Appendices

Appendix. A1

(a). From Eq.(85), we could not obtain an explicit closed-form solution for T^*. For simplicity, we use Taylor's series approximation of $e^{\theta T}$ to obtain the closed-form solution. Utilizing the fact that $e^{\theta T} \approx 1 + \theta T + (\theta T)^2/2$ as θT is small, we obtain

$$\theta T e^{\theta T} - e^{\theta T} + 1 \approx (\theta T)^2/2 \tag{A1.1}$$

Substituting (A1.1) in Eq.(85), we get

$$(h_1 + c\theta + sI_e\alpha)\lambda(s)T^2 \approx 2A$$

Consequently we obtain the optimal value of $T_1^*(s)$ as

$$T_1^*(s) = \sqrt{\frac{2A}{\lambda(s)(h_1 + c\theta + sI_e\alpha)}} \tag{A1.2}$$

To ensure $T_1^*(s) \leq t_1$, substituting $(A1.2)$ into this inequality we have

$$2A \leq t_1^2\lambda(s)(h_1 + c\theta + sI_e\alpha)$$

and so $\delta_{11}(s) \geq 2A$ implies that $T^* = T_1^*(s)$.

(b). Following the similar approach given above, from Eq.(87) we get

$$T_2^*(s) = \sqrt{\frac{2A + \lambda(s)(h_2 - h_1)t_1^2}{\lambda(s)(h_2 + c\theta + sI_e\alpha)}} \tag{A1.3}$$

To ensure that $t_1 < T < M$, substituting (A1.3) into this inequality, we have

$$\lambda(s)(h_1 + c\theta + sI_e\alpha)t_1^2 < 2A < \lambda(s)[(h_2 + c\theta + sI_e\alpha)M^2 - (h_2 - h_1)t_1^2]$$

This implies that if $\delta_{11}(s) < 2A < \delta_{12}(s)$ then $T^* = T_2^*(s)$.

(c). By Taylor's series expansion,

$$\theta T e^{\theta(T-M)} - e^{\theta(T-M)} + 1 \approx \frac{(\theta T)^2}{2} - \frac{(\theta M)^2}{2} + \theta M$$

as θT is small (chang et al. [18]) Therefore, the Eq.(89) gives that

$$\lambda(s)(h_2 + c\theta + cI_k)T^2 \approx 2A + \lambda(s)[(h_2 - h_1)t_1^2 + (cI_k - sI_e\alpha)M^2]$$

It implies that

$$T_3^*(s) = \sqrt{\frac{2A + \lambda(s)[(h_2 - h_1)t_1^2 + (cI_k - sI_e\alpha)M^2]}{\lambda(s)(h_2 + c\theta + cI_k)}} \quad \text{(A1.4)}$$

To ensure that $T \geq M$, substituting (A1.4) into this inequality, we have

$$2A \geq \lambda(s)[(h_2 + c\theta + sI_e\alpha)M^2 - (h_2 - h_1)t_1^2]$$

It implies that if $\delta_{12}(s) \leq 2A$, then $T^* = T_3^*(s)$.

Appendix. A2

Solving Eq.(90) by using Taylor's series approximations, we get

$$T_4^*(s) = \sqrt{\frac{2A + (cI_k - sI_e\alpha)\lambda(s)M^2}{\lambda(s)(h_1 + c\theta + cI_k)}} \quad \text{(A1.5)}$$

Following the similar approach as in appendix.A1, we prove the Theorem 4.2 by using the inequalities $0 < T_1^*(s) \leq M, M < T_4^*(s) < t_1$ and $T_3^*(s) \geq t_1$.

Appendix. A3

Here, we use the Taylor's series approximation as in appendix A1. Solving the Eqs. (91), (93),(95) and (97), we get

$$\begin{aligned} T_5^*(s) &= \sqrt{\frac{2A}{\lambda(s)(h_1 + c\theta + sI_e\alpha)}} \\ T_6^*(s) &= \sqrt{\frac{2A + \lambda(s)(h_2 - h_1)t_1^2}{\lambda(s)(h_2 + c\theta + sI_e\alpha)}} \\ T_7^*(s) &= \sqrt{\frac{2A + \lambda(s)[(h_2 - h_1)t_1^2 + sI_eN^2(1-\alpha)]}{\lambda(s)(h_2 + c\theta + sI_e)}} \\ T_8^*(s) &= \sqrt{\frac{2A + \lambda(s)[(h_2 - h_1)t_1^2 + cI_kM^2 - sI_e(M^2 - N^2(1-\alpha))]}{\lambda(s)(h_2 + c\theta + cI_k)}} \end{aligned}$$

Using the above solutions and the inequalities $0 < T_5^*(s) \leq t_1, t_1 < T_6^*(s) < N, N \leq T_7^*(s) < M$ and $T_8^*(s) \geq M$, we easily get the required results in Theorem 4.3.

