Chapter 4

Symmetric Left Bi-Derivations in Prime and Semiprime Rings
In Chapter 4, we proved some results on symmetric left bi-derivations on semiprime rings and lie ideals with symmetric left bi-derivations in prime rings. In section 4.1, we proved that let $R$ be a 2-torsion and 3-torsion free semiprime ring. Let $D(\cdot, \cdot): R \times R \to R$ and $B(\cdot, \cdot): R \times R \to R$ be a symmetric left bi-derivation and symmetric bi-additive mapping. If $D(d(x), x) = 0$ and $(d(d(x)) - f(x))^n = 0$, holds for all $x$ in $R$, where $d$ be a trace of $D$ and $f$ be a trace of $B$. In this case $D$ is central mapping on $R$. In section 4.2, we proved that let $R$ be a prime ring and $U$ be a nonzero lie ideal of $R$. A symmetric bi-additive mapping $D(\cdot, \cdot): R \times R \to R$ is called a symmetric bi-derivation and $d$ is a trace of $D$. We shall show that $U \subseteq Z(R)$ such that $R$ admitting the trace $d$ satisfying the several conditions of symmetric left bi-derivation.
4.1 Symmetric left Bi-Derivations on Semiprime Rings:

The concept of a symmetric bi-derivation has been introduced by Maksa.Gy in [26,27]. A classical result in the theory of centralizing mappings is a theorem first proved by Posner.E [32].Vukman.J [36,37] has studied some results concerning symmetric bi-derivations in prime and semiprime rings. In [31] Ozturk and Jun have introduced the concept of a symmetric bi-derivation of near ring and studied some properties. Bresar.M [11] proved that, if R is a non-commutative 2-torsion free prime ring and \( D: R \times R \rightarrow R \) is a symmetric bi-derivation, then \( D = 0 \).Jaya Subba Reddy.C.et al [23] has studied some results on symmetric left bi-derivations on semiprime rings. Atteya.M.J [5] has studied some results concerning a bi-derivation on prime and semiprime rings. Motivated by the above results we proved some results in symmetric left bi-derivations on semiprime rings.

Throughout this section \( R \) will be represent an associative ring. We shall denote by \( Z(R) \) the center of a ring \( R \). Recall that a ring \( R \) is semiprime if \( aRa = (0) \) implies that \( a = 0 \). We shall write \([x,y]\) for \( xy - yx \) and use the identities

\[
[x,y,z] = [x,z]y + x[y,z], [x,yz] = [x,y]z + y[x,z].
\]

An additive map \( d: R \rightarrow R \) is called derivation if \( d(xy) = d(x)y + xd(y) \), for all \( x,y \in R \). A mapping \( B(\cdot,\cdot): R \times R \rightarrow R \) is said to be symmetric if \( B(x,y) = B(y,x) \) holds for all \( x,y \in R \). A mapping \( f: R \rightarrow R \) defined by \( f(x) = B(x,x) \), where \( B(\cdot,\cdot): R \times R \rightarrow R \) is a symmetric mapping, is called a trace of B. It is obvious that, in case \( B(\cdot,\cdot): R \times R \rightarrow R \) is symmetric mapping which is also bi-additive (i.e. additive in both arguments) the trace of B satisfies the relation

\[
f(x + y) = f(x) + f(y) + 2B(x,y),
\]

for all \( x,y \in R \). We shall use the fact that the trace of a symmetric bi-
additive mapping is an even function. A symmetric bi-additive mapping $D(\ldots): R \times R \to R$ is called a symmetric bi-derivation if $D(xy, z) = D(x, z)y + XD(y, z)$, for all $x, y, z \in R$. Obviously, in this case also the relation $D(xy, yz) = D(x, y)z + yD(x, z)$, for all $x, y, z \in R$. A symmetric bi-additive mapping $D(\ldots): R \times R \to R$ is called a symmetric left bi-derivation if $D(xy, z) = xD(y, z) + yD(x, z)$ for all $x, y, z \in R$. Obviously, in this case also the relation $D(x, yz) = yD(x, z) + zD(x, y)$ for all $x, y, z \in R$. A mapping $f: R \to R$ is said to be commuting on $R$ if $[f(x), x] = 0$, for all $x \in R$. A mapping $f: R \to R$ is said to be centralizing on $R$ if $[f(x), x] \in Z(R)$, for all $x \in R$. A ring $R$ is said to be n-torsion free if whenever $na = 0$, with $a \in R$, then $a = 0$, where $n$ is nonzero integer.

