Chapter 4

Partial trade credit financing in a supply chain with perishable items, two warehouses and selling price dependent demand

4.1 Introduction

In many industrial settings, managers face the problem of establishing the pricing and lot-sizing policies that maximize the revenue from selling a given inventory of items by a fixed deadline, with the full inventory of items being available for sale from the beginning of the selling period. As selling price influences the demand, here we consider selling price dependent demand rather a constant demand. Examples of industries where this proposed inventory problem arises fall broadly into two categories: those where the product is a manufactured good with a limited shelf life (such as food items or fashion garments), and those where the product is a service (such as flight seats or hotel rooms).

Perishable products are commonly found in commerce and industry. Sometimes the rate of deterioration is too low, for items such as steel, hardware, glassware and toys, to cause consideration of deterioration in the determination of economic lot sizes. However, some items have a significant rate of deterioration, such as fruits, fresh fishes, perfumes, alcohol, gasoline and photographic films that deteriorate rapidly over time, which can not be ignored in the decision making process of ordering lot size.

Allowing customers to delay payment for the items is a common business practice.

The retailers in small business rely on trade credit as a source of short-term funds. Once a trade credit has been offered, the amount of time the retailer's capital tied up in stock is reduced and that leads to a reduction in retailer's holding cost. Hence, the retailer may purchase more goods than that can be stored in his own warehouse (W_1) with limited capacity. These excess quantities are stored in a rented warehouse (W_2). The proposed model is applicable for the business of small and medium sized retailers since their storage capacity are small and limited. Especially, countries like Taiwan and India have traditionally relied on its small and medium sized firms to compete in international markets since the 1950s. In general, the inventory holding charges in W_2 are higher than those in W_1. When the demand occurs, first it is replenished from the W_2 which storages those exceeding items. This is done to reduce the inventory costs. It is further assumed that the transportation costs between warehouses are negligible.

To the best of our knowledge, there is no inventory model for a supply chain with perishable items, two-warehouse facility and selling price dependent demand at the retailer under partial trade credit financing. So the chapter fulfills this gap in the literature.

The rest of the chapter is organized as follows. In the next section, the problem is described under various assumptions. In section 4.3, we formulate the model by considering the possible costs and revenues. Section 4.4 proves that the optimal replenishment policy not only exists but also it is unique. In section 4.5, we derive the decision rules to find optimal cycle time. Section 4.6 provides the optimal selling price when cycle time is fixed. In section 4.7, several numerical examples are presented to illustrate the theory. Finally, section 4.8 concludes the chapter.

4.2 Problem description

We consider supplier-retailer supply chain in which supplier fulfills the retailer's demand and retailer fulfills market customers' demands. The retailer is having two warehouses (own warehouse W_1 and rented warehouse W_2) in order to store more purchased items due to trade credit promotional effect offered by the supplier. The inventory holding charge (h_2) in W_2 is higher than the holding charge (h_1) in W_1. When demand occurs at the retailer, he first uses the warehouse W_2. This is done to reduce inventory holding cost. During the time period of length t_1, the demand is met from warehouse W_1. The selling items are perishable such as fruits, fresh fishes, gasoline, photographic films, etc. As pricing is an obvious strategy to influence demand, the price-dependent demand has been considered rather than a constant demand pattern. Supplier is willing to provide the retailer full trade credit period (M) for payments and the retailer offers partial trade credit(up to time N). The retailer's trade credit period offered by the supplier is not necessarily longer than the customer's trade credit period offered by the retailer. Interest payable rate is not necessarily higher than the interest earned rate. The proposed model focuses on the profit maximization instead of the cost minimization.

In addition to the notations in chapter 1, the following notations are used at the retailer of a supply chain. $\lambda(s)$ is annual demand as a function of s; t_1 is the time period during which the W_1 is used to meet demands at the retailer; $TP(s,T)$ is the annual total profit, which is a function of s and T. We describe various assumptions at the retailer of a supply chain.

1. Demand, $\lambda(s)$, is a downward sloping function of s.
2. The sales profit, $(s-c)\lambda(s)$, is a concave function of s.
3. Deterioration time of a product follows the exponential distribution with parameter θ, i.e., the deterioration rate is a constant fraction of the on-hand inventory. It is assumed that the deterioration rate in W_1 is the same as in W_2.
4. Before the settlement of an account, the retailer can use sales revenue to earn the interest. At the end of period M, the credit is settled and the retailer starts paying the interest charges for the items in stock with an annual rate I_k.
5. The retailer offers partial payment scheme at the rate of α to his customer. Then his customer must pay off the remaining balance at the end of period N. Hence the retailer can earn interest with rate I_e from the partial payment up to the time N.

6. Time horizon is infinite.

7. Inventory holding cost is charged only on the amount of undecayed stock.

8. Shortages are not allowed and lead time is negligible.

Under these conditions, we model the retailer's inventory system to determine optimal pricing and lot sizing policies.

4.3 Model formulation

The inventory situation is described as follows. During the time period $[0, T]$, the inventory level is decreased owing to both the selling price dependent demand and deterioration. The inventory level, $I(t)$, at time t can be described by the following differential equation:

$$\frac{dI(t)}{dt} = -\lambda(s) - \theta I(t), \qquad 0 \le t \le T \tag{65}$$

with the condition that $I(t) = 0$ when $t = T$. The solution to Eq.(65) is

$$I(t) = \frac{\lambda(s)}{\theta}\left[e^{\theta(T-t)} - 1\right], \quad 0 \le t \le T \tag{66}$$

Consequently, the order quantity is

$$Q = I(0) = \frac{\lambda(s)}{\theta}\left[e^{\theta T} - 1\right] \tag{67}$$

The capacity of W_1,

$$Z = \frac{\lambda(s)}{\theta}\left[e^{\theta t_1} - 1\right] \tag{68}$$

and

$$t_1(s) = \frac{1}{\theta}\ln\left(1 + \frac{\theta Z}{\lambda(s)}\right) \tag{69}$$

where ln(.) is natural logarithmic function.

Lemma 4.1 $Q \le Z$ if and only if $T \le t_1$.

Proof. From Eqs.(67) and (68), it is easy to prove the result. □

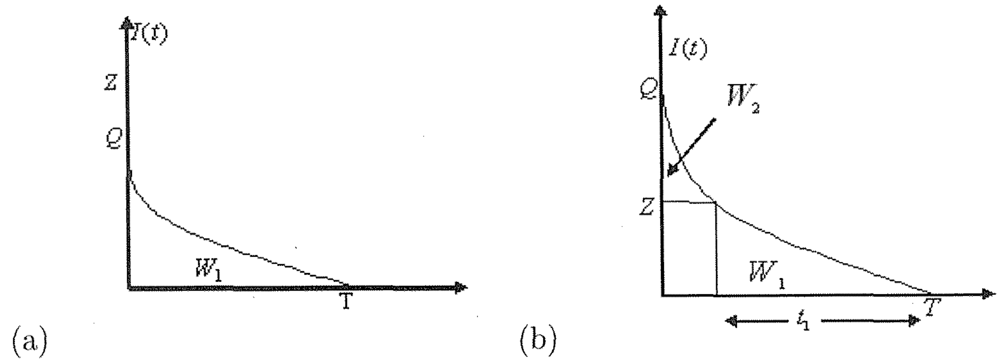

Figure 8: (a) Inventory level when $T \leq t_1$, (b) Inventory level when $T > t_1$

We now derive sales profit, cost of placing orders, deteriorating and stock-holding.

(1) Sales profit (SP)$= (s-c)\lambda(s)$

(2) Cost of placing orders (OC) $= A/T$.

(3) Cost of deteriorated units (DC)

$$
\begin{aligned}
&= \frac{c}{T}[Q-\lambda(s)T] \\
&= \frac{c\theta\lambda(s)}{T\theta^2}\left[e^{\theta T}-\theta T-1\right]
\end{aligned}
$$

(4) Cost of carrying inventory (HC_1) in the warehouse W_1,

Case 1: $T \leq t_1$, shown in Fig. 8 (a)

$$
\begin{aligned}
HC_1 &= \frac{h_1}{T}\int_0^T I(t)\ \mathrm{dt} \\
&= \frac{h_1\lambda(s)}{\theta^2 T}\left[e^{\theta T}-\theta T-1\right]
\end{aligned}
$$

Case 2: $T > t_1$, shown in Fig. 8 (b)

$$
\begin{aligned}
HC_1 &= \frac{h_1}{T}\left[Z(T-t_1)+\int_{T-t_1}^{T} I(t)\ \mathrm{dt}\right] \\
&= \frac{h_1\lambda(s)}{T\theta^2}\left[e^{\theta t_1}(\theta T-\theta t_1+1)-(1+\theta T)\right]
\end{aligned}
$$

(5) Cost of carrying inventory (HC_2) in the warehouse W_2, shown in Figs. 8 (a)-(b)

Case 1: $T \leq t_1$

There is no need of having W_2 since $Q \leq Z$

Case 2: $T > t_1$

$$\begin{aligned} HC_2 &= \frac{h_2}{T}\int_0^{T-t_1} (I(t) - Z)\ \mathrm{dt} \\ &= \frac{h_2\lambda(s)}{T\theta^2}\left[e^{\theta T} + e^{\theta t_1}(\theta t_1 - \theta T - 1)\right] \end{aligned}$$

The above costs are same in both the cases $M < N$ and $M \geq N$. But there are significant differences in interest earned and interest payable.

