Chapter 1

Introduction

It is a common practice to use ordinary differential equations to describe the evolution of physical, engineering or biological system. A dynamical system is a way of describing the passage in time of all points of a given space. In the sequel, dynamical system is denoted by system of differential equations. Though such a mathematical notion is inadequate for modeling many real time systems, such study of dynamical system revealed the inner phenomena and characteristics of a real system. In Newtonian dynamics, a system’s state can be fully determined by its initial conditions. Moreover, linear system always exhibit straightforward predictable behaviour and can be solved analytically. In contrast, most nonlinear dynamical systems are impossible to solve analytically. Higher dimensional systems may exhibit unpredictable behaviour that makes a particular explicit solution essentially worthless in the large scheme of understanding the behaviour of the system. Many research developments have been devoted to the study of qualitative phenomena for better understanding of the behavior of a dynamical system. In this view, this thesis mainly focused to study synchronization, a universal phenomena. This chapter focuses a brief overview of thesis, motivation for the research, fundamentals behind synchronization of dynamical systems, the mathematical notions and methods involved.
1.1 A Brief Overview

In 1963, Lorenz’s strange attractor of his simplified atmospheric model [84] truly opened up new horizons in all areas of science and engineering, and famously known as ‘butterfly effect’, more scientifically ‘chaos theory’. In fact, the essence of the chaos theory is that two very near initial conditions does not give transient behaviour rather lead to diverge exponentially. That is called sensitive dependence of initial condition which is one main feature of chaotic systems. After relativity and quantum mechanics, chaos has become the 20th century’s third great revolution in physical sciences.

One of the fundamental nonlinear phenomena observed in nature is synchronization [97] and it is also basic mechanisms of self-organization in many complex systems. In general, synchronization can be understood as an adjustment of some relations between characteristic times, frequencies or phases of two or more dynamical systems during their interactions. Even for two identical chaotic systems, the behavior of solutions tend to diverge at larger time, further exploits more complexity in studying systems’ qualitative properties. Hence, the use of external force along with different control techniques came into play for synchronizing the chaotic dynamics.

From the pioneering work on the synchronization [18], researchers dwell on the study of synchronization phenomena in various contexts due to its vast applications in biology and secure communications [8]. Over the past two decades, with great deal of interests, various types of chaotic synchronization such as, complete synchronization, anti synchronization, lag synchronization and projective synchronization have been discovered. In recent years, many control techniques such as nonlinear control, linear feedback control, and adaptive control have been exhibited for synchronization (see for e.g. [9, 107, 126] and references therein).

On the other hand, nature is full of transparent delays. Systems with delays abound in the world. Such systems often known as time-delay systems or hereditary systems
or systems with memory effects. In fact, time-delay systems are often used to model a large class of engineering systems where propagation and transmissions of information or material involved. Time-delay/lag is a property of a physical system by which the response to an applied force is delayed in its effect [40].

Time-delays can produce dynamical instability which may lead to deterministic chaos. Even a scalar differential equation with time-delays has an infinite-dimensional phase space which favors chaotic solutions. Indeed, delay-differential equations or more generally functional differential equations are often needed for modeling of such delayed dynamics. Thus studying synchronization phenomena in delayed nonlinear systems has become an important research problem. In fact, representing a nonlinear systems in Lur’e form with sector bounded nonlinear conditions has become an innovative approach for studying chaotic dynamics in aspect of control theory.

Using linear matrix inequalities via Lyapunov-Krasovskii stability theory [40] for studying of synchronization problem has shed new light on Master-Slave synchronization schemes [42]. If a given synchronization criteria is independent of time-delay, then one can choose arbitrarily any delay to a given chaotic dynamics. This leads to instability in the synchronization and increases the conservatism on the obtained criteria. When the synchronization criteria depends on the time-delay, the conditions are more effective even for small delays and can find the maximum allowable delay for stable synchronization. Delay-dependent synchronization criteria for chaotic Lur’e systems with time-varying delay is a new complex problem in synchronization study [125]. In particular, in existing results, there were quite restrictions on the derivative of time-varying delay and the synchronization criteria are very conservative. It is important to stress from analyzing the literature that there is still a room for improvement on studying synchronization problems in delayed dynamical system. Moreover, in digital analog circuits, the control is designed as samples [88, 130], hence the study of synchronization using sample data control needed a new attention. It is vital important to consider the
random effects as noise perturbations [1, 93]. Noise perturbed chaotic synchronization is an enhancing problem of current research interest. Also, the parameters involved in the system dynamics may be uncertain/unknown, and its estimation in the Slave dynamics is a practically required problem. Hence, synchronization of system with noise perturbation along with parameter estimation has been posed [8]. Artificial neural networks with learning algorithms used as a tool for many real time applications. It has become a common practice to represent the dynamics of neural networks as differential equations to study its properties. In relatively large scale neural networks, the neuron states are not often completely available in the network outputs and the need to estimate the neuron states through available measurements is critically acclaimed problem [114, 116]. State estimation for Markovian jumping delayed neural networks has not been dealt in the literature completely.

