(B(p)\). By lemma 4, there exists \(N\) and a sequence \((z_0^n, z_1^n)\) such that \((z_0^n, z_1^n) \in B(p_n)\) for \(n > N\) and \(\lim(z_0^n, z_1^n) = (z_0, z_1)\).

Since \((x_0^n, x_1^n) = d(p_n)\) is the optimal bundle in \(B(p_n)\) and it is the case that \((z_0^n, z_1^n) \in B(p_n)\) for \(n > N\), it follows that for \(n > N\),

\[
u(x_0^n, x_1^n) \geq u(z_0^n, z_1^n)\]
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Taking limits on both sides and using the continuity of $u(\cdot)$, we have

$$u(x_0, x_1) \geq u(z_0, z_1)$$

Since $(z_0, z_1)$ was chosen to be an arbitrary point in $B(p)$ is follows that $(x_0, x_1)$ is the optimal point in $B(p)$, i.e. $d(p) = (x_0, x_1)$. \hfill $\Box$

Lemma 6. The demand function $d(p)$ is continuous.

Proof. Let $p^n > 0$ be a sequence of prices converging to $p$. We shall prove that $\lim d(p^n)$ exists and is equal to $d(p)$.

We proceed by contradiction. Suppose $d(p^{n_k})$ does not converge to $d(p)$. Then, there must be a $\varepsilon > 0$, and a subsequence $p^{n_k}$ of $p^n$ such that,

$$|d(p^{n_k}) - d(p)| > \varepsilon \quad (4.5)$$

We choose an arbitrary $0 < \delta < \min(p, \eta(p))$. Since $\lim p^n = p$ and $\eta(\cdot)$ and $M(\cdot)$ are continuous, there must exist a $N$ such that for $n > N$,

$$p_n \geq p - \delta$$

$$\eta(p_n) \geq \eta(p) - \delta$$

$$M(p_n) \leq M(p) + \delta$$

Since $x_1^n \geq 0$, $x_0^n \leq (M(p) + \delta)/(p - \delta)$ for $n > N$. Similarly, since $x_0^n \geq 0$, $x_1^n \leq (M(p) + \delta)/(\eta(p) - \delta)$ for $n > N$. This means that $d(p^n)$ and, hence $d(p^{n_k})$, must be bounded. Being a bounded subset of an Euclidean space it
must have a convergent subsequence. Let us call this subsequence \( d(p^{n_i}) \). Since, \( p^{n_i} \) is a subsequence of \( p^n \) and hence converges to \( p \), \( d(p^{n_i}) \) satisfies the conditions of lemma 3 and hence must converge to \( d(p) \). However, as \( d(p^{n_i}) \) is a subsequence of \( d(p^{n_k}) \), this contradicts (4.5)

\( \square \)

**Lemma 7.** The excess demand function \( d_0(p) \) is continuous.

**Proof.** Since \( d_0(p) \) is just the first component of the continuous function \( d(p) \), it must be continuous. The excess demand function is given by,

\[
\zeta^h(p) = d_0^h(p) - \omega_0^h
\]

and hence it must be continuous too. \( \square \)

### 4.8 Aggregate demand

**Lemma 8.** As \( p \to 0 \), \( \zeta(p) \to \infty \).

**Proof.** We know that there is at least one agent with \( D^h < 0 \). We begin by analysing the aggregate demand of such agents.

Suppose \( p^n \) is a sequence with \( \lim p^n = 0 \). Since \( 0 < L \leq \eta(p^n) < U \) by assumption 2, \( \eta(p^n) \) is a bounded sequence and must have a convergent subsequence. We assume that such a convergent subsequence has been chosen so that,

\[
\lim \eta(p_n) = p_1 \quad \text{(say)}
\]

where

\[
0 < L \leq p_1 \leq U
\]
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Let \((x_0^n, x_1^n) = d(p_0^n)\) be the corresponding optimal bundles. We first show that the sequence \((x_0^n, x_1^n)\) must be unbounded.