**Lemma 4.1.1:** Let $d: R \to R$ be a derivation, where $R$ is a prime ring. Suppose that either (i) $ad(x) = 0$, for all $x \in R$ or (ii) $d(x)a = 0$, for all $x \in R$ holds. In both the cases we have $a = 0$ or $d = 0$.

**Proof:**

(i) We have $ad(x) = 0$, for all $x \in R$.  

We replace $x$ by $xy$ in (4.1.1), we get $ad(xy) = 0$

$ad(x)y + axd(y) = 0$, for all $x, y \in R$.

By using (4.1.1) in the above equation we get $axd(y) = 0$

$aRd(y) = 0$, for all $y \in R$.

Since $R$ is prime which implies that either $a = 0$ or $d = 0$.

(ii) We have $d(x)a = 0$, for all $x \in R$.  

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We replace \( x \) by \( xy \) in (4.1.2), we get \( d(xy)a = 0 \)

\[
d(xy)y + xd(y)a = 0, \text{ for all } x, y \in R.
\]

By using (4.1.2) in the above equation we get \( d(xy)a = 0 \)

\[
d(x)y + xd(y)a = 0, \text{ for all } x, y \in R.
\]

Since \( R \) is prime which implies that either \( d = 0 \) or \( a = 0 \).

**Lemma 4.1.2:** Let \( R \) be a prime ring of characteristic not two and let \( a, b \in R \) be a fixed elements. If \( axb + bxa = 0 \), for all \( x \in R \), then either \( a = 0 \) or \( b = 0 \).

**Proof:** We have \( axb + bxa = 0 \), for all \( x \in R \).

\[
axb = -bxa, \text{ for all } x \in R. \tag{4.1.3}
\]

We replace \( x \) by \( xbrax \) in (4.1.3), we get

\[
a(xbrax)b = -b(xbrax)a
\]

\[
axbraxb = -(b(xbr)a)xa
\]

By using (4.1.3) in the above equation, we get

\[
axbraxb = (a(xbr)b)xa
\]

\[
(axb)(axb) = (axb)(bxr)
\]

Again by using (4.1.3) in the above equation, we get

\[
(axb)(axb) = (axb)(-axb)
\]

\[
(axb)(axb) = -(axb)(bxr)
\]

\[
2(axb)(axb) = 0, \text{ for all } r \in R.
\]
Since $R$ is prime ring of characteristic not 2, we get

$$axb = 0,$$

for all $x \in R$ and hence $a = 0$ or $b = 0$.

**Lemma 4.1.3:** [7] The center of semiprime ring contains no non-zero nilpotent elements.

**Theorem 4.1.1:** Let $n$ be a positive integer and $R$ be a 2-torsion free semiprime ring. Let $D(\cdot, \cdot): R \times R \to R$ be a symmetric left bi-derivation and $d$ is a trace of $D$ such that $D(d(x), x)^n = 0$, for all $x \in R$, then $D$ is central mapping on $R$.

**Proof:** We have $D(d(x), x)^n = 0$, for all $x \in R$.

If $n = 1$, we have $D(d(x), x) = 0$, for all $x \in R$. \hfill (4.1.4)

We replace $d(x)$ by $d(x)y$ in (4.1.4), we get

$$D(d(x)y, x) = 0$$

$$d(x)D(y, x) + yD(d(x), x) = 0$$

By using (4.1.4) in the above equation, we get

$$d(x)D(y, x) = 0$$

$$d(x)D(x, y) = 0,$$ for all $x, y \in R$. \hfill (4.1.5)

We replace $x$ by $x^2$ in (4.1.5), we get

$$d(x^2)D(x^2, y) = 0$$

$$4x^2d(x)2xD(x, y) = 0$$

$$8x^2d(x)xD(x, y) = 0$$
If $x = 0$ it is trivial, if $x \neq 0$ then $d(x)xD(x, y) = 0$, for all $x, y \in R$. \hfill (4.1.6)

By the linearization of (4.1.4), we get

$D(d(x + y), x + y) = 0$

$D(d(x) + d(y) + 2D(x, y), x + y) = 0$

$D(d(x), x) + D(d(x), y) + D(d(y), x) + D(d(y), y) + D(2D(x, y), x) + D(2D(x, y), y) = 0$

By using (4.1.4) in the above equation, we get

$D(d(x), y) + D(d(y), x) + 2D(D(x, y), x) + 2D(D(x, y), y) = 0$, for all $x, y \in R$. \hfill (4.1.7)

We replace $x$ by $-x$ in (4.1.7), we get

$D(d(-x), y) + D(d(y), -x) + 2D(D(-x, y), -x) + 2D(D(-x, y), y) = 0$

$D(d(x), y) - D(d(y), x) + 2D(D(x, y), x) - 2D(D(x, y), y) = 0$, for all $x, y \in R$. \hfill (4.1.8)