1.1.1 Objective of thesis

Motivated by the above discussion, proposing new synchronization schemes along with corresponding synchronization conditions for delayed dynamics is taken for the research study. Various synchronization conditions with parameter estimations as well as state estimation scheme are obtained in this thesis. In addition, different control approaches such as delayed feedback control, sampled-data control and nonlinear observer control are designed. Further, projective synchronization and adaptive synchronization with noise perturbation are taken into account. In parallel, the synchronization framework is applied to estimate the unknown parameters of chaotic system and state of a delayed neural networks. This thesis presents the above discussed problems along with newly obtained synchronization conditions. In view of the theoretical results, numerical simulations have been corroborated to show the effectiveness of the obtained conditions. The obtained sufficient conditions are represented in LMI form which are universally applicable, and effectively solvable using various optimization algorithms such as interior point
algorithms for semi-definite programming. By taking various real time models such as Ohua system, Ikeda systems, Lorenz-Stenflo systems, etc., the sufficient conditions have been solved numerically using Matlab and the strengths of the proposed methods and results are exhibited. The most interesting results are synchronization of nonlinear system using time-delayed feedback control and sampled-data control, projective synchronization for time-delay system and newly proposed scheme for secure communication.

1.1.2 Organization of thesis

The core part of this study can be given as follows. Firstly, synchronization problem between two chaotic system/estimation of true state of neural networks is defined. A suitable control term is proposed for desired output. Studying the stability of closed-loop error dynamical system between two system is equivalent to the main objective of the proposed problem. The error dynamics are given as nonlinear DDEs and the stability of delayed error system is analyzed. Using Lyapunov-Krasovskii stability theory, new LKFs are introduced accordingly and the corresponding conditions for stability of error dynamics are obtained in terms of LMIs. The required controllers are designed and MAUBs of delay has been found by solving the obtained conditions. Numerical simulations of the synchronization scheme using Matlab are corroborated to show the effectiveness of the proposed results.

The thesis consists of the following 5 chapters other than Introduction and Conclusion.

Chapter 2 is concerned with delayed feedback control for coupling between Master-Slave synchronization of nonlinear systems. The corresponding synchronization criteria are obtained. Moreover, synchronization in a network of chaotic systems with both variable delay coupling and hybrid coupling are presented.

Chapter 3 is dedicated for sampled-data control of synchronization of nonlinear systems with continuous and discontinuous Lyapunov-Krasovskii functional, and secure
communication application of the proposed scheme.

In Chapter 4, projective synchronization problem for chaotic system with more
general case of time-varying delay is discussed. The synchronization conditions have
been further exploited to multiple time-delay systems and the numerical results are
corroborated.

In Chapter 5, the adaptive control techniques are used for synchronization of a
deterministic chaotic system with noise perturbed system. Further proposed adaptive
synchronization scheme is applied to estimate the unknown parameters.

Chapter 6 is devoted to study the state estimation of Markovian jumping neural
networks with time-varying delay. Delay dependent LMI conditions are obtained for
state estimation of MJNNs.

1.1.3 Motivating Examples

In the following, some of physical time time-delay systems modeled in different research
areas are given for motivating the theoretical analysis.

Mackey-Glass time-delay systems
The Mackey-Glass time-delay system [91] has been a well studied nonlinear DDE equa-
tion in the literature for its chaotic dynamics. It has received a central importance in
recent studies on synchronization in view of its hyperchaotic behavior. Analog version of
the Mackey-Glass system has also been realized experimentally using electronic circuits.
The Mackey-Glass system, which is originally deduced as a model for blood production
in patients with leukemia, can be represented by the first order nonlinear DDE

\[
\dot{x}(t) = -bx(t) + \frac{ax(t-\tau)}{1 + x(t-\tau)^c},
\]

where \(a, b\) and \(c\) are positive constants. Here, \(x(t)\) represents the concentration of blood
at time $t$ (density of mature cells in bloodstream), when it is produced, $x(t - \tau)$ is the concentration when the request for more blood is made and $\tau$ is the time-delay between the production of immature cells in the bone marrow and their maturation for release in circulating bloodstream. In patients with leukemia, the time $t$ may become excessively large, and the concentration of blood will oscillate, or if $t$ is even larger, the concentration can vary chaotically, as demonstrated by Mackey and Glass. This model is often used as a prototype model in the literature for nonlinear delay systems exhibiting chaotic attractors and even hyperchaotic attractors for large values of delay time.

**Delayed logistic equations**

A more realistic logistic delay equation [3] for single species dynamics by assuming egg formation to occur $\tau$ time units before hatching can be represented as follows,

$$\frac{dx}{dt} = r x(t) \left[1 - \frac{x(t - \tau)}{K}\right],$$

where $x(t)$ denotes the population size at time $t$, $r > 0$ is the intrinsic growth rate and $K > 0$ is the carrying capacity of the population. This equation is often referred to as the Hutchinson's equation or delayed logistic equation. This form of delayed dynamics is known as system with constant delay.