If \((x_0^n, x_1^n) = d(p^n)\) is bounded then it must have a convergent subsequence \((x_0^{n_k}, x_1^{n_k})\). Let the limit of this convergent subsequence be \((x_0, x_1)\). Since \(D < 0\), this particular agent can never go bankrupt and therefore his budget constraint is always:

\[
p^n x_0 + \eta(p^n) x_1 \leq p^n \omega_0 + \eta(p^n) \omega_1 - D
\]

Taking limits we have,

\[
p_1 x_1 \leq p_1 \omega_1 - D
\]  \hspace{1cm} (4.6)

Choose some \(\lambda \in (0, 1)\) and let

\[
x_0^\lambda = \lambda(x_0 + 1), \quad x_1^\lambda = \lambda x_1
\]

Note that since \(D < 0\), the right hand side of (4.6) is strictly positive. Therefore if (4.6) holds with equality then \(p_1 x_1 > 0\) and hence

\[
p_1 x_1^\lambda = \lambda p_1 x_1 < p_1 x_1 = p_1 \omega_1 - D
\]

On the other hand if (4.6) holds with strict inequality then,

\[
p_1 x_1^\lambda = \lambda p_1 x_1 \leq p_1 x_1 < p_1 \omega_1 - D
\]
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In either case,

\[ p_1 x_1^\lambda < p_1 \omega_1 - D \]

or,

\[ p_1 (x_1^\lambda - \omega_1) < -D \]

Since \( p^n \to 0 \) and \( \eta(p^n) \to p_1 \), we can use continuity to argue from the above that there is some \( K \) such that for \( k > K \),

\[ p^{n_k}(x_0^\lambda - \omega_0) + \eta(p^{n_k})(x_1^\lambda - \omega_1) < -D \]

Which means that \((x_0^\lambda, x_1^\lambda) \in B(p^{n_k})\). From the definition of \((x_0^{n_k}, x_1^{n_k}) = d(p^{n_k})\) it follows that,

\[ u(x_0^{n_k}, x_1^{n_k}) \geq u(x_0^\lambda, x_1^\lambda) \]

Taking limits and using the continuity of \( u(\cdot) \) we have,

\[ u(x_0, x_1) \geq u(x_0^\lambda, x_1^\lambda) \]

or

\[ u(x_0, x_1) \geq u(\lambda(x_0 + 1), \lambda x_1) \]

Since this is true for all \( \lambda \in (0, 1) \), we substitute \( \lambda_i = (1 - 1/i) \) for \( i = 2, 3, \ldots \), in the above inequality and take limits to get,

\[ u(x_0, x_1) \geq u(x_0 + 1, x_1) \]

This contradicts our assumption that \( u(\cdot) \) is strictly monotonic. Hence it
must be the case that \((x_0^n, x_1^n)\) is unbounded.

Now we show that it is in fact \(x_0\) which must become unbounded.

Recall the form of the budget constraint,

\[
px_0 + \eta(p)x_1 \leq p\omega_0 + \eta(p)\omega_1 - D
\]

Since \(L \leq \eta(p) \leq U\), and \(x_1 \geq 0, \omega_1 > 0,\)

\[
px_0 + Lx_1 \leq p\omega_0 + B\omega_1 - D
\]

Or, as \(x_0 \geq 0,\)

\[
x_1 \leq \frac{p\omega_0 + B\omega_1 - D}{L}
\]

Hence \(x_1\) cannot become unbounded as \(p^n\) tends to 0 and it must be \(x_0\) which becomes unbounded.

We have seen that the excess demand of agents with \(D_h < 0\) goes to infinity as \(p\) tends to 0. On the other hand the excess demand of all agents are bounded below since consumption cannot be non-negative. Hence the aggregate excess demand \(\zeta(\cdot)\) must go to infinity. \(\square\)

**Lemma 9.** As \(p \to \infty, d_1(p) \to \infty.\)

**Proof.** Suppose \(p^n\) is a sequence with \(\lim p^n = \infty.\) Let \((x_0^n, x_1^n) = d(p^n)\) be the corresponding optimal bundles. We first show that the sequence \((x_0^n, x_1^n)\) must be unbounded.