4.3.1 Total cost of the inventory when $M < N$

When $T \geq M$ the account is settled at time M and after that the interest is paid for the items in stock; If $T < M$, the retailer does not need to pay any interest charge. The annual interest payable (IP) is calculated for the above cases.

Case 1: $M \leq T$

$$\begin{aligned} IP &= \frac{cI_k}{T}\int_M^{T} I(t)\ \mathrm{dt} \\ &= \frac{cI_k\lambda(s)}{T\theta^2}\left[e^{\theta(T-M)} - \theta(T-M) - 1\right] \end{aligned}$$

Case 2: $M > T$

In this case, $IP = 0$.

According to assumption (5), there are two cases that occur in costs of interest earned (IE) per year

Case 1: $M \leq T$, shown in Fig. 7 (a) of chapter 3

$$\begin{aligned} IE &= \frac{sI_e\alpha}{T}\int_0^{M} \lambda(s)t\ \mathrm{dt} \\ &= \frac{sI_e\alpha\lambda(s)M^2}{2T} \end{aligned}$$

Case 2: $M > T$, shown in Fig 7(b) of chapter 3

$$\begin{aligned} IE &= \frac{sI_e\alpha}{T}\left[\int_0^{T} \lambda(s)t\ \mathrm{dt} + \int_T^{M} \lambda(s)T\ \mathrm{dt}\right] \\ &= sI_e\alpha\lambda(s)[M - T/2] \end{aligned}$$

The annual total profit gained by the retailer as a function of s and T,

$$\begin{aligned} TP(s,T) &= \text{Sales profit} - \text{Ordering cost} - \text{Cost of deteriorated items} \\ &\quad - \text{Holding cost at } W_1 - \text{ Holding cost at } W_2 - \text{Interest payable} \\ &\quad + \text{Interest earned} \\ &= (SP) - (OC) - (DC) - (HC_1) - (HC_2) - (IP) + (IE) \end{aligned} \tag{70}$$

From the above arguments, we obtain the following results:

(I) When $t_1 \leq M$

$$TP(s,T) = \begin{cases} TP_1(s,T) & \text{if } 0 < T \leq t_1 \\ TP_2(s,T) & \text{if } t_1 < T < M \\ TP_3(s,T) & \text{if } T \geq M \end{cases} \tag{71}$$

where

$$TP_1(s,T) = (s-c)\lambda(s) - \frac{A}{T} - \frac{(c\theta + h_1)\lambda(s)}{T\theta^2}\left[e^{\theta T} - \theta T - 1\right] + sI_e\alpha\lambda(s)[M - T/2] \tag{72}$$

$$\begin{aligned} TP_2(s,T) &= (s-c)\lambda(s) - \frac{A}{T} - \frac{(c\theta + h_2)\lambda(s)}{T\theta^2}\left[e^{\theta T} - \theta T - 1\right] \\ &\quad + \frac{(h_2 - h_1)\lambda(s)}{\theta^2 T}\left[(\theta T - \theta t_1 + 1)e^{\theta t_1} - (1 + \theta T)\right] + sI_e\alpha\lambda(s)[M - T/2] \end{aligned} \tag{73}$$

$$\begin{aligned} TP_3(s,T) &= (s-c)\lambda(s) - \frac{A}{T} - \frac{(c\theta + h_2)\lambda(s)}{T\theta^2}\left[e^{\theta T} - \theta T - 1\right] \\ &\quad + \frac{(h_2 - h_1)\lambda(s)}{\theta^2 T}\left[(\theta T - \theta t_1 + 1)e^{\theta t_1} - (1 + \theta T)\right] \\ &\quad - \frac{cI_k\lambda(s)}{\theta^2 T}\left[e^{\theta(T-M)} - \theta(T - M) - 1\right] + sI_e\alpha\lambda(s)M^2/2T \end{aligned} \tag{74}$$

It is easy to see that $TP_1(s,t_1) = TP_2(s,t_1)$ and $TP_2(s,M) = TP_3(s,M)$. Hence, $TP(s,T)$ is continuous and well-defined.

(II) When $t_1 > M$

$$TP(s,T) = \begin{cases} TP_1(s,T) & \text{if } 0 < T \leq M \\ TP_4(s,T) & \text{if } M < T < t_1 \\ TP_3(s,T) & \text{if } T \geq t_1 \end{cases} \tag{75}$$

where

$$
\begin{aligned}
TP_4(s,T) &= (s-c)\lambda(s) - \frac{A}{T} - \frac{(c\theta + h_1)\lambda(s)}{T\theta^2}\left[e^{\theta T} - \theta T - 1\right] \\
&\quad -\frac{cI_k\lambda(s)}{\theta^2 T}\left[e^{\theta(T-M)} - \theta(T-M) - 1\right] + sI_e\alpha\lambda(s)M^2/2T
\end{aligned}
$$

It is easy to see that $TP_1(s,M) = TP_4(s,M)$ and $TP_4(s,t_1) = TP_3(s,t_1)$. Hence, $TP(s,T)$ is continuous and well-defined.

4.3.2 Total cost of the inventory when $M \geq N$

There are three cases that occur in costs of interest payable (IP) for the items kept in stock per year.

Case 1: $M \leq T$

$$
\begin{aligned}
IP &= \frac{cI_k}{T}\int_M^T I(t)\ \mathrm{dt} \\
&= \frac{cI_k\lambda(s)}{\theta^2 T}\left[e^{\theta(T-M)} - \theta(T-M) - 1\right]
\end{aligned}
$$

Case 2: $N < T < M$

In this case, $IP = 0$

Case 3: $T \leq N$

In this case, $IP = 0$

According to assumption (5), there are three cases that occur in costs of interest earned (IE) per year.

Case 1: $M \leq T$, shown in Fig. 5 (a) of chapter 3

$$
\begin{aligned}
IE &= \frac{sI_e}{T}\left[\alpha\int_0^N \lambda(s)t\ \mathrm{dt} + \int_N^M \lambda(s)t\ \mathrm{dt}\right] \\
&= \frac{sI_e\lambda(s)}{2T}\left[M^2 - (1-\alpha)N^2\right]
\end{aligned}
$$

Case 2: $N < T < M$, shown in Fig. 5 (b) of chapter 3

$$
\begin{aligned}
IE &= \frac{sI_e}{T}\left[\alpha\int_0^N \lambda(s)t\ \mathrm{dt} + \int_N^T \lambda(s)t\ \mathrm{dt} + \int_T^M \lambda(s)T\ \mathrm{dt}\right] \\
&= \frac{sI_e\lambda(s)}{2T}\left[2MT - (1-\alpha)N^2 - T^2\right]
\end{aligned}
$$

Case 3: $T \leq N$, shown in Fig. 6 of chapter 3

$$\begin{aligned} IE &= \frac{sI_e}{T}\left[\alpha \int_0^T \lambda(s)t \text{ dt } + \alpha \int_T^N \lambda(s)T \text{ dt } + \int_N^M \lambda(s)T \text{ dt}\right] \\ &= sI_e\lambda(s)\left[M - (1-\alpha)N - \frac{\alpha T}{2}\right] \end{aligned}$$

From the above arguments, we obtained the following results.