**Population model**

In [38], authors proposed a single species population model exhibiting the Allee effect [100] in which the per capita growth rate is a quadratic function of the density and is subject to more than one identical time-delay terms represented as

$$\frac{dx}{dt} = x(t) \left[a + bx(t - \tau) - cx^2(t - \tau)\right],$$
where $a > 0$, $c > 0$, $\tau > 0$ and $b$ are real constants. In this model, when the density of the population is not small, the positive feedback effects of aggregation and cooperation are dominated by density-dependent stabilizing negative feedback effects due to intraspecific competition. In other words, intraspecific mutualism dominates at low densities and intraspecific competition dominates at higher densities.

**Stem-cell equation**

The stem-cell equation [63] can be put in the following form with time-varying delay:

$$
\dot{S}(t) = 2M(t - \tau(t))S(t - \tau(t)) - S(t) [M(t) + \omega],
$$

where $S(t)$ is the available stem-cell population. The rate $M(t)S(t)$ at which stem-cells enter the mitotic channel is controlled by the mitotic operator, $M(t)$, acting on the stem-cell population and the rate at which they return after dividing is $2M(t - \tau(t))S(t - \tau(t))$, assuming that there are no losses. $\tau(t)$ represents the time-varying delay between cells leaving the stem-cell population to enter the mitotic cycle and the return of two daughter cells.

**Ikeda time-delay system**

The Ikeda system [54] had been introduced to describe the dynamics of an optical bistable resonator and it had been shown that the transmitted light from a ring cavity containing a nonlinear dielectric medium undergoes transition from a stationary state to periodic and nonperiodic states, when the intensity of the incident light is increased. The Ikeda system is well known for delay induced chaotic behavior and it is also receiving focus on synchronization studies in recent times. The model is specified by the state equation

$$
\dot{x} = -\alpha x(t) - \beta \sin x(t - \tau),
$$
where $\alpha > 0$ and $\beta > 0$ are the parameters and $\tau$ is the delay time. Physically $x(t)$ is the phase lag of the electric field across the resonator and thus may clearly assume both positive and negative values, $\alpha$ is the relaxation coefficient, $\beta$ is the laser intensity injected into the system and $\tau$ is the round-trip time of the light in the resonator.

1.2 Nonlinear Dynamical Systems

In this section, the following definitions are given which are subsequently used in the sequel.

- **Dynamical system:** In nonlinear dynamics, it usually means to speak about an abstract mathematical system which is a model for a real-world system. Mathematically, a dynamical system is defined by its state and by its dynamics. A pendulum is an example for a dynamical system.

- **State of a system:** A number or a vector (i.e., a list of numbers) defining the state of the dynamical system uniquely.

- **Phase space:** All possible states of the system. Each point in the phase-space corresponds to a unique state.

- **Dynamics or equation of motion:** The causal relation between the present state and the next state in the future. It is a deterministic rule which tells us what happens in the next time step. In the case of a continuous time, the time step is infinitesimally small. Thus, the equation of motion is an ordinary differential equation (ODE) (or a system of ODEs):

$$\dot{x} = f(x),$$

where $x$ is the state and $t$ is the time variable (overdot is the time derivative -
as always). \( f(x) \) is the nonlinear vector field. According to the problem defined, initial and boundary conditions must be given.

- **Nonlinear:** The dynamics is linear if the causal relation between the present state and the next state is linear. Otherwise it is nonlinear.

### 1.2.1 Time-delay systems

Delay effects in nonlinear systems are ubiquitous in various fields of physics, chemistry, biology, engineering, and even in social and economic systems. They may arise due to processing and latency times or the finite propagation speed of information between the constituents of a complex system, for instance in electronic or optical systems or neural networks (NNs). Time-delay has two complementary, counterintuitive and almost contradicting facets. On the one hand, delay is able to induce instabilities, bifurcations of periodic and more complicated orbits, multi-stability and chaotic motion. On the other hand, delay can suppress instabilities, stabilize unstable stationary or periodic states and may control complex chaotic dynamics. In this thesis, delay has been considered either in system definition or in the control inputs. The sufficient LMI conditions are novel fundamental results on synchronization and estimation of delayed dynamical systems represented by means of LMIs depending the information on the delay.

Even a simple first order scalar time-delay system with appropriate nonlinearity can exhibit hyperchaotic attractors for suitable values of parameters. Time-delay is also ubiquitous in many physical systems due to finite switching speeds of amplifiers, finite lengths of vehicles in traffic flows, finite signal propagation time in biological networks and circuits, memory effects and so on. Therefore the study of chaos synchronization in coupled time-delay systems and the effect of time-delay feedback control has been taken into account.
1.2.2 Sector nonlinearities and Lur’e system

Nonlinear dynamical analysis has emerged as a novel method for the study of complex systems in the past few decades. From extensive study on the linear system and its properties, the focus has been turned to more complex nonlinear models of nature. Mathematically, the nonlinearity has been characterized, in particular suppressed to various restrictions to suit the requirement. The famous Lipschitz condition has been used to classify the nonlinearity involved in a deterministic system. In the recent development of the control theory, sector bounded nonlinearity condition has been widely used to study a large class of nonlinear systems including chaotic and hyperchaotic systems.