If \((x_0^n, x_1^n) = d(p^n)\) is bounded then it must have a convergent subsequence \((x_0^{n_k}, x_1^{n_k})\). Let the limit of this convergent subsequence be \((x_0, x_1)\).
For a sufficiently high price no agent goes bankrupt and therefore we can write the budget constraint as:

\[ p^{n_k}x_0 + \eta(p^{n_k})x_1 \leq p^{n_k}w_0 + \eta(p^{n_k})\omega_1 - D \]

or,

\[ x_0 + \frac{\eta(p^{n_k})}{p^{n_k}}x_1 \leq \omega_1 + \frac{\eta(p^{n_k})}{p^{n_k}}\omega_1 - \frac{D}{p^{n_k}} \]

Taking limits and using the fact that \( \eta(p) < B \),

\[ x_0 \leq \omega_0 \tag{4.7} \]

Choose some \( \lambda \in (0,1) \) and let

\[ x_0^\lambda = \lambda x_0, \quad x_1^\lambda = \lambda (x_1 + 1) \]

Note that since by assumption \( \omega_0 > 0 \), the right hand side of (4.7) is strictly positive. Therefore if (4.7) holds with equality then \( x_0 > 0 \) and hence

\[ x_0^\lambda = \lambda x_0 < x_0 = \omega_0 \]

On the other hand if (4.7) holds with strict inequality then,

\[ x_0^\lambda = \lambda x_0 \leq x_0 < \omega_0 \]
In either case,
\[ x_0^\lambda \leq \omega_0 \]
or,
\[ (x_0^\lambda - \omega_0) < 0 \]

Since \( \eta(p^n_k) / p^n_k \to 0 \), we can use continuity to argue from the above that there is some \( K \) such that for \( k > K \),

\[ (x_0^\lambda - \omega_0) + \frac{\eta(p^n_k)}{p^n_k}(x_1^\lambda - \omega_1) < -\frac{D}{p^n_k} \]

Which means that \( (x_0^\lambda, x_1^\lambda) \in B(p^n_k) \). From the definition of \( (x_0^n_k, x_1^n_k) = d(p^n_k) \) it follows that,
\[ u(x_0^n_k, x_1^n_k) \geq u(x_0^\lambda, x_1^\lambda) \]

Taking limits and using the continuity of \( u(\cdot) \) we have,
\[ u(x_0, x_1) \geq u(x_0^\lambda, x_1^\lambda) \]
or
\[ u(x_0, x_1) \geq u(\lambda x_0, \lambda(x_1 + 1)) \]

Since this is true for all \( \lambda \in (0, 1) \), we substitute \( \lambda_i = (1 - 1/i) \) for \( i = 2, 3, \ldots \), in the above inequality and take limits to get,
\[ u(x_0, x_1) \geq u(x_0, x_1 + 1) \]

This contradicts our assumption that \( u(\cdot) \) is strictly monotonic. Hence it
must be the case that \((x^n_0, x^n_1)\) is unbounded.

Now we show that it is in fact \(x_1\) which must become unbounded.

Recall the form of the budget constraint,

\[
p x_0 + \eta(p) x_1 \leq p \omega_0 + \eta(p) \omega_1 - D
\]

Since \(L \leq \eta(p) \leq U\), and \(x_1 \geq 0, \omega_1 > 0\),

\[
p x_0 + L x_1 \leq p \omega_0 + B \omega_1 - D
\]

Or, as \(x_1 \geq 0\),

\[
x_0 \leq \omega_0 + \frac{B \omega_1 - D}{pL}
\]

Hence \(x_0\) cannot become unbounded as \(p^n \to \infty\) and it must be \(x_1\) which becomes unbounded.

Hence the result is proved. \(\square\)

**Lemma 10.** There is a \(p > 0\) such that \(\zeta(p) \leq 0\).

**Proof.** For sufficiently high prices no agent goes bankrupt and we can write the budget constraint as,

\[
p x_0 + \eta(p) x_1 \leq p \omega_0 + \eta(p) \omega_1 - D
\]

or,

\[
p(x_0 - \omega_0) \leq \eta(p)(\omega_1 - x_1) - D
\]

We know from lemma 9 that the right hand side becomes negative as
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\( p \to \infty \). Hence for a sufficiently large \( \bar{p} \) it must be the case that,

\[ x_0 < \omega_0 \]

Repeating this analysis for all the \( H \) households and taking \( p \) greater than the largest of \( \bar{p} \) for all households, the result is proved.

*Proof of Theorem 1.* Follows from using lemmas 8 and 9 to apply the intermediate value theorem to the continuous function \( \zeta(\cdot) \). □