(I) When $t_1 < N$

$$TP(s,T) = \begin{cases} TP_5(s,T) & \text{if } 0 < T \leq t_1 \\ TP_6(s,T) & \text{if } t_1 < T \leq N \\ TP_7(s,T) & \text{if } N < T \leq M \\ TP_8(s,T) & \text{if } T > M \end{cases} \tag{76}$$

where

$$TP_5(s,T) = (s-c)\lambda(s) - \frac{A}{T} - \frac{(c\theta + h_1)\lambda(s)}{T\theta^2}\left[e^{\theta T} - \theta T - 1\right] + sI_e\lambda(s)[M - (1-\alpha)N - \alpha T/2] \tag{77}$$

$$\begin{aligned} TP_6(s,T) &= (s-c)\lambda(s) - \frac{A}{T} - \frac{(c\theta + h_2)\lambda(s)}{T\theta^2}\left[e^{\theta T} - \theta T - 1\right] + sI_e\lambda(s)[M - (1-\alpha)N - \alpha T/2] \\ &\quad + \frac{(h_2 - h_1)\lambda(s)}{\theta^2 T}\left[(\theta T - \theta t_1 + 1)e^{\theta t_1} - (\theta T + 1)\right] \end{aligned} \tag{78}$$

$$\begin{aligned} TP_7(s,T) &= (s-c)\lambda(s) - \frac{A}{T} - \frac{(c\theta + h_2)\lambda(s)}{T\theta^2}\left[e^{\theta T} - \theta T - 1\right] + \frac{sI_e\lambda(s)}{2T}[2MT - (1-\alpha)N^2 - T^2] \\ &\quad + \frac{(h_2 - h_1)\lambda(s)}{\theta^2 T}\left[(\theta T - \theta t_1 + 1)e^{\theta t_1} - (\theta T + 1)\right] \end{aligned} \tag{79}$$

$$\begin{aligned} TP_8(s,T) &= (s-c)\lambda(s) - \frac{A}{T} - \frac{(c\theta + h_2)\lambda(s)}{T\theta^2}\left[e^{\theta T} - \theta T - 1\right] + \frac{sI_e\lambda(s)}{2T}[M^2 - (1-\alpha)N^2] \\ &\quad + \frac{(h_2 - h_1)\lambda(s)}{\theta^2 T}\left[(\theta T - \theta t_1 + 1)e^{\theta t_1} - (\theta T + 1)\right] - \frac{cI_k\lambda(s)}{\theta^2 T}\left[e^{\theta(T-M)} - \theta(T-M) - 1\right] \end{aligned} \tag{80}$$

It is easy to see that $TP_5(s,t_1) = TP_6(s,t_1)$, $TP_6(s,N) = TP_7(s,N)$ and $TP_7(s,M) = TP_8(s,M)$. Hence, $TP(s,T)$ is continuous and well-defined.

(II) When $N \leq t_1 \leq M$

$$TP(s,T) = \begin{cases} TP_5(s,T) & \text{if } 0 < T \leq N \\ TP_9(s,T) & \text{if } N < T \leq t_1 \\ TP_7(s,T) & \text{if } t_1 < T \leq M \\ TP_8(s,T) & \text{if } T > M \end{cases} \tag{81}$$

where

$$TP_9(s,T) = (s-c)\lambda(s) - \frac{A}{T} - \frac{(c\theta + h_1)\lambda(s)}{T\theta^2}\left[e^{\theta T} - \theta T - 1\right] + \frac{sI_e\lambda(s)}{2T}[2MT - (1-\alpha)N^2 - T^2] \tag{82}$$

It is easy to see that $TP_5(s,N) = TP_9(s,N)$, $TP_9(s,t_1) = TP_7(s,t_1)$ and $TP_7(s,M) = TP_8(s,M)$. Hence, $TP(s,T)$ is continuous and well-defined.

(III) When $M < t_1$

$$TP(s,T) = \begin{cases} TP_5(s,T) & \text{if } 0 < T \le N \\ TP_9(s,T) & \text{if } N < T \le M \\ TP_{10}(s,T) & \text{if } M < T \le t_1 \\ TP_8(s,T) & \text{if } T > t_1 \end{cases} \tag{83}$$

where

$$\begin{aligned} TP_{10}(s,T) \quad = \quad & (s-c)\lambda(s) - \frac{A}{T} - \frac{(c\theta + h_1)\lambda(s)}{T\theta^2}\left[e^{\theta T} - \theta T - 1\right] - \frac{cI_k\lambda(s)}{\theta^2 T}\left[e^{\theta(T-M)} - \theta(T-M) - 1\right] \\ & + \frac{sI_e\lambda(s)}{2T}[M^2 - (1-\alpha)N^2] \end{aligned} \tag{84}$$

It is easy to see that $TP_5(s,N) = TP_9(s,N)$, $TP_9(s,M) = TP_{10}(s,M)$ and $TP_{10}(s,t_1) = TP_8(s,t_1)$. Hence, $TP(s,T)$ is continuous and well-defined.

4.4 Existence of unique optimal solution T^* when s is fixed

Lemma 4.2.

(a) $x^2e^x - 2xe^x + 2e^x - 2 > 0$ is an increasing function of x on $(0,\infty)$.

(b) $2e^x - x^2 - 2x - 2 > 0$ is an increasing function of x on $(0,\infty)$.

(c) $xe^x - e^x + 1 > 0$ is an increasing function of x on $(0,\infty)$.

Proof. (a),(b) and (c) can be proved from the first and second derivatives of the functions. Since the function's value at $x = 0$ is zero, these functions are positive on the interval $(0,\infty)$. □

4.4.1 When $M < N$

The first order and second order partial derivatives of $TP_1(s,T)$ with respect to

T are

$$\frac{\partial TP_1(s,T)}{\partial T} = \frac{A}{T^2} - \frac{\lambda(s)(h_1+c\theta)}{\theta^2 T^2}\left[\theta T e^{\theta T} - e^{\theta T} + 1\right] - \frac{sI_e\alpha\lambda(s)}{2} \quad (85)$$

$$\frac{\partial^2 TP_1(s,T)}{\partial T^2} = -\left[\frac{2A}{T^3} + \frac{\lambda(s)(h_1+c\theta)}{\theta^2 T^3}\left[(\theta T)^2 e^{\theta T} - 2\theta T e^{\theta T} + 2e^{\theta T} - 2\right]\right] < 0 \quad (86)$$

Eq.(86) holds from (a) of Lemma 4.2. Hence, $TP_1(T|s)$ is a concave function on $(0,\infty)$ and $\frac{\partial TP_1}{\partial T} = 0$ has unique solution $T_1^*(s)$. The first order and second order partial derivatives of $TP_2(s,T)$ with respect to T are

$$\frac{\partial TP_2(s,T)}{\partial T} = \frac{A}{T^2} - \frac{\lambda(s)(h_2+c\theta)}{\theta^2 T^2}\left[\theta T e^{\theta T} - e^{\theta T} + 1\right] + \frac{\lambda(s)(h_2-h_1)}{\theta^2 T^2}\left[\theta t_1 e^{\theta t_1} - e^{\theta t_1} + 1\right] - \frac{sI_e\alpha\lambda(s)}{2} \quad (87)$$

$$\frac{\partial^2 TP_2(s,T)}{\partial T^2} = -\left[2AT^3 + \frac{\lambda(s)(h_2+c\theta)}{\theta^2 T^3}\left[(\theta T)^2 e^{\theta T} - 2\theta T e^{\theta T} + 2e^{\theta T} - 2\right] + \frac{2\lambda(s)(h_2-h_1)}{\theta^2 T^3}\left[\theta t_1 e^{\theta t_1} - e^{\theta t_1} + 1\right]\right] \quad (88)$$

From (a) and (c) of Lemma 4.2, we have $\frac{\partial^2 TP_2(s,T)}{\partial T^2} < 0$. Therefore, $TP_2(T|s)$ is a concave function on $(0,\infty)$ and $\frac{\partial TP_2(s,T)}{\partial T} = 0$ has unique solution $T_2^*(s)$ on $(0,\infty)$. The first order partial derivative of $TP_3(s,T)$ with respect to T is

$$\frac{\partial TP_3(s,T)}{\partial T} = \frac{A}{T^2} - \frac{\lambda(s)(h_2+c\theta)}{\theta^2 T^2}\left[\theta T e^{\theta T} - e^{\theta T} + 1\right] + \frac{\lambda(s)(h_2-h_1)}{\theta^2 T^2}\left[\theta t_1 e^{\theta t_1} - e^{\theta t_1} + 1\right] - \frac{cI_k\lambda(s)}{\theta^2 T^2}\left[\theta T e^{\theta(T-M)} - e^{\theta(T-M)} - \theta M + 1\right] - \frac{sI_e\alpha\lambda(s)M^2}{2T^2} \quad (89)$$

For a fixed value of s, let

$$f_1(T) = 2A\theta^2 - 2\lambda(s)(h_2+c\theta)\left[\theta T e^{\theta T} - e^{\theta T} + 1\right] + 2\lambda(s)(h_2-h_1)\left[\theta t_1 e^{\theta t_1} - e^{\theta t_1} + 1\right] - 2cI_k\lambda(s)\left[\theta T e^{\theta(T-M)} - e^{\theta(T-M)} - \theta M + 1\right] - sI_e\alpha\lambda(s)\theta^2 M^2$$

We see that $f_1(T)$ and $\frac{\partial TP_3}{\partial T}$ have same sign and domain. The first order derivative $f_1'(T) = -2\lambda(s)(h_2+c\theta)\theta^2 T e^{\theta T} - 2\lambda(s)cI_k\theta^2 T e^{\theta(T-M)} < 0$. Hence, $f_1(T)$ is decreasing on $[M,\infty)$ and $\lim_{T\to\infty} f_1(T) = -\infty$. Suppose that $\frac{\partial TP_3}{\partial T}|_{T=M} > 0$, we have $f_1(M) > 0$ then the Intermediate Value Theorem yields that $f_1(T) = 0$ has unique solution $T_3^*(s)$ on $[M,\infty)$. Suppose that $\frac{\partial TP_3}{\partial T}|_{T=M} \le 0$, we have $f_1(M) \le 0$. Since $f_1(T)$ is decreasing on $[M,\infty)$, we have $f_1(T) \le f_1(M) \le 0$ for $T \ge M$; that is $\frac{\partial TP_3}{\partial T} \le 0$ in the interval $[M,\infty)$. Hence $TP_3(T)$ is decreasing on $[M,\infty)$. So $T_3^* = M$.