A function \( \phi : \mathbb{R} \rightarrow \mathbb{R} \) is said to be in sector \([k, \ l]\), if for all \( q \in \mathbb{R} \), \( p = \phi(q) \) lies between \( kq \) and \( lq \). Then the function is said to be sector bounded. This kind of nonlinearity can be used to study the qualitative properties of chaotic systems. On the other hand, Lur’ë system is nonlinear feedback system formed by a linear dynamical system as shown in Fig. 1.1. Nowadays, it is a common method for describing time-varying nonlinearity. Absolute stability of the Lur’ë nonlinear system using Popov criterion have
been discussed enormously in \cite{41, 60}. In this thesis, the generalized Lur'e representation of nonlinear system is used to derive the main results. Formal definitions of system as per problem formulation is given in the respective chapters. The generalized sector condition is defined below:

**Definition 1.2.1** (Generalized sector bounded nonlinearity \cite{60}) $f(\cdot)$ denotes the vector-valued nonlinear function satisfying $f(0) = 0$ and satisfies the following sector bounded nonlinearity condition. The nonlinear function $f(\cdot) = \left[ f_1, f_2, \ldots, f_l \right]^T : \mathbb{R}^l \to \mathbb{R}^l$ satisfies the following inequality

$$
\begin{bmatrix}
  f_i(\xi) - \sigma^+_i \xi \\
  f_i(\xi) - \sigma^-_i \xi
\end{bmatrix} \begin{bmatrix}
  f_i(\xi) - \sigma^+_i \xi \\
  f_i(\xi) - \sigma^-_i \xi
\end{bmatrix} \leq 0, \quad \forall \xi \in \mathbb{R},
$$

(1.1)

for all $i = 1, 2, \ldots, l$. Thus the nonlinear function $f_i(\cdot)$ is said to belong in the sector $[\sigma^-_i, \sigma^+_i]$.

### 1.3 Chaos Theory

Chaos theory is considered to be the third revolution in physics following the relativity theory and quantum mechanics. It has been studied extensively in the past thirty years. A lot of chaotic phenomena have been found and enormous mathematical strides have been taken. Nowadays, it has been agreed by scientists and engineers that chaos is ubiquitous in natural sciences, social sciences and so on. In modern scientific term, deterministic chaos depicts an irregular and unpredictable time evolution of many (simple) deterministic dynamical systems, characterized by nonlinear coupling of its variables. Wherever nonlinearity exists, chaos may be found. For a long time, chaos had been thought of as a harmful behavior that could decrease the performance of a system and therefore should be avoided when the system is running. Generally, a chaotic system is a nonlinear deterministic system that possesses complex and unpredictable behavior.
One remarkable feature of a chaotic system distinguishing itself from other nonchaotic systems is that the system is extremely sensitive to initial conditions. Any tiny perturbation of the initial conditions will significantly alter the long-term dynamics of the system.

Two trajectories of a chaotic system starting close to each other will diverge after some time (so-called sensitive dependence on initial conditions). Mathematically, chaotic systems are characterized by local instability and global boundedness of the trajectories. Since local instability of a linear system implies unboundedness (infinite growth) of its solutions, a chaotic system should be described by a nonlinear mathematical model. In the following, two important chaotic models that made revolution in the science are discussed.
1.3.1 Lorenz system

In 1960s, Ed Lorenz created a simple weather model in which small changes in starting conditions led to a marked changes in outcome, called sensitive dependence on initial conditions, or popularly, the butterfly effect. That is, long-range prediction of imprecisely measured systems becomes an impossibility. The celebrated Lorenz system is given as follows.

\[
\begin{align*}
\dot{x}_1 &= -\sigma (x_1 - x_2) \\
\dot{x}_2 &= -x_1 x_3 + r x_1 - x_2 \\
\dot{x}_3 &= x_1 x_2 - bx_3
\end{align*}
\] (1.2)

This system is related to the Rayleigh-Bernard convection under very crude approximations. The quantity \( x_1 \) is proportional to the circulatory fluid particle velocity; the quantities \( x_2 \) and \( x_3 \) are related to the temperature profile; \( \sigma, b, r \) are dimensionless parameters. Lorenz [84] has studied the behaviour of (1.2) for the case with \( \sigma = 10 \) and \( b = 8/3 \) at varying \( r \) (which is proportional to the Rayleigh number). The chaotic attractor formed by Lorenz system in the phase space and 3D are given in Fig. 1.2. In fact, there is massive literature on the chaotic properties of Lorenz system which revolutionized the science. The 'strange', in other words, chaotic regime in the system has been formed by so called sensitive dependence on initial conditions.

1.3.2 Chua system

Chua’s circuit is a simple electronic circuit that exhibits classic chaotic behaviour. First introduced in 1983 by Leon Chua, its ease of construction has made it an ubiquitous real-world example of a chaotic system, leading some to declare it a ‘paradigm for chaos’. The Chua’s circuit consists of two linear capacitors, two linear resistors, one linear inductor,
Figure 1.3: Chua’s circuit diagram and its realization using two Operational Amplifiers and six linear resistors to implement the Chua diode NR

and a nonlinear resistor. Let us take the following representation of Chua’s circuit system

\[
\begin{align*}
\dot{x}_1(t) &= a(x_2(t) - \phi(x_1(t))) \\
\dot{x}_2(t) &= x_1(t) - x_2(t) + x_3(t) \\
\dot{x}_3(t) &= -bx_2(t)
\end{align*}
\] (1.3)