Appendix. A4

Solving the Eq.(98) using Taylor series approximations, we get

$$T_9^*(s) = \sqrt{\frac{2A + sI_e\lambda(s)N^2(1-\alpha)}{\lambda(s)(h_1 + c\theta + sI_e)}}$$

Following the approach given in appendix A1 and using the inequalities $0 < T_5^*(s) \leq N$, $N < T_9^*(s) < t_1, t_1 \leq T_7^*(s) < M$ and $T_8^*(s) \geq M$, we easily derive the Theorem 4.4.

Appendix. A5

Solving the Eq.(99), we get

$$T_{10}^*(s) = \sqrt{\frac{2A + \lambda(s)[cI_kM^2 - sI_e(M^2 - N^2(1-\alpha)]}{\lambda(s)(h_1 + c\theta + cI_k)}}$$

Following the approach given in appendix A1 and using the inequalities $0 < T_5^*(s) \leq N$, $N < T_9^*(s) < M, M \leq T_{10}^*(s) < t_1$ and $T_8^*(s) \geq t_1$, we easily derive the Theorem 4.5.

Appendix. B1

(a). From Eq.(131), we could not obtain an explicit closed-form solution for T^*. For simplicity, we use Taylor's series approximation of $e^{\theta T}$ to obtain the closed-form solution. Utilizing the fact that $e^{\theta T} \approx 1 + \theta T + (\theta T)^2/2$ as θT is small, we obtain

$$\theta T e^{\theta T} - e^{\theta T} + 1 \approx (\theta T)^2/2 \tag{B1.1}$$

By Taylor's series expansion,

$$\theta T e^{\theta(T-M)} - e^{\theta(T-M)} + 1 \approx \frac{(\theta T)^2}{2} - \frac{(\theta M)^2}{2} + \theta M \tag{B1.2}$$

as θT is small (for details, see chang et al. [18])

Substituting (B1.1) and (B1.2) in $f_1(T) = 0$ (see Eq.(131)), we get

$$2A \approx (1-\beta_1)\lambda_1(h + c\theta)T^2 + cI_k(1-\beta_1)\lambda_1[T^2 - M^2] + sI_e\alpha(1-\beta_1)\lambda_1 M^2$$

Consequently we obtain the optimal value $T_1^*(\eta)$ as

$$T_1^*(\eta) = \sqrt{\frac{2A - (sI_e\alpha - cI_k)(1-\beta_1)\lambda_1 M^2}{(1-\beta_1)\lambda_1(h + c\theta + cI_k)}} \tag{B1.3}$$

To ensure the condition that $M \leq T_1^*(\eta)$, substituting (B1.3) into this inequality, we have

$$(1-\beta_1)\lambda_1 M^2(h + c\theta + sI_e\alpha) \leq 2A$$

and so $\Delta_1(\eta) \leq 2A$ implies that $T^* = T_1^*(\eta)$.

(b). Following the similar approach as given above and from Eq.(132), we get

$$T_2^*(\eta) = \sqrt{\frac{2A}{(1-\beta_1)\lambda_1(h + c\theta + sI_e\alpha)}} \tag{B1.4}$$

To ensure the condition that $M > T_2^*(\eta)$, substituting (B1.4) into this inequality, we have

$$(1-\beta_1)\lambda_1 M^2(h + c\theta + sI_e\alpha) > 2A$$

and so $\Delta_1(\eta) > 2A$ implies that $T^* = T_2^*(\eta)$.

Appendix. B2

(a) Solving Eq.(133) by using the Taylor's series approximation, we get

$$T_3^*(\eta) = \sqrt{\frac{2A + (1-\beta_1)\lambda_1[cI_k M^2 - sI_e(M^2 - (1-\alpha)N^2)]}{(1-\beta_1)\lambda_1(h + c\theta + cI_k)}} \tag{B2.1}$$

Following the similar approach as in appendix B1 and ensuring the condition that $M \le T_3^*(\eta)$, we have

$$(1-\beta_1)\lambda_1[(c\theta + h)M^2 + sI_e(M^2 - (1-\alpha)N^2)] \le 2A$$

i.e., $\Delta_2(\eta) \le 2A$ implies that $T^* = T_3^*(\eta)$.