The first order partial derivative of $TP_4(s,T)$ with respect to T is

$$\frac{\partial TP_4(s,T)}{\partial T} = \frac{A}{T^2} - \frac{\lambda(s)(h_1+c\theta)}{\theta^2 T^2}\left[\theta T e^{\theta T} - e^{\theta T} + 1\right] - \frac{cI_k\lambda(s)}{\theta^2 T^2}\left[\theta T e^{\theta(T-M)} - e^{\theta(T-M)} - \theta M + 1\right] - \frac{sI_e\alpha\lambda(s)M^2}{2T^2} \quad (90)$$

For a fixed value of s, let

$$f_2(T) = 2A\theta^2 - 2\lambda(s)(h_1 + c\theta)\left[\theta T e^{\theta T} - e^{\theta T} + 1\right] - 2cI_k\lambda(s)\left[\theta T e^{\theta(T-M)} - e^{\theta(T-M)} - \theta M + 1\right]$$
$$- sI_e\alpha\lambda(s)\theta^2 M^2$$

By the similar argument as above, we can prove the existence of unique solution $T_4^*(s)$ which is obtained by solving the equation $f_2(T) = 0$.

4.4.2 When $M \geq N$

The first order and second order partial derivatives of $TC_5(s,T)$ with respect to T are

$$\frac{\partial TP_5(s,T)}{\partial T} = \frac{A}{T^2} - \frac{\lambda(s)(h_1 + c\theta)}{\theta^2 T^2}\left[\theta T e^{\theta T} - e^{\theta T} + 1\right] - \frac{sI_e\alpha\lambda(s)}{2} \quad (91)$$
$$\frac{\partial^2 TP_5(s,T)}{\partial T^2} = \frac{-2A}{T^3} - \frac{\lambda(s)(h_1 + c\theta)}{\theta^2 T^3}\left[(\theta T)^2 e^{\theta T} - 2\theta T e^{\theta T} + 2e^{\theta T} - 2\right] < 0 \quad (92)$$

Eq.(92) holds from (a) of Lemma 4.2. Hence, $TP_5(T|s)$ is a concave function on $(0,\infty)$ and $\frac{\partial TP_5}{\partial T} = 0$ has unique solution $T_5^*(s)$ on $(0,\infty)$. The first order and second order partial derivatives of $TP_6(s,T)$ with respect to T are

$$\frac{\partial TP_6(s,T)}{\partial T} = \frac{A}{T^2} - \frac{\lambda(s)(h_2 + c\theta)}{\theta^2 T^2}\left[\theta T e^{\theta T} - e^{\theta T} + 1\right] + \frac{\lambda(s)(h_2 - h_1)}{\theta^2 T^2}\left[\theta t_1 e^{\theta t_1} - e^{\theta t_1} + 1\right]$$
$$- \frac{sI_e\alpha\lambda(s)}{2} \quad (93)$$
$$\frac{\partial^2 TP_6(s,T)}{\partial T^2} = -\left[\frac{2A}{T^3} + \frac{\lambda(s)(h_2 + c\theta)}{\theta^2 T^3}\left[(\theta T)^2 e^{\theta T} - 2\theta T e^{\theta T} + 2e^{\theta T} - 2\right]\right.$$
$$\left. + \frac{2\lambda(s)(h_2 - h_1)}{\theta^2 T^3}\left[\theta t_1 e^{\theta t_1} - e^{\theta t_1} + 1\right]\right] \quad (94)$$

From (a) and (c) of Lemma 4.2, we have $\frac{\partial^2 TP_6(s,T)}{\partial T^2} < 0$. Hence, $TP_6(T|s)$ is a concave function on $(0,\infty)$ and $\frac{\partial TP_6(s,T)}{\partial T} = 0$ has unique solution $T_6^*(s)$ on $(0,\infty)$. The first and second order partial derivatives of $TP_7(s,T)$ with respect to T are

$$\frac{\partial TP_7(s,T)}{\partial T} = \frac{A}{T^2} - \frac{\lambda(s)(h_2 + c\theta)}{\theta^2 T^2}\left[\theta T e^{\theta T} - e^{\theta T} + 1\right] + \frac{\lambda(s)(h_2 - h_1)}{\theta^2 T^2}\left[\theta t_1 e^{\theta t_1} - e^{\theta t_1} + 1\right]$$
$$- \frac{sI_e\lambda(s)}{2} + \frac{sI_e\lambda(s)N^2(1-\alpha)}{2T^2} \quad (95)$$
$$\frac{\partial^2 TP_7(s,T)}{\partial T^2} = -\left[\frac{2A}{T^3} + \frac{\lambda(s)(h_2 + c\theta)}{\theta^2 T^3}\left[(\theta T)^2 e^{\theta T} - 2\theta T e^{\theta T} + 2e^{\theta T} - 2\right]\right.$$
$$\left. + \frac{2\lambda(s)(h_2 - h_1)}{\theta^2 T^3}\left[\theta t_1 e^{\theta t_1} - e^{\theta t_1} + 1\right] + \frac{\lambda(s)sI_e N^2(1-\alpha)}{T^3}\right] \quad (96)$$

From (a) and (c) of Lemma 4.2, we have $\frac{\partial^2 TP_7(s,T)}{\partial T^2} < 0$. Hence, $TP_7(T|s)$ is a concave function on $(0,\infty)$ and $\frac{\partial TP_7(s,T)}{\partial T} = 0$ has the unique solution $T_7^*(s)$ on $(0,\infty)$.

The first order partial derivatives of $TP_8(s,T)$ with respect to T is

$$\frac{\partial TP_8(s,T)}{\partial T} = \frac{A}{T^2} - \frac{\lambda(s)(h_2+c\theta)}{\theta^2 T^2}\left[\theta T e^{\theta T} - e^{\theta T} + 1\right] + \frac{\lambda(s)(h_2-h_1)}{\theta^2 T^2}\left[\theta t_1 e^{\theta t_1} - e^{\theta t_1} + 1\right]$$
$$- \frac{cI_k\lambda(s)}{\theta^2 T^2}\left[\theta T e^{\theta(T-M)} - e^{\theta(T-M)} - \theta M + 1\right] - \frac{sI_e\lambda(s)}{2T^2}\left[M^2 - (1-\alpha)N^2\right] \quad (97)$$

For a fixed value of s, let

$$f_3(T) = 2A\theta^2 - 2\lambda(s)(h_2+c\theta)\left[\theta T e^{\theta T} - e^{\theta T} + 1\right] + 2\lambda(s)(h_2-h_1)\left[\theta t_1 e^{\theta t_1} - e^{\theta t_1} + 1\right]$$
$$- 2cI_k\lambda(s)\left[\theta T e^{\theta(T-M)} - e^{\theta(T-M)} - \theta M + 1\right] - sI_e\lambda(s)\theta^2[M^2 - (1-\alpha)N^2]$$

then we have $f_3'(T) = -2\lambda(s)(h_2+c\theta)\theta^2 T e^{\theta T} - 2\lambda(s)cI_k\theta^2 T e^{\theta(T-M)} < 0$. Hence $f_3(T)$ is decreasing on $[M,\infty)$ and $\lim_{T\to\infty} f_3(T) = -\infty$. Suppose that $\frac{\partial TP_8}{\partial T}|_{T=M} > 0$, we have $f_3(M) > 0$ then the Intermediate Value Theorem yields that $f_3(T) = 0$ has the unique solution $T_8^*(s)$ on $[M,\infty)$. Suppose that $\frac{\partial TP_8}{\partial T}|_{T=M} \le 0$, we have $f_3(M) \le 0$. Since $f_3(T)$ is decreasing on $[M,\infty)$, we have $f_3(T) < f_3(M) \le 0$ for $T \ge M$. Since $f_3(T)$ and $\frac{\partial TP_8}{dT}$ have same sign and domain, $\frac{\partial TP_8}{dT} \le 0$. Hence $TP_8(T)$ is decreasing on $[M,\infty)$. So $T_8^* = M$.