with nonlinear characteristic of Chua’s diode

\[
\phi(x(t)) = m_1 x_1(t) + \frac{1}{2}(m_0 - m_1)(|x_1(t) + c| - |x_1(t) - c|),
\]

and parameters \(a = 9, b = 14.28, c = 1, m_0 = -(1/7), m_1 = 2/7\). With this parameters Chua system produces the double scroll attractor which is given in Fig. 1.4 and its circuit realization using two operational amplifiers and six linear resistors to implement the Chua diode NR is shown in Fig. 1.3. In this thesis, Chua’s system has been used widely to verify the obtained synchronization conditions. Due to its applicability, effectiveness and generalization of LMI conditions can be verified through the numerical results.
1.4 Synchronization

The word “synchronous” is often encountered in both scientific and everyday language. Originating from the Greek words χρόνος (chronos, meaning time) and συν (syn, meaning the same, common), in a direct translation “synchronous” means “sharing the common time”, “occurring in the same time”. This term, as well as the related words “synchronization” and “synchronized”, refers to a variety of phenomena in almost all branches of natural sciences, engineering and social life, phenomena that appear to be rather different but nevertheless often obey universal laws [97]. The time series plot of Lorenz chaotic system (1.2) is given in Fig. 1.5. The dotted lines represents the time series of the same dynamics with identical parameters but small change in the initial conditions. It can be clearly seen from Fig. 1.5 that the trajectories are deviating from the previous solutions as the time progress. If one can incorporate different mechanisms to control this behaviour that exploit certain defining characteristics of chaos, the approach becomes
more interesting and, hopefully, more efficient and effective to be applicable practically. Thus synchronization in chaotic system using different controllers came into play. In addition, due to the inherent association of nonlinearity and chaos to various issues, the scope of chaos synchronization is much more diverse. Third important class of control goals corresponds to synchronization (more accurately, controlled synchronization as opposed to autosynchronization or self-synchronization).

Generally speaking, synchronization is understood as concordance or concurrent change of the states of two or more systems or, perhaps, concurrent change of some quantities related to the systems, for example, alignment of oscillation frequencies. If the required relation is established only asymptotically, one may speak about asymptotic synchronization. If synchronization does not exist in the system without any control, finding a control function ensuring synchronization in the closed-loop system can be proposed as synchronization problem.
1.4.1 Importance of synchronization

Chaos synchronization is a counterintuitive phenomenon. A chaotic system moves irregularly and unpredictably, and two chaotic systems, starting from almost identical initial states, end in completely uncorrelated trajectories. Hence it came as a surprise when Pecora and Carroll showed in 1990 that two chaotic systems which are coupled by some of their internal variables can synchronize to a common identical chaotic motion [18, 96]. The dynamics is still irregular and unpredictable, but both sides have identical trajectories. The combination of synchronization and unpredictability leads to an interesting application for secure communication.

Since the pioneering work by Carroll and Pecora [18] on the Master-Slave (drive-response) concept for achieving the synchronization of coupled chaotic systems, several researchers have proposed a variety of alternative schemes for ensuring the control and synchronization of such systems (see, e.g., [26, 130]). Synchronization of coupled chaotic systems have potential applications in various fields, including chaos generator design, secure communication, chemical reaction, biological systems, and information science.

On the other hand, a number of Master-Slave synchronization schemes for chaotic system with Lur’e representation have been proposed [13, 41, 58, 71, 72, 89, 108, 113]. The controllers such as observer-based control [13, 58], adaptive control [61, 108], sampled-data controllers [10, 130], intermittent control [113], and state delayed feedback control [102, 125], have been used to synchronize chaos in continuous-time systems. In control theory view, synchronization can be viewed as stability of the closed-loop error dynamics which is formed between the Master and Slave systems. Thus, the results are obtained in terms of delay-dependent conditions to suffice the stability of error dynamics. In the following sections, stability concepts and methodologies used in the proposed thesis are discussed.
1.5 Stability Analysis

Stability is the most important qualitative property of automatic control systems. Unstable systems have no practical significance. The concept of stability is so important since every control system must be primarily stable and only then other properties can be studied. The theory of stability is the preoccupation of scientists from the beginning of the theory of differential equations. The key problem is to obtain information on the system’s behavior (state trajectory) without solving the differential equation. The theory considers the system’s behavior over a long period of time. One of the first scientists who investigates the stability of conservative mechanical systems in the “modern” sense is Lagrange - his observation is that the equilibrium state of an unforced system is stable if it has a minimum of potential energy. A basic step was made by Russian Mathematician Lyapunov, who defined the general concepts of stability, for both linear and nonlinear systems. Formal definition of stability concepts are given below:

Definition 1.5.1 For the system \( \dot{x}(t) = f(t, x(t)) \) and \( x(t_0) = x_0 \), the trivial solution \( x(t) = 0 \) is said to be stable if for any \( t_0 \in \mathbb{R} \) and \( \epsilon > 0 \), there exists a \( \delta = \delta(t_0, \epsilon) > 0 \) such that \( \|x_{t_0}\| < \delta \) implies \( \|x(t)\| < \epsilon \) for \( t \geq t_0 \). It is said to be asymptotically stable, if for any \( t_0 \in \mathbb{R} \) and any \( \epsilon > 0 \), there exists a \( \delta_a = \delta_a(t_0, \epsilon) > 0 \) such that \( \|x_{t_0}\| < \delta_a \) implies \( \lim_{t \to \infty} x(t) = 0 \). It is said to be uniformly stable if it is stable and \( \delta(t_0, \epsilon) \) can be chosen independently of \( t_0 \). It is uniformly asymptotically stable if there exists a \( T = T(\delta_a, \eta) \), such that \( \|x(t)\| < \eta \) for \( t \geq t_0 + T \) and \( t_0 \in \mathbb{R} \). It is globally (uniformly) asymptotically stable if it is (uniformly) asymptotically stable and \( \delta_a \) can be an arbitrarily large, finite number.

1.5.1 Lyapunov stability

Having defined stability and asymptotic stability of an equilibrium state of a dynamical system, the next issue to be considered is that of determining stability. One may
obviously execute to find all possible solutions to the state-space equation of the given system; however, such an approach is often difficult. A more elegant approach in modern stability theory has been founded by Lyapunov. Specifically, the stability problem can be investigated by applying the direct method of Lyapunov, which makes use of a continuous scalar function of the state vector, called a Lyapunov function.

Lyapunov’s theorem on the stability and asymptotic stability are given below and they are famously known as Lyapunov first and second method respectively.

**Theorem 1.5.2** The equilibrium state $\bar{x}$ is stable if in a small neighborhood of $\bar{x}$ there exists a positive definite function $V(x)$ such that its derivative with respect to time is negative semidefinite or identically equal to zero in that region.

**Theorem 1.5.3** The equilibrium state $\bar{x}$ is asymptotically stable if in a small neighborhood of $\bar{x}$ there exists a positive definite function $V(x)$ such that its derivative with respect to time is negative definite in that region.

The scalar function $V(x)$ is called as Lyapunov function for the equilibrium state $\bar{x}$.

The above theorems require the Lyapunov function $V(x)$ to be a positive definite function. Such a function is defined as: The function $V(x)$ is positive definite in the state space $\mathcal{L}$ if, for all $x$ in $\mathcal{L}$ it satisfies the following requirements:

- The function $V(x)$ has continuous partial derivatives with respect to the elements of the state vector $x$.
- $V(\bar{x}) = 0$.
- $V(x) > 0$ if $x \neq \bar{x}$.

The important point of this discussion is that Lyapunov’s theorems can be applied without having to solve the state-space equation of the system. Unfortunately, the theorems give no indication of how to find a Lyapunov function; it is a matter of ingenuity,
trial and error in each case. In many problems of interest, the energy function can serve as a Lyapunov function. The inability to find a suitable Lyapunov function does not however prove instability of the system. The existence of a Lyapunov function is sufficient but not necessary for stability.

Lyapunov stability method has more theoretical importance than practical value and can be used to derive and prove other stability results. Its final statement for linear time invariant system is elegant and easily tested using Matlab. However, it is computationally more involved than the other methods for examining the stability of linear systems. Its importance lies in its generality since it can be applied to all nonlinear and linear systems without taking into account whether or not these systems are time invariant or time varying. More about the general study of Lyapunov stability can be found in several books on nonlinear systems [14, 60].

Failure of a Lyapunov function candidate to satisfy the conditions for stability or asymptotic stability does not mean that the equilibrium is not stable or asymptotically stable. The Lyapunov approach allows to assess the stability of equilibrium points of a system without solving the differential equations that describe the system. There are no generally applicable methods for finding Lyapunov functions.

1.5.2 Stability analysis of time-delay systems

The stability of time-delay systems has been widely investigated in the last two decades. Many interesting stability conditions for time-delay systems have been reported in the literature. In recent book [40], Gu et al. presented an in-depth overview of the recent stability results of time-delay systems obtained by using the celebrated LMI technique. In the time-domain, two approaches have been proposed to discuss the stability analysis problem of time-delay systems. One is the Razumikhin Theorem approach and the other is the Lyapunov-Krasovskii functional approach. However, the Lyapunov-Krasovskii functional approach can generally provide less conservative results than the
Razumikhin Theorem approach. This is because more information of time-delay can be utilized in the Lyapunov–Krasovskii functional approach. The existing synchronization criteria / stability criteria for error dynamics are classified into two categories: delay-independent criteria [125] and delay-dependent criteria [42]. The delay-independent ones are irrespective of the size of time-delay. While the delay-dependent ones are concerned with the size of time-delay. Generally speaking, the delay-dependent conditions are regarded to be less conservative than the delay-independent ones, especially when the size of time-delay is small. Much effort has been paid to develop less conservative delay-dependent stability conditions for time-delay systems using different approaches such as delay decomposition approach [131], free-weighting matrices [89]. In this thesis, delay-dependent stability criteria is obtained for the closed loop error system in order to derive very less conservative synchronization conditions.