(b). Using the similar approach as in (a) and from the Eq.(135), we can get $T_4^*(\eta)$ as

$$T_4^*(\eta) = \sqrt{\frac{2A + (1-\beta_1)\lambda_1 sI_e(1-\alpha)N^2}{(1-\beta_1)\lambda_1(h + c\theta + sI_e)}} \tag{B2.2}$$

To ensure the condition that $N < T_4^*(\eta) < M$, we get the inequality : $\Delta_3(\eta) < 2A < \Delta_2(\eta)$.

(c). Solving the Eq.(137), we get

$$T_5^*(\eta) = \sqrt{\frac{2A}{(1-\beta_1)\lambda_1(h + c\theta + sI_e\alpha)}} \tag{B2.3}$$

To ensure the condition that $T_5^*(\eta) \le N$, we get the inequality : $\Delta_3(\eta) \ge 2A$.

Appendix. B3

(a). Let

$$\begin{aligned} z_1(\eta) = \frac{\partial TC_1}{\partial \eta} &= \frac{-a(c\theta + h)\lambda_1}{T\theta^2}\left[e^{\theta T} - \theta T - 1\right] - \frac{acI_k\lambda_1}{T\theta^2}\left[e^{\theta(T-M)} - \theta(T-M) - 1\right] \\ &\quad -sI_e\left[\frac{-a\lambda_1\alpha M^2}{2T} + (1-2\eta)\delta(a\lambda_1 + b\lambda_2)M\right] \end{aligned}$$

Then

$$\begin{aligned} z_1(0) &= \frac{-a(c\theta + h)\lambda_1}{T\theta^2}\left[e^{\theta T} - \theta T - 1\right] - \frac{acI_k\lambda_1}{T\theta^2}\left[e^{\theta(T-M)} - \theta(T-M) - 1\right] \\ &\quad -sI_e\left[\frac{-a\lambda_1\alpha M^2}{2T} + \delta(a\lambda_1 + b\lambda_2)M\right] < 0 \text{ for the case } M \le T \end{aligned}$$

and

$$
\begin{aligned}
z_1(1) = \; = \; & \frac{-a(c\theta+h)\lambda_1}{T\theta^2}\left[e^{\theta T}-\theta T-1\right]-\frac{acI_k\lambda_1}{T\theta^2}\left[e^{\theta(T-M)}-\theta(T-M)-1\right] \\
& -sI_e\left[\frac{-a\lambda_1\alpha M^2}{2T}-\delta(a\lambda_1+b\lambda_2)M\right]
\end{aligned}
$$

If $z_1(1) > 0$ then, by Intermediate value theorem, there exists unique $\eta_1^*(T) \in (0, 1)$.

(b) Similar to (a).

Appendix. C

Proof for Theorem 6.1.

For the given values of s and N, $TP_2(s,T,N)$ and $TP_3(s,T,N)$ are concave on $T>0$. However, $TP_1(s,T,N)$ is concave on $T>0$ if $A-\frac{sI_e\lambda(s,N)}{2}(M-N)^2>0$. Thus there exists a unique value of T (say $T_1^*(s,N)$) which maximizes $TP_1(T|s,N)$ and T_1^* would satisfy the inequality $0\le M-N\le T$. Since $Z_1(T|s,N)$ is non-increasing on $(0,\infty)$, we have that $Z_1(M-N)\ge Z_1(T_1^*)=0$. That is, $Z_1(M-N)\ge 0$. Now,

$$\begin{aligned}Z_1(M-N) &= A-\frac{sI_e\lambda(s,N)(M-N)^2}{2}-\frac{(c\theta+h)P}{\theta}\left[\frac{(M-N)\lambda(s,N)e^{\theta(M-N)}}{P+\lambda(s,N)(e^{\theta(M-N)}-1)}\right.\\&\left.-\frac{1}{\theta}\log\left(1+\frac{\lambda(s,N)}{P}(e^{\theta(M-N)}-1)\right)\right]\end{aligned}$$

So, $\Delta_1(s,N)=Z_1(M-N)$. If $\Delta_1(s,N)\ge 0$, then $T^*(s,N)=T_1^*$ is the optimal solution of $TP(T|s,N)$.