By similar approach as above, we can show the existence of unique solutions $T_9^*(s)$ and $T_{10}^*(s)$ for the equations $\frac{\partial TP_9(s,T)}{\partial T} = 0$ and $\frac{\partial TP_{10}(s,T)}{\partial T} = 0$ respectively. The first order partial derivatives of $TP_9(s,T)$ and $TP_{10}(s,T)$ are

$$\frac{\partial TP_9(s,T)}{\partial T} = \frac{A}{T^2} - \frac{\lambda(s)(h_1+c\theta)}{\theta^2 T^2}\left[\theta T e^{\theta T} - e^{\theta T} + 1\right] - \frac{sI_e\lambda(s)}{2} + \frac{sI_e\lambda(s)N^2(1-\alpha)}{2T^2} \quad (98)$$

$$\frac{\partial TP_{10}(s,T)}{\partial T} = \frac{A}{T^2} - \frac{\lambda(s)(h_1+c\theta)}{\theta^2 T^2}\left[\theta T e^{\theta T} - e^{\theta T} + 1\right]$$
$$- \frac{cI_k\lambda(s)}{\theta^2 T^2}\left[\theta T e^{\theta(T-M)} - e^{\theta(T-M)} - \theta M + 1\right] - \frac{sI_e\lambda(s)}{2T^2}\left[M^2 - (1-\alpha)N^2\right] \quad (99)$$

4.5 Decision rules to find the optimal cycle time T^* when s is fixed

4.5.1 When $M < N$

Let

$$\delta_{11}(s) = \lambda(s)t_1^2(h_1 + c\theta + sI_e\alpha)$$
$$\delta_{12}(s) = \lambda(s)[(h_2 + c\theta + sI_e\alpha)M^2 - (h_2 - h_1)t_1^2]$$
$$\delta_{21}(s) = \lambda(s)M^2(h_1 + c\theta + sI_e\alpha)$$
$$\delta_{22}(s) = \lambda(s)[(h_1 + c\theta)t_1^2 + cI_k(t_1^2 - M^2) + sI_e\alpha M^2].$$

It is easy to see that the inequalities $\delta_{11}(s) < \delta_{12}(s)$ and $\delta_{21}(s) < \delta_{22}(s)$ are satisfied for the cases $t_1 \leq M$ and $t_1 > M$ respectively.

Theorem 4.1 For a fixed value of s and $t_1 \leq M$,

(a) if $\delta_{11}(s) \geq 2A$ then $T^* = T_1^*(s)$,

(b) if $\delta_{11}(s) < 2A < \delta_{12}(s)$ then $T^* = T_2^*(s)$,

(c) if $\delta_{12}(s) \leq 2A$ then $T^* = T_3^*(s)$.

Proof. Please refer to appendix A1. □

Theorem 4.2 For a fixed value of s and $t_1 > M$,

(a) if $\delta_{21}(s) \geq 2A$ then $T^* = T_1^*(s)$,

(b) if $\delta_{21}(s) < 2A < \delta_{22}(s)$ then $T^* = T_4^*(s)$,

(c) if $\delta_{22}(s) \leq 2A$ then $T^* = T_3^*(s)$.

Proof. Please refer to appendix A2. □

4.5.2 When $M \geq N$

Let

$$\begin{aligned}
\delta_{31}(s) &= \lambda(s)t_1^2(h_1 + c\theta + sI_e\alpha) \\
\delta_{32}(s) &= \lambda(s)[h_2(N^2 - t_1^2) + h_1 t_1^2 + N^2(c\theta + sI_e\alpha)] \\
\delta_{33}(s) &= \lambda(s)[c\theta M^2 + sI_e(M^2 - N^2(1-\alpha)) + h_2(M^2 - t_1^2) + h_1 t_1^2] \\
\delta_{41}(s) &= \lambda(s)N^2(h_1 + c\theta + sI_e\alpha) \\
\delta_{42}(s) &= \lambda(s)[t_1^2(h_1 + c\theta) + sI_e(t_1^2 - (1-\alpha)N^2)] \\
\delta_{43}(s) &= \lambda(s)[c\theta M^2 + sI_e(M^2 - N^2(1-\alpha)) + h_2(M^2 - t_1^2) + h_1 t_1^2]
\end{aligned}$$

It is easy to see that the inequalities $\delta_{31}(s) < \delta_{32}(s) < \delta_{33}(s)$ and $\delta_{41}(s) < \delta_{42}(s) < \delta_{43}(s)$ are satisfied for the cases $t_1 < N$ and $N \leq t_1 \leq M$ respectively. Let

$$\begin{aligned}
\delta_{51}(s) &= \lambda(s)N^2(h_1 + c\theta + sI_e\alpha) \\
\delta_{52}(s) &= \lambda(s)[M^2(h_1 + c\theta) + sI_e(M^2 - N^2(1-\alpha))] \\
\delta_{53}(s) &= \lambda(s)[t_1^2(h_1 + c\theta) + cI_k(t_1^2 - M^2) + sI_e(M^2 - N^2(1-\alpha))]
\end{aligned}$$

It is easy to see that the inequality $\delta_{51}(s) < \delta_{52}(s) < \delta_{53}(s)$ when $t_1 > M$.

Theorem 4.3 For a fixed value of s and $t_1 < N$,

(a) if $\delta_{31}(s) \geq 2A$ then $T^* = T_5^*(s)$,

(b) if $\delta_{31}(s) < 2A < \delta_{32}(s)$ then $T^* = T_6^*(s)$,

(c) if $\delta_{32}(s) \leq 2A < \delta_{33}(s)$ then $T^* = T_7^*(s)$,

(d) $\delta_{33}(s) \leq 2A$ implies $T^* = T_8^*(s)$.

Proof. Please refer to appendix A3. □

Theorem 4.4 For a fixed value of s and $N \leq t_1 \leq M$,

(a) if $\delta_{41}(s) \geq 2A$ then $T^* = T_5^*(s)$,

(b) if $\delta_{41}(s) < 2A < \delta_{42}(s)$ then $T^* = T_9^*(s)$,

(c) if $\delta_{42}(s) \leq 2A < \delta_{43}(s)$ then $T^* = T_7^*(s)$,

(d) $\delta_{43}(s) \leq 2A$ implies $T^* = T_8^*(s)$.

Proof. Please refer to appendix A4. □

Theorem 4.5. For a fixed value of s and $t_1 > M$, we have

(a) If $\delta_{51}(s) \geq 2A$ then $T^* = T_5^*(s)$,

(b) If $\delta_{51}(s) < 2A < \delta_{52}(s)$ then $T^* = T_9^*(s)$,

(c) If $\delta_{52}(s) \leq 2A < \delta_{53}(s)$ then $T^* = T_{10}^*(s)$,

(d) $\delta_{53}(s) \leq 2A$ implies $T^* = T_8^*(s)$.

Proof. Please refer to appendix A5. □

4.6 Determination of optimal selling price s^* for the given value of T

For any given value of T, the optimal value of s can be determined by solving the first order necessary condition (i.e., $\frac{\partial TP}{\partial s} = 0$) and examining the second order sufficient condition for the concavity (i.e., $\frac{\partial^2 TP}{\partial s^2} < 0$).