To describe time-delay systems, functional differential equations (FDE) are used. Let $y(t)$ be a solution of the functional differential equation

$$\dot{x}(t) = f(t, x_t),$$

where $x(t) \in \mathbb{R}^n$ and $f: \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ depends on $x_t = x(\xi)$ for $t - r \leq \xi \leq t$, where $r$ is the maximum delay and $x_0$ (initial condition) such that $x(t_0 + \theta) = \phi(\theta) \in \mathcal{C}([t_0, t]; \mathbb{R}^n)$. The stability of the solution concerns the system’s behavior when the system trajectory $x(t)$ deviates from $y(t)$. Assume that the functional differential equation (1.4) admits the solution $x(t) = 0$, which will be referred to as the trivial solution. For quantifying the delay free system, one can use the Lyapunov function $V(t, x(t))$ which uses only system’s future evolution. On the other hand for time-delay system, the state $x(t)$ in the interval $[t - r, t]$ or $x_t$ (that is, system’s past states or memory effects) is required. Hence corresponding new functional $V(t, x_t)$ is required for studying the stability of time-delay systems. Such a function is famously known as Lyapunov - Krasovskii functional. The
following Lyapunov - Krasovskii stability theorem is used throughout the thesis to derive the stability criterion.

**Theorem 1.5.4** Consider a system given in (1.4), and that \( u, v, \omega : \mathbb{R}_+ \to \mathbb{R}_+ \) are continuous nondecreasing functions, where additionally \( u(s) \) and \( v(s) \) are positive for \( s > 0 \), and \( u(0) = v(0) = 0 \). If there exists a continuous differentiable functional \( V : \mathbb{R} \times C \to \mathbb{R} \) such that

\[
u(\|\phi(0)\|) \leq V(t, \phi) \leq v(\|\phi\|_c)\]

and

\[
\dot{V}(t, \phi) \leq -w(\|\phi(0)\|),
\]

then the trivial solution of (1.4) is uniformly stable. If \( w(s) > 0 \) for \( s > 0 \), then it is uniformly asymptotically stable. If, in addition, \( \lim_{s \to \infty} u(s) = \infty \), then it is globally uniformly asymptotically stable.

On constructing the LKFs, the sufficient conditions are obtained in terms of LMIs for asymptotic stability of error dynamical systems.

### 1.5.3 Linear matrix inequalities (LMIs)

LMIs and LMI techniques have emerged as powerful design tools in areas ranging from control engineering to system identification and structural design. The following three factors make the technique appealing:

- A variety of design specifications and constraints can be expressed as LMIs.
- Once, the conditions formulated in terms of LMIs, a problem can be solved exactly by efficient convex optimization algorithms (the "LMI solvers") numerically in
Matlab. The major numerical tools are LMI lab in Optimization Toolbox, and SEDUMI via YALMIP.

- While most problems with multiple constraints or objectives lack analytical solutions in terms of matrix equations, they often remain tractable in the LMI framework. This makes LMI-based design a valuable alternative to classical "analytical" methods.

1.5.4 Markovian jumping systems (MJS)

Markovian jumping linear systems were first introduced by Krasovskii and Lidskii in 1961. Since then, more and more attentions have been devoted to the study of this class of jump linear systems. The information latching phenomenon usually happens in NNs, see ( [1, 5, 6, 93]) which can be dealt by extracting finite state representation from trained network. In other words, the NNs may have finite modes and the modes may switch from one to another at different times and it is shown that the switching between different NNs modes can be governed by a Markov chain with finite state space $S = \{1, 2, \cdots, s\}$ and with uncertain switching probabilities [93]. MJS are a special class of hybrid systems, which are specified by two components in the state. The first one refers to the mode, which is described by a continuous time finite-state Markovian process, and the second one refers to the state which is represented by a system of differential equations. MJS have the advantage of modeling the different types of dynamic systems subject to abrupt variation in their structures, and have many applications such as target tracking problems, state estimation problem, manufactory processes and fault-tolerant systems. The definitions for Markov chain and stability of MJS are given below.

**Definition 1.5.5** (Markov chain [1, 93]) Let $r(t), t \geq 0$ be a right-continuous Markov process on the probability space which takes values in the finite space $S = 1, 2, \ldots, N$ with generator $\Gamma = (\gamma_{ij}) (i, j \in S)$ given by
\[ P(r(t + \Delta) = j | r(t) = i) = \begin{cases} 
\gamma_{ij} \Delta + o(\Delta), & i \neq j, \\
1 + \gamma_{ii} \Delta + o(\Delta) & i = j, 
\end{cases} \] (1.5)

where \( \Delta > 0 \), \( \lim_{\Delta \to 0} \left( \frac{0(\Delta)}{\Delta} \right) = 0 \) and \( \gamma_{ij} \) is transition rate from mode \( i \) to mode \( j \) satisfying \( \gamma_{ij} \geq 0 \) for \( i \neq j \) with \( \gamma_{ii} = -\sum_{j=1,j \neq i}^{N} \gamma_{ij}, i, j \in S \).

**Definition 1.5.6 [1, 93]** For any \( \varphi \in L_{x_0}^{2}([-\tau, 0]; \mathbb{R}^n) \) and initial mode \( \rho_0 \in S \), the equilibrium solution of Markov-type dynamical system is said to be asymptotically stable in the mean square if

\[
\lim_{t \to \infty} \mathbb{E}\|x(t, \varphi, \rho_0)\|^2 = 0,
\]

where \( x(t, \varphi, \rho_0) \) is the solution of the system at time \( t \).