For the fixed values of s and N, there exists a unique value T (say $T_2^*(s,N)$) which maximizes $TP_2(T|s,N)$ and $T_2^*(s,N)$ would satisfy the condition $0\le T\le M-N$. Since $Z_2(T|s,N)$ is non-increasing on $(0,\infty)$, $Z_2(M-N)\le Z_2(T_2^*)=0$ i.e. $Z_2(M-N)\le 0$.

Now,

$$\begin{aligned}Z_2(M-N) &= A-\frac{sI_e\lambda(s,N)(M-N)^2}{2}-\frac{(c\theta+h)P}{\theta}\left[\frac{(M-N)\lambda(s,N)e^{\theta(M-N)}}{P+\lambda(s,N)(e^{\theta(M-N)}-1)}\right.\\&\left.-\frac{1}{\theta}\log\left(1+\frac{\lambda(s,N)}{P}(e^{\theta(M-N)}-1)\right)\right]\\&= \Delta_1(s,N)\end{aligned}$$

Hence, if $\Delta_1(s,N)\le 0$, then $T^*(s,N)=T_2^*$ is the optimal solution of $TP(T|s,N)$.

For the fixed values of s and N, there exists a unique value T (say $T_3^*(s,N)$) which maximizes $TP_3(T|s,N)$ and T_3^* would satisfy the condition $M-N\le 0\le T$. Since $Z_3(T|s,N)$ is non-increasing on $(0,\infty)$, $Z_3(M-N)\ge Z_3(T_3^*)=0$ i.e. $Z_3(M-N)\ge 0$.

Now,

$$\begin{aligned}Z_3(M-N) = A-\frac{P}{\theta}\left[(c\theta+h)+cI_k\theta(N-M)\right]\left[\frac{(M-N)\lambda(s,N)e^{\theta(M-N)}}{P+D(e^{\theta(M-N)}-1)}\right.\\\left.-\frac{1}{\theta}\log\left(1+\frac{\lambda(s,N)}{P}(e^{\theta(M-N)}-1)\right)\right]-\frac{cI_k\lambda(s,N)}{\theta^2}\left[\theta(M-N)e^{\theta(M-N)}-e^{\theta(M-N)}+1\right]\end{aligned}$$

So, $\Delta_2(s,N)=Z_3(M-N)$. Hence, if $\Delta_2(s,N)\ge 0$, $T^*(s,N)=T_3^*$ is the optimal solution of $TP(T|s,N)$. Combining the three possible cases, we obtain the Theorem.

Appendix D.

The exact solutions for the equations in section 8.2.1 when lead time is exponentially distributed

For the given value of $\delta > 0$ and $\lambda > 0$, let $\rho = \frac{\lambda}{\lambda+\delta}$. The steady state probability that x units on hand at any time t,

$$\begin{aligned} \psi_1(x) &= \sum_{j=1}^{Q} \frac{1}{Q}\left(\frac{\delta}{\delta+\lambda}\right)\left(\frac{\lambda}{\delta+\lambda}\right)^{R+j-x} \quad \text{for } 0 \le x < R+1 \\ &= \frac{1}{Q}\left[\rho^{R-x+1} - \rho^{R+Q-x+1}\right] \end{aligned} \tag{D.1}$$

$$\begin{aligned} \psi_1(x) &= \sum_{j=x-R}^{Q} \frac{1}{Q}\left(\frac{\delta}{\delta+\lambda}\right)\left(\frac{\lambda}{\delta+\lambda}\right)^{R+j-x} \quad \text{for } R+1 \le x \le R+Q \\ &= \frac{1}{Q}\left[1 - \rho^{R+Q-x+1}\right] \end{aligned} \tag{D.2}$$

The steady state probability that y units are backordered

$$\begin{aligned} \psi_2(y) &= \sum_{j=1}^{Q} \frac{1}{Q}\left(\frac{\delta}{\delta+\lambda}\right)\left(\frac{\lambda}{\delta+\lambda}\right)^{R+j+y} \quad \text{for } y \ge 0 \\ &= \frac{1}{Q}\left[\rho^{R+y+1} - \rho^{R+Q+y+1}\right] \end{aligned} \tag{D.3}$$