4.6.1 When $M < N$

The first and second order partial derivatives of TP_i $(i = 1, 2, 3, 4)$ are as follows.

$$\frac{\partial TP_1}{\partial s} = (s-c)\lambda'(s) + \lambda(s) - \frac{(c\theta + h_1)\lambda'(s)}{\theta^2 T}[e^{\theta T} - \theta T - 1] + I_e\alpha(M - T/2)[s\lambda'(s) + \lambda(s)] \quad (100)$$

$$\frac{\partial^2 TP_1}{\partial s^2} = (s-c)\lambda''(s) + 2\lambda'(s) - \frac{(c\theta + h_1)\lambda''(s)}{\theta^2 T}[e^{\theta T} - \theta T - 1] + I_e\alpha(M - T/2)[s\lambda''(s) + 2\lambda'(s)] \quad (101)$$

$$
\begin{aligned}
\frac{\partial TP_2}{\partial s} &= (s-c)\lambda'(s) + \lambda(s) - \frac{(c\theta + h_2)\lambda'(s)}{\theta^2 T}[e^{\theta T} - \theta T - 1] \\
&+ \frac{(h_2 - h_1)\lambda'(s)}{\theta^2 T}[(\theta T - \theta t_1 + 1)e^{\theta t_1} - (1 + \theta T)] + \frac{(h_2 - h_1)\lambda(s)}{T}\left[(T - t_1)e^{\theta t_1}\frac{\partial t_1}{\partial s}\right] \\
&+ I_e\alpha(M - T/2)[s\lambda'(s) + \lambda(s)] \qquad (102) \\
\frac{\partial^2 TP_2}{\partial s^2} &= (s-c)\lambda''(s) + 2\lambda'(s) - \frac{(c\theta + h_2)\lambda''(s)}{\theta^2 T}[e^{\theta T} - \theta T - 1] \\
&+ \frac{(h_2 - h_1)\lambda''(s)}{\theta^2 T}[(\theta T - \theta t_1 + 1)e^{\theta t_1} - (1 + \theta T)] \\
&+ \frac{2(h_2 - h_1)\lambda'(s)}{T}\left[(T - t_1)e^{\theta t_1}\frac{\partial t_1}{\partial s}\right] + I_e\alpha(M - T/2)[s\lambda''(s) + 2\lambda'(s)] \\
&+ \frac{(h_2 - h_1)\lambda(s)}{T}\left[(T - t_1)e^{\theta t_1}\left(\frac{\partial^2 t_1}{\partial s^2} + \theta(\frac{\partial t_1}{\partial s})^2\right) - e^{\theta t_1}(\frac{\partial t_1}{\partial s})^2\right] \qquad (103) \\
\frac{\partial TP_3}{\partial s} &= (s-c)\lambda'(s) + \lambda(s) - \frac{(c\theta + h_2)\lambda'(s)}{\theta^2 T}[e^{\theta T} - \theta T - 1] \\
&+ \frac{(h_2 - h_1)\lambda'(s)}{\theta^2 T}[(\theta T - \theta t_1 + 1)e^{\theta t_1} - (1 + \theta T)] + \frac{(h_2 - h_1)\lambda(s)}{T}\left[(T - t_1)e^{\theta t_1}\frac{\partial t_1}{\partial s}\right] \\
&- \frac{cI_k\lambda'(s)}{\theta^2 T}\left[e^{\theta(T-M)} - \theta(T - M) - 1\right] + \frac{I_e\alpha M^2}{2T}[s\lambda'(s) + \lambda(s)] \qquad (104)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 TP_3}{\partial s^2} &= (s-c)\lambda''(s) + 2\lambda'(s) - \frac{(c\theta + h_2)\lambda''(s)}{\theta^2 T}[e^{\theta T} - \theta T - 1] \\
&+ \frac{(h_2 - h_1)\lambda''(s)}{\theta^2 T}[(\theta T - \theta t_1 + 1)e^{\theta t_1} - (1 + \theta T)] + \frac{I_e\alpha M^2}{2T}[s\lambda''(s) + 2\lambda'(s)] \\
&+ \frac{2(h_2 - h_1)\lambda'(s)}{T}\left[(T - t_1)e^{\theta t_1}\frac{\partial t_1}{\partial s}\right] - \frac{cI_k\lambda''(s)}{\theta^2 T}\left[e^{\theta(T-M)} - \theta(T - M) - 1\right] \\
&+ \frac{(h_2 - h_1)\lambda(s)}{T}\left[(T - t_1)e^{\theta t_1}\left(\frac{\partial^2 t_1}{\partial s^2} + \theta(\frac{\partial t_1}{\partial s})^2\right) - e^{\theta t_1}(\frac{\partial t_1}{\partial s})^2\right] \qquad (105) \\
\frac{\partial TP_4}{\partial s} &= (s-c)\lambda'(s) + \lambda(s) - \frac{(c\theta + h_1)\lambda'(s)}{\theta^2 T}[e^{\theta T} - \theta T - 1] \\
&- \frac{cI_k\lambda'(s)}{\theta^2 T}\left[e^{\theta(T-M)} - \theta(T - M) - 1\right] + \frac{I_e\alpha M^2}{2T}[s\lambda'(s) + \lambda(s)] \qquad (106) \\
\frac{\partial^2 TP_4}{\partial s^2} &= (s-c)\lambda''(s) + 2\lambda'(s) - \frac{(c\theta + h_1)\lambda''(s)}{\theta^2 T}[e^{\theta T} - \theta T - 1] \\
&- \frac{cI_k\lambda''(s)}{\theta^2 T}\left[e^{\theta(T-M)} - \theta(T - M) - 1\right] + \frac{I_e\alpha M^2}{2T}[s\lambda''(s) + 2\lambda'(s)] \qquad (107)
\end{aligned}
$$

4.6.2 When $M \geq N$

The first and second order partial derivatives of $TP_i (i = 5, 6, .., 10)$ are as follows.

$$
\begin{aligned}
\frac{\partial TP_5}{\partial s} &= (s-c)\lambda'(s) + \lambda(s) - \frac{(c\theta + h_1)\lambda'(s)}{\theta^2 T}[e^{\theta T} - \theta T - 1] \\
&+ I_e\left(M - (1-\alpha)N - \frac{\alpha T}{2}\right)[s\lambda'(s) + \lambda(s)] \qquad (108)
\end{aligned}
$$

$$\frac{\partial^2 TP_5}{\partial s^2} = (s-c)\lambda''(s)+2\lambda'(s)-\frac{(c\theta+h_1)\lambda''(s)}{\theta^2 T}[e^{\theta T}-\theta T-1] + I_e\left(M-(1-\alpha)N-\frac{\alpha T}{2}\right)[s\lambda''(s)+2\lambda'(s)] \quad (109)$$

$$\frac{\partial TP_6}{\partial s} = (s-c)\lambda'(s)+\lambda(s)-\frac{(c\theta+h_2)\lambda'(s)}{\theta^2 T}[e^{\theta T}-\theta T-1] + \frac{(h_2-h_1)\lambda'(s)}{\theta^2 T}[(\theta T-\theta t_1+1)e^{\theta t_1}-(1+\theta T)] + \frac{(h_2-h_1)\lambda(s)}{T}\left[(T-t_1)e^{\theta t_1}\frac{\partial t_1}{\partial s}\right]+I_e\left[M-(1-\alpha)N-\frac{\alpha T}{2}\right](s\lambda'(s)+\lambda(s)) \quad (110)$$

$$\frac{\partial^2 TP_6}{\partial s^2} = (s-c)\lambda''(s)+2\lambda'(s)-\frac{(c\theta+h_2)\lambda''(s)}{\theta^2 T}[e^{\theta T}-\theta T-1] + \frac{(h_2-h_1)\lambda''(s)}{\theta^2 T}[(\theta T-\theta t_1+1)e^{\theta t_1}-(1+\theta T)] + \frac{2(h_2-h_1)\lambda'(s)}{T}\left[(T-t_1)e^{\theta t_1}\frac{\partial t_1}{\partial s}\right]+I_e\left[M-(1-\alpha)N-\frac{\alpha T}{2}\right](s\lambda''(s)+2\lambda'(s)) + \frac{(h_2-h_1)\lambda(s)}{T}\left[(T-t_1)e^{\theta t_1}\left(\frac{\partial^2 t_1}{\partial s^2}+\theta(\frac{\partial t_1}{\partial s})^2\right)-e^{\theta t_1}(\frac{\partial t_1}{\partial s})^2\right] \quad (111)$$

$$\frac{\partial TP_7}{\partial s} = (s-c)\lambda'(s)+\lambda(s)-\frac{(c\theta+h_2)\lambda'(s)}{\theta^2 T}[e^{\theta T}-\theta T-1] + \frac{(h_2-h_1)\lambda'(s)}{\theta^2 T}[(\theta T-\theta t_1+1)e^{\theta t_1}-(1+\theta T)] + \frac{(h_2-h_1)\lambda(s)}{T}\left[(T-t_1)e^{\theta t_1}\frac{\partial t_1}{\partial s}\right]+\frac{I_e}{2T}\left[2MT-(1-\alpha)N^2-T^2\right](s\lambda'(s)+\lambda(s)) \quad (112)$$