### 1.6 Useful Lemmas

Some useful lemmas are recalled in this section in order to derive main results of the thesis which are presented below.

**Lemma 1.6.1 (Schur complement [14]).** The following LMI

\[
\begin{bmatrix}
Q(x) & S(x) \\
S^T(x) & R(x)
\end{bmatrix} > 0,
\]

where \( Q(x) = Q^T(x) \), \( R(x) = R^T(x) \) and \( S(x) \) depend affinely on \( x \), is equivalent to one of the following conditions

(i) \( Q(x) > 0, \ R(x) - S^T(x)Q^{-1}S(x) > 0 \);

(ii) \( R(x) > 0, \ Q(x) - S(x) - R^{-1}(x)S^T(x) > 0 \).

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Lemma 1.6.2 (Jensen's inequality [40]). For any matrix $N \in \mathbb{R}^{m \times m}$, $N = N^T$, scalar $\gamma > 0$, vector function $\omega : [0, \gamma] \to \mathbb{R}^m$ such that the integrations concerned are well defined, then

$$\gamma \int_0^\gamma \omega^T(s)N\omega(s)ds \geq \left(\omega(s)ds\right)^T N\left(\omega(s)ds\right).$$

Lemma 1.6.3 [41, 42] (i) Let $U, V(t), W$ and $Q$ be real matrices of appropriate dimensions with $Q$ satisfying $Q = Q^T$, then

$$Q + UV(t)W + W^T V^T(t)U^T < 0$$

for all $V^T(t)V(t) \leq I$, if and only if there exists a scalar $\epsilon > 0$ such that

$$Q + \epsilon^{-1} U^T U + \epsilon W^T W < 0.$$

(ii) Given any real matrices $K_1, K_2, Q$ of appropriate dimensions and a number $\nu > 0$ such that $0 < Q = Q^T$, then the following inequality holds:

$$K_1^T K_2 + K_2^T K_1 \leq \nu K_1^T Q K_1 + \nu^{-1} K_2^T Q^{-1} K_2.$$

(iii) For any real matrix $W \in \mathbb{R}^{n \times n}$, $W = W^T > 0$, a nonsingular matrix $U \in \mathbb{R}^{n \times n}$, and a scalar $\alpha > 0$, then

$$-U^{-1} W (U^{-1})^T \leq \alpha^2 W^{-1} - \alpha U^{-1} - \alpha (U^{-1})^T$$

Lemma 1.6.4 (S-procedure [14]) Let $T_i \in \mathbb{R}^{n \times n}$ or $T_i \in \mathbb{R}^{m \times m} (i = 0, 1, 2, \ldots, p)$ be symmetric matrices. The conditions on $T_i \quad (i = 0, 1, 2, \ldots, p)$

$$\zeta^T T_0 \zeta > 0, \quad \forall \quad \zeta \neq 0 \quad \text{subject to} \quad \zeta^T T_i \zeta > 0 \quad (i = 1, 2, \ldots, p)$$
hold if there exist \( \tau_i \geq 0 \) \( (i = 1, 2, \ldots, p) \) such that

\[
T_0 - \sum_{i=1}^{p} \tau_i T_i > 0.
\]

**Lemma 1.6.5** By the definition of Kronecker product, the following properties hold:

\[
\begin{align*}
(\alpha A) \otimes B &= A \otimes (\alpha B); \\
(A + B) \otimes C &= A \otimes C + B \otimes C; \\
(A \otimes B)(C \otimes D) &= (AC) \otimes (BD).
\end{align*}
\]

**Lemma 1.6.6** [41] For any constant matrix \( X \in \mathbb{R}^{n \times n} \), \( X = X^T > 0 \), there exists positive scalar \( \tau_M \) such that \( 0 \leq \tau(t) \leq \tau_M \), and a vector-valued function \( \dot{x} : [-\tau_M, 0] \to \mathbb{R}^n \), the integration

\[
-\tau_M \int_{t-\tau_M}^{t} \dot{x}^T(s)X\dot{x}(s)ds
\]

is well defined.

**Lemma 1.6.7** [78] (Extended Wirtinger Inequality). Let \( z(t) \in W[a, b] \) and \( z(a) = 0 \).

Then for any matrix \( R > 0 \), the following inequality holds:

\[
\int_{a}^{b} z^T(s)Rz(s)ds \leq \frac{4(b-a)^2}{\pi^2} \int_{a}^{b} \dot{z}^T(s)R\dot{z}(s)ds.
\]

**Lemma 1.6.8** For any \( n \times n \) constant matrix \( M > 0 \), any scalars \( a \) and \( b \) with \( a < b \) and a vector function \( x(t) : [a, b] \to \mathbb{R}^n \) such that integrations concerned are well defined, then the following inequality holds

\[
\left[ \int_{a}^{b} x(s)ds \right]^T M \left[ \int_{a}^{b} x(s)ds \right] \leq (b-a) \left[ \int_{a}^{b} x(s)^T M x(s)ds \right].
\]