Expected number of units that are backordered at any time t,

$$\begin{aligned} B(Q,R) &= \sum_{y=0}^{\infty} y\psi_2(y) \\ &= \frac{1}{Q}(\rho^{R+2} - \rho^{R+Q+2})(1-\rho)^{-2} \end{aligned} \tag{D.4}$$

Average stock level

$$\begin{aligned} D(Q,R) &= \sum_{x=0}^{R+Q} x\psi_1(x) \\ &= R + \frac{Q+1}{2} + \frac{1}{Q}\left[(1-\rho^Q)\sum_{j=0}^{R} j\rho^{R-j+1} - R\rho\left(\frac{1-\rho^Q}{1-\rho}\right) - \sum_{j=0}^{Q} j\rho^{Q-j+1}\right] \end{aligned} \tag{D.5}$$

The average stock level in the supplier in terms of Q_r

$$D(Q_0/Q_r, R_0/Q_r) = \frac{R_0}{Q_r} + \frac{1}{2}\left(\frac{Q_0}{Q_r+1}\right) + \frac{1}{Q_0/Q_r}\Bigg[(1-\rho^{Q_0/Q_r})\sum_{j=0}^{R_0/Q_r} j\rho^{\frac{R_0}{Q_r}-j+1}$$
$$-\frac{R_0}{Q_r}\rho\left(\frac{1-\rho^{Q_0/Q_r}}{1-\rho}\right) - \sum_{j=0}^{Q_0/Q_r} j\rho^{Q_0/Q_r-j+1}\Bigg] \quad \text{(D.6)}$$

The average number of backorders in the supplier in terms of Q_r

$$B\left(\frac{Q_0}{Q_r}, \frac{R_0}{Q_r}\right) = \frac{Q_r}{Q_0}\left(\rho^{\frac{R_0}{Q_r}+2} - \rho^{\frac{R_0}{Q_r}+\frac{Q_0}{Q_r}+2}\right)(1-\rho)^{-2} \quad \text{(D.7)}$$

Appendix E.

Finding the left limit TC_l and right limit TC_r of the interval valued supply chain cost function TC in section 8.3.4

Let, $\widetilde{\lambda}_r = (\lambda_{r1}, \lambda_{r2}, \lambda_{r3}) \equiv [\lambda_{rL}, \lambda_{rR}]$, $\widetilde{L}_r = (L_{r1}, L_{r2}, L_{r3}) \equiv [L_{rL}, L_{rR}]$, $\widetilde{L}_0 = (L_{01}, L_{02}, L_{03}) \equiv [L_{0L}, L_{0R}]$, $\widetilde{h}_r = (h_{r1}, h_{r2}, h_{r3}) \equiv [h_{rL}, h_{rR}]$, $\widetilde{h}_0 = (h_{01}, h_{02}, h_{03}) \equiv [h_{0L}, h_{0R}]$, $\widetilde{\pi}_r = (\pi_{r1}, \pi_{r2}, \pi_{r3}) \equiv [\pi_{rL}, \pi_{rR}]$, $\widetilde{\pi}_p = (\pi_{p1}, \pi_{p2}, \pi_{p3}) \equiv [\pi_{pL}, \pi_{pR}]$.

From these input data we have,

$$w \equiv [w_1, w_2]$$

where

$$\begin{aligned}
w_1 &= \lambda_{rL} L_{rL} P_L(R_r + b) - (R_r + b) P_R(R_r + b + 1) \\
w_2 &= \lambda_{rR} L_{rR} P_R(R_r + b) - (R_r + b) P_L(R_r + b + 1) \\
P_L(x) &= 1 - e^{-\lambda_{rL} L_{rL}} \sum_{i=0}^{x-1} \frac{(\lambda_{rR} L_{rR})^i}{i!} \\
P_R(x) &= 1 - e^{-\lambda_{rR} L_{rR}} \sum_{i=0}^{x-1} \frac{(\lambda_{rL} L_{rL})^i}{i!}
\end{aligned}$$

The demand rate at the supplier

$$\widetilde{\lambda}_0 \equiv [\lambda_{0L}, \lambda_{0R}], \text{ where, } \lambda_{0L} = \frac{n\lambda_{rL}}{Q_r + w_2}, \ \lambda_{0R} = \frac{n\lambda_{rR}}{Q_r + w_1}$$