$$\frac{\partial^2 TP_7}{\partial s^2} = (s-c)\lambda''(s)+2\lambda'(s)-\frac{(c\theta+h_2)\lambda''(s)}{\theta^2 T}[e^{\theta T}-\theta T-1] + \frac{(h_2-h_1)\lambda''(s)}{\theta^2 T}[(\theta T-\theta t_1+1)e^{\theta t_1}-(1+\theta T)] + \frac{2(h_2-h_1)\lambda'(s)}{T}\left[(T-t_1)e^{\theta t_1}\frac{\partial t_1}{\partial s}\right]+\frac{I_e}{2T}\left[2MT-(1-\alpha)N^2-T^2\right](s\lambda''(s)+2\lambda'(s)) + \frac{(h_2-h_1)\lambda(s)}{T}\left[(T-t_1)e^{\theta t_1}\left(\frac{\partial^2 t_1}{\partial s^2}+\theta(\frac{\partial t_1}{\partial s})^2\right)-e^{\theta t_1}(\frac{\partial t_1}{\partial s})^2\right] \quad (113)$$

$$\frac{\partial TP_8}{\partial s} = (s-c)\lambda'(s)+\lambda(s)-\frac{(c\theta+h_2)\lambda'(s)}{\theta^2 T}[e^{\theta T}-\theta T-1] + \frac{(h_2-h_1)\lambda'(s)}{\theta^2 T}[(\theta T-\theta t_1+1)e^{\theta t_1}-(1+\theta T)] + \frac{(h_2-h_1)\lambda(s)}{T}\left[(T-t_1)e^{\theta t_1}\frac{\partial t_1}{\partial s}\right]-\frac{cI_k\lambda'(s)}{\theta^2 T}\left[e^{\theta(T-M)}-\theta(T-M)-1\right] + \frac{I_e}{2T}\left[M^2-(1-\alpha)N^2\right](s\lambda'(s)+\lambda(s)) \quad (114)$$

$$\frac{\partial^2 TP_8}{\partial s^2} = (s-c)\lambda''(s)+2\lambda'(s)-\frac{(c\theta+h_2)\lambda''(s)}{\theta^2 T}[e^{\theta T}-\theta T-1] + \frac{(h_2-h_1)\lambda''(s)}{\theta^2 T}[(\theta T-\theta t_1+1)e^{\theta t_1}-(1+\theta T)] + \frac{2(h_2-h_1)\lambda'(s)}{T}\left[(T-t_1)e^{\theta t_1}\frac{\partial t_1}{\partial s}\right]-\frac{cI_k\lambda''(s)}{\theta^2 T}\left[e^{\theta(T-M)}-\theta(T-M)-1\right] + \frac{I_e}{2T}\left[M^2-(1-\alpha)N^2\right](s\lambda''(s)+2\lambda'(s)) + \frac{(h_2-h_1)\lambda(s)}{T}\left[(T-t_1)e^{\theta t_1}\left(\frac{\partial^2 t_1}{\partial s^2}+\theta(\frac{\partial t_1}{\partial s})^2\right)-e^{\theta t_1}(\frac{\partial t_1}{\partial s})^2\right] \quad (115)$$

$$\frac{\partial TP_9}{\partial s} = (s-c)\lambda'(s) + \lambda(s) - \frac{(c\theta + h_1)\lambda'(s)}{\theta^2 T}[e^{\theta T} - \theta T - 1]$$
$$+\frac{I_e}{2T}\left[2MT - (1-\alpha)N^2 - T^2\right](s\lambda'(s) + \lambda(s)) \quad (116)$$

$$\frac{\partial^2 TP_9}{\partial s^2} = (s-c)\lambda''(s) + 2\lambda'(s) - \frac{(c\theta + h_1)\lambda''(s)}{\theta^2 T}[e^{\theta T} - \theta T - 1]$$
$$+\frac{I_e}{2T}\left[2MT - (1-\alpha)N^2 - T^2\right](s\lambda''(s) + 2\lambda'(s)) \quad (117)$$

$$\frac{\partial TP_{10}}{\partial s} = (s-c)\lambda'(s) + \lambda(s) - \frac{(c\theta + h_1)\lambda'(s)}{\theta^2 T}[e^{\theta T} - \theta T - 1]$$
$$-\frac{cI_k\lambda'(s)}{\theta^2 T}\left[e^{\theta(T-M)} - \theta(T-M) - 1\right] + \frac{I_e}{2T}\left[M^2 - (1-\alpha)N^2\right](s\lambda'(s) + \lambda(s)) \quad (118)$$

$$\frac{\partial^2 TP_{10}}{\partial s^2} = (s-c)\lambda''(s) + 2\lambda'(s) - \frac{(c\theta + h_1)\lambda''(s)}{\theta^2 T}[e^{\theta T} - \theta T - 1]$$
$$-\frac{cI_k\lambda''(s)}{\theta^2 T}\left[e^{\theta(T-M)} - \theta(T-M) - 1\right] + \frac{I_e}{2T}\left[M^2 - (1-\alpha)N^2\right](s\lambda''(s) + 2\lambda'(s)) \quad (119)$$

4.7 Computational analysis

The purposes of the computational analysis are as follows:

1. To obtain the optimal solutions for different cases when demand is a linearly decreasing function of selling price, i.e., $\lambda(s) = a - bs$ where a and b are positive constants.

2. To use sensitivity analysis to highlight the influence of model parameters.

Note that $TP_i(s|T)$ (i=1,2,...,10) are continuous functions of s over the compact set $[0, s_u]$, where s_u is an extremely large number. Hence $TP_i(s|T)$ has a maximum value. It is clear that $TP_i(s|T)$ is not maximum if $s = 0$ or s_u. As a result, the optimal selling price s must be an interior point between 0 and s_u. We can easily calculate the Hessain matrix of TP as a negative definite although it is not straightforward to find a closed form solution when both the price and replenishment time are decision variables. Following the solution approach given in Chang et al. [20], the solution of $(s-c)\lambda'(s) + \lambda(s) = 0$, say s_l, is the lower bound for the optimal selling price s^*. The solution s_l has been taken as a initial value for s in algorithm 4.1. To find the optimal solution (s^*, T^*), we run algorithm 4.1 using Matlab 7.0.

Algorithm 4.1

To find the optimal value of s and T, we adopt the following steps.

Step 1. Let $j = 1$. Find s_l by solving the equation $(s - c)\lambda'(s) + \lambda(s) = 0$.

Step 2. (a) If $M < N$, calculate $\delta_{11}(s_l), \delta_{12}(s_l), \delta_{21}(s_l)$ and $\delta_{22}(s_l)$. Otherwise go to (b). Using Theorems 4.1 and 4.2, find optimal cycle time which belongs to the set $\{T_1^*, T_2^*, T_3^*, T_4^*\}$. Let T_{ij} be the corresponding T_i^*, $i \in \{1, 2, 3, 4\}$. Let $s_{ij} = s_l$.

(b) If $M \geq N$, calculate $\delta_{31}(s_l), \delta_{32}(s_l), \delta_{33}(s_l), \delta_{41}(s_l), \delta_{42}(s_l), \delta_{43}(s_l), \delta_{51}(s_l), \delta_{52}(s_l)$ and $\delta_{53}(s_l)$. Using Theorems 4.3, 4.4 and 4.5, find optimal cycle time which belongs to the set $\{T_5^*, T_6^*, T_7^*, T_8^*, T_9^*, T_{10}^*\}$. Let T_{ij} be the corresponding T_i^*, $i \in \{5, 6, 7, 8, 9, 10\}$. Let $s_{ij} = s_l$.

Step 3. Substitute $T = T_{ij}$ in the equation $\frac{\partial TP_i}{\partial s} = 0$ and solve it for s. Find $\hat{s}_i$ such that $\frac{\partial^2 TP_i}{\partial s^2}|_{(T=T_{ij}, s=\hat{s}_i)} < 0$ and let $s_{i,j+1} = \hat{s}_i$.

Step 4. Substitute $s = s_{i,j+1}$ in the equation $\frac{\partial TP_i}{\partial T} = 0$ and solve to obtain T_i^*. Let $T_{i,j+1}$ be the solution T_i^*.

Step 5. If $|T_{ij} - T_{i,j+1}| < \epsilon$ and $|s_{ij} - s_{i,j+1}| < \epsilon$, where ϵ is any small positive number, then $s^* = s_{i,j+1}$ and $T^* = T_{i,j+1}$. Otherwise let j = j+1, go to step 3.

4.7.1 Optimal solutions for different cases

Example 1. When $M < N$ and $t_1 \leq M$

Let $a = 500, b = 1.21$, $Z = 150$ units, $A = \$240$ per order, $\theta = 0.05$, $\alpha = 0.7$, $h_1 = \$1$ per unit, $h_2 = \$3$ per unit, $c = \$120$ per unit, $I_k = 0.15$ per dollar, $I_e = 0.12$ per dollar, $N = 0.32$ year and $M = 0.86$ year. For these input parameters, we get $s^* = \$\,263.7392, T^* = 0.3017$ year, $Q^* = 54.9845$ units and $TP^* = \$\,27853$ using the Theorem 4.1 and the proposed algorithm 4.1.

Example 2. When $M < N$ and $t_1 > M$

Let $a = 250, b = 1.21$, $Z = 170$ units,$A = \$240$ per order, $\theta = 0.05$, $\alpha = 0.7$, $h_1 = \$1$ per unit, $h_2 = \$3$ per unit, $c = \$120$ per unit, $I_k = 0.15$ per dollar, $I_e = 0.12$

per dollar, $N = 0.32$ year and $M = 0.86$ year. For these input parameters, we get $s^* = \$\,162.0922, T^* = 0.6972$ year, $Q^* = 38.2210$ units and $TP^* = \$\,2160$ using the Theorem 4.2 and the proposed algorithm 4.1.

Example 3. When $M \geq N$ and $t_1 < N$

Let $a = 500, b = 1.21$, $Z = 150$ units, $A = \$1240$ per order, $\theta = 0.05, \alpha = 0.7$, $h_1 = \$1$ per unit, $h_2 = \$3$ per unit, $c = \$100$ per unit, $I_k = 0.15$ per dollar, $I_e = 0.12$ per dollar, $N = 0.90$ year and $M = 0.94$ year. For these input parameters, we get $s^* = \$\,255.1536, T^* = 0.6875$ year, $Q^* = 133.7804$ units and $TP^* = \$\,29987$ using the Theorem 4.3 and the proposed algorithm 4.1.

Example 4. When $M \geq N$ and $N \leq t_1 \leq M$

Let $a = 500, b = 1.21$, $Z = 150$ units, $A = \$1240$ per order, $\theta = 0.05, \alpha = 0.7$, $h_1 = \$1$ per unit, $h_2 = \$3$ per unit, $c = \$100$ per unit, $I_k = 0.15$ per dollar, $I_e = 0.12$ per dollar, $N = 0.60$ year and $M = 0.94$ year. For these input parameters, we get $s^* = \$\,254.5941, T^* = 0.6662$ year, $Q^* = 130.0161$ units and $TP^* = \$\,30507$ using the Theorem 4.4 and the proposed algorithm 4.1.

Example 5. When $M \geq N$ and $M < t_1$

Let $a = 500, b = 1.21$, $Z = 150$ units, $A = \$1240$ per order, $\theta = 0.05, \alpha = 0.7$, $h_1 = \$1$ per unit, $h_2 = \$3$ per unit, $c = \$100$ per unit, $I_k = 0.15$ per dollar, $I_e = 0.12$ per dollar, $N = 0.90$ year, and $M = 0.74$ year. For these input parameters, we get $s^* = \$\,255.7058, T^* = 0.6671$ year, $Q^* = 129.2918$ units and $TP^* = \$\,29336$ using the Theorem 4.5 and the proposed algorithm 4.1.

Table 3: Sensitivity analysis for various inventory model parameters

parameter		s^*	T^*	Q^*	TP^*
α	0.1	255.8087	0.7911	153.7008	29638
	0.3	255.4306	0.7518	146.2641	29911
	0.5	255.0278	0.7102	138.3960	30200
	0.9	254.1207	0.6189	121.0150	30837
h_2	3	254.5941	0.6662	130.0161	30507
	5	254.7521	0.6221	128.9765	30432
	8	254.9342	0.5992	127.0742	30305
	11	255.0992	0.5801	126.3210	30297
N	0.02	253.8658	0.5940	116.2518	31011
	0.04	253.8685	0.5942	116.3006	31009
	0.06	253.8728	0.5947	116.3820	31006
	0.08	253.8790	0.5953	116.4957	31002
M	0.78	255.4794	0.6669	129.4393	29570
	0.84	255.1436	0.6666	129.6580	29921
	0.90	254.8124	0.6663	129.8738	30273
	0.96	254.4857	0.6661	130.0867	30625
Z	150	253.8658	0.5940	116.2518	31011
	250	251.7652	0.5968	118.8253	31123
	350	250.1910	0.5985	120.0871	31256
	450	248.9670	0.6012	121.9875	31345
θ	0.05	254.5941	0.6662	130.0161	30507
	0.15	255.7858	0.5914	117.8166	29885
	0.25	256.8302	0.5378	108.9296	29316
	0.35	257.7758	0.4969	102.0803	28788
A	1240	254.5941	0.6662	130.0161	30507
	1440	254.7414	0.7078	138.1587	30216

	1640	254.8839	0.7471	145.8527	29940
	1840	255.0219	0.7845	153.1654	29678
c	100	254.5941	0.6662	130.0161	30507
	120	264.1590	0.6662	122.1762	26722
	140	273.7338	0.6640	114.6942	23171
	160	283.3236	0.6626	107.5099	19856
I_e	0.12	254.5941	0.6662	130.0161	30507
	0.14	254.0048	0.6341	124.1288	31028
	0.16	253.4335	0.6080	119.3538	31560
	0.1800	252.8789	0.5861	115.3996	32101

4.7.2 Sensitivity analysis

In order to study the effects of various inventory parameters on s^*, T^*, Q^* and TP^*, we consider the case: $N \leq t_1 \leq M$ when $M \geq N$. Here, we consider the same data set as in numerical example 4. For different values of $\alpha, h_2, N, M, Z, \theta, A, c$ and I_e, the optimal solutions are obtained as in Table 3. The following results can be obtained from Table 3.

1. As α increases, s^*, T^* and Q^* decrease whereas TP^* increases. It indicates the following managerial phenomena: when the customer's fraction of the total amount owed payable at the time of placing an order is increasing, the retailer may order less quantity and he increases the order frequency. Since the retailer can accumulate more interest under higher order frequency and higher 'α', he marginally reduces the selling price; consequently the profit is increased.

2. As h_2 increases, T^*, Q^*, TP^* decrease whereas s^* increases. This shows that if the holding cost is high in rented warehouse W_2, then the retailer should order less to avoid using rented warehouse. In order to balance the decline in profit, the retailer increases marginally the selling price.

3. It is observed that when N increases, s^*, T^* and Q^* increase; but TP^* decreases. It implies that the retailer orders more quantity and increases the selling price of the commodity to accumulate more interest in order to compensate the loss of

interest earned when longer trade credit period (N) is offered to his customers; so the retailer should offer lower credit period to the customers in order to increase his profit.

4. A higher value of M causes the higher values of Q^* and TP^*; but the lower values of s^* and T^*. It shows the following managerial phenomena: when the supplier provides a longer credit period, the retailer replenishes the goods more often. In other words, the retailer will shorten the cycle time and reduce marginally the selling price in order to induce more sales and take the advantages of the longer credit period.

5. As Z increases, s^* marginally decreases; but T^*, Q^* and TP^* increase. When the retailer increases his own warehouse capacity, he may order more quantity due to less holding cost comparing to the rented warehouses. He marginally reduces the selling price in order to increase the sales volume and so the profit.

6. As the value of θ increases, TP^*, Q^* and T^* decrease whereas s^* increases. That is, when the items are starting to deteriorate, it is optimal to rise marginally the selling price in order to manage the decline in profit due to deterioration.

7. As ordering cost, A, increases, the replenishment cycle time T^* and Q^* significantly increase; but optimal selling price marginally increases. The optimal profit decreases as A increases. It indicates the following managerial effect. If the ordering cost is higher, it is reasonable that the retailer lengthens the cycle time to reduce the frequency of replenishment and he marginally increases the selling price.

8. When purchase cost, c, increases s^* increases whereas T^*, Q^* and TP^* decrease. That is, if the unit purchase cost is increased then the optimal cycle time will be decreased and so the order quantity. On the other hand, a higher value of c results in TP^* being decreased.

9. As I_e increases, T^*, Q^* and s^* are marginally decreased. Profit TP^* increases as interest earned rate of the retailer increases. It implies that if the retailer increases

his interest earned rate, then he can shorten his cycle time with marginally less selling price.

4.8 Conclusion

In this chapter, we developed an EOQ model with exponentially deteriorating items and two warehouse facility in a supply chain to investigate retailer's decision making right under partial trade credit policy and price dependent demand. We developed effective and easy-to-use theorems to help the decision maker to find the optimal replenishment policy and optimal selling price. Finally, numerical examples are given to illustrate the proposed theorems and we obtain: (1) the retailer should order more quantity only if he owns larger storage space in W_1 to store more items; (2) the retailer should order less quantity in order to avoid renting expensive warehouse W_2 to store these exceeding items when the unit stock-holding cost h_2 is expensive; (3) the retailer should order less quantity to take the benefits of the trade credit more frequently when the larger the differences between the unit selling price and the purchasing price per item.