The cost incurred at the supplier

$$\begin{aligned}
\widetilde{C}_0 &\equiv [C_{0L}, C_{0R}] \\
C_{0L} &= h_{0L} \left[(Q_0 + 1)/2 + R_0 - \lambda_{0R} L_{0R} + \frac{\beta_L(R_0) - \beta_R(R_0 + Q_0)}{Q_0} \right] \\
C_{0R} &= h_{0R} \left[(Q_0 + 1)/2 + R_0 - \lambda_{0L} L_{0L} + \frac{\beta_R(R_0) - \beta_L(R_0 + Q_0)}{Q_0} \right] \\
\beta_L(v) &= \frac{(\lambda_{0L} L_{0L})^2}{2} P_{0L}(v) - \lambda_{0R} L_{0R}\, v P_{0R}(v) + \frac{v(v+1)}{2} P_{0L}(v+1) \\
\beta_R(v) &= \frac{(\lambda_{0R} L_{0R})^2}{2} P_{0R}(v) - \lambda_{0L} L_{0L}\, v P_{0L}(v) + \frac{v(v+1)}{2} P_{0R}(v+1)
\end{aligned}$$

$$
\begin{aligned}
P_{0L}(v) &= 1 - e^{-\lambda_{0L}L_{0L}} \sum_{i=0}^{x-1} \frac{(\lambda_{0R}L_{0R})^i}{i!} \\
P_{0R}(v) &= 1 - e^{-\lambda_{0R}L_{0R}} \sum_{i=0}^{x-1} \frac{(\lambda_{0L}L_{0L})^i}{i!}
\end{aligned}
$$

The total cost of the supply chain system,

$$
\begin{aligned}
TC &\equiv [TC_l, TC_r] \\
TC_l &= C_{0L} + n(h_{rL}D_{rL} + \pi_{pL}B_{rL} + \pi_{rL}E_{rL}) \\
TC_r &= C_{0R} + n(h_{rR}D_{rR} + \pi_{pR}B_{rR} + \pi_{rR}E_{rR})
\end{aligned}
$$

where

$$
\begin{aligned}
D_{rL} &= \frac{\lambda_{rL}}{Q_r + w_2}\left[\frac{Q_r(Q_r+1)}{2\lambda_{rR}} + \frac{Q_rR_r}{\lambda_{rR}} - Q_rL_{rR} + \frac{b(b-1) - 2Q_r(R_r+b)}{2\lambda_{rR}}P_L(R_r+b)\right. \\
&\quad + Q_rL_{rL}\,P_L(R_r+b-1) + R_rL_{rL}\,[P_L(R_r+b-1) - P_R(R_r)] \\
&\quad \left. + \frac{\lambda_{rL}L_{rL}^2}{2}[P_L(R_r-1) - P_R(R_r+b-2)] + \frac{R_r(R_r+1)}{2\lambda_{rR}}[P_L(R_r+1) - P_R(R_r+b)]\right] \\
D_{rR} &= \frac{\lambda_{rR}}{Q_r + w_1}\left[\frac{Q_r(Q_r+1)}{2\lambda_{rL}} + \frac{Q_rR_r}{\lambda_{rL}} - Q_rL_{rL} + \frac{b(b-1) - 2Q_r(R_r+b)}{2\lambda_{rL}}P_R(R_r+b)\right. \\
&\quad + Q_rL_{rR}\,P_R(R_r+b-1) + R_rL_{rR}\,[P_R(R_r+b-1) - P_L(R_r)] \\
&\quad \left. + \frac{\lambda_{rR}L_{rR}^2}{2}[P_R(R_r-1) - P_L(R_r+b-2)] + \frac{R_r(R_r+1)}{2\lambda_{rL}}[P_R(R_r+1) - P_L(R_r+b)]\right] \\
B_{rL} &= \frac{\lambda_{rL}}{Q_r + w_2}\left[\lambda_{rL}L_{rL}\,[P_L(R_r) - P_R(R_r+b-1)] - R_rP_L(R_r+1) + (R_r+b)P_L(R_r+b)\right] \\
B_{rR} &= \frac{\lambda_{rR}}{Q_r + w_1}\left[\lambda_{rR}L_{rR}\,[P_R(R_r) - P_L(R_r+b-1)] - R_rP_R(R_r+1) + (R_r+b)P_R(R_r+b)\right] \\
E_{rL} &= \frac{w_1\lambda_{rL}}{Q_r + w_2} \\
E_{rR} &= \frac{w_2\lambda_{rR}}{Q_r + w_1}
\end{aligned}
$$