By adding (4.1.7) and (4.1.8), we get

$2D(d(x), y) + 4D(D(x, y), x) = 0$

$D(d(x), y) + 2D(D(x, y), x) = 0$, for all $x, y \in R$. \hfill (4.1.9)

We replace $y$ by $xy$ in (4.1.9), we get

$D(d(x), xy) + 2D(D(x, xy), x) = 0$
\[ xD(d(x), y) + yD(d(x), x) + 2D(xD(x, y) + yD(x, x), x) = 0 \]
\[ xD(d(x), y) + yD(d(x), x) + 2D(xD(x, y), x) + 2D(yD(x, x), x) = 0 \]
\[ xD(d(x), y) + yD(d(x), x) + 2xD(D(x, y), x) + 2D(x, y)D(x, x) \]
\[ + 2yD(D(x, x), x) + 2D(x, x)D(y, x) = 0 \]
\[ xD(d(x), y) + yD(d(x), x) + 2xD(D(x, y), x) + 2D(x, y)d(x) + 2yD(d(x), x) \]
\[ + 2d(x)D(y, x) = 0 \]

By using (4.1.4) and (4.1.9) in the above equation, we get
\[ 2D(x, y)d(x) + 2d(x)D(y, x) = 0 \]
\[ D(x, y)d(x) + d(x)D(x, y) = 0, \text{ for all } x, y \in R. \] (4.1.10)

By using (4.1.5) in (4.1.10), we get
\[ D(x, y)d(x) = 0, \text{ for all } x, y \in R. \] (4.1.11)

We replace \( y \) by \( x \) in (4.1.10), we get
\[ D(x, x)d(x) + d(x)D(x, x) = 0 \]
\[ d(x)d(x) + d(x)d(x) = 0 \]
\[ 2d(x)d(x) = 0 \]
\[ d(x)d(x) = 0, \text{ for all } x \in R. \] (4.1.12)

We replace \( y \) by \( yx \) in (4.1.10), we get
\[ D(x, yx)d(x) + d(x)D(x, yx) = 0 \]
\[ yD(x, x)d(x) + xD(x, y)d(x) + d(x)yD(x, x) + d(x)xD(x, y) = 0 \]
By using (4.1.6), (4.1.11), (4.1.12) in above equation, we get

\[ d(x)yd(x) = 0, \text{ for all } x, y \in R. \]

Which implies that \( d(x) = 0 \), for all \( x \in R \), by semiprimeness of \( R \), which means that \( D(x, y) = 0 \), for all \( x, y \in R \).

Assume that \( n > 1 \), we have \( D(d(x), x)^n = 0 \), for all \( x \in R \). According to Lemma 4.1.3 we get \( D(d(x), x) = 0 \), for all \( x \in R \), so by the same technique in first part of the proof we can complete the proof of the theorem.

**Theorem 4.1.2:** Let \( n \) be a positive integer and \( R \) be a 2-torsion and 3-torsion free semiprime ring. Let \( D(., .): R \times R \rightarrow R \) and \( B(., .): R \times R \rightarrow R \) be a symmetric left bi-derivation and symmetric bi-additive mapping respectively. Suppose that \( (d(d(x)) - f(x))^n = 0 \), for all \( x \in R \), where \( d \) be a trace of \( D \) and \( f \) be a trace of \( B \), then \( D \) is central on \( R \).

**Proof:** We have\( (d(d(x)) - f(x))^n = 0 \), for all \( x \in R \).

If \( n = 1 \), we have \( d(d(x)) = f(x) \), for all \( x \in R \). \hspace{1cm} (4.1.13)

By the linearization of (4.1.13), we get

\[ d(d(x + y)) = f(x + y) \]

\[ d(d(x) + d(y) + 2D(x, y)) = f(x) + f(y) + 2B(x, y) \]

\[ d(d(x)) + d(d(y)) + d(2D(x, y)) + 2D(d(x), d(y)) + 2D(d(x), 2D(x, y)) \]
\[ + 2D(d(y), 2D(x, y)) = f(x) + f(y) + 2B(x, y) \]
\[ d(d(x)) + d(d(y)) + 4d(D(x,y)) + 2D(d(x),d(y)) + 4D(d(x),D(x,y)) \]
\[ + 4D(d(y),D(x,y)) = f(x) + f(y) + 2B(x,y) \]

By using (4.1.13) in the above equation, we get

\[ 4d(D(x,y)) + 2D(d(x),d(y)) + 4D(d(x),D(x,y)) + 4D(d(y),D(x,y)) \]
\[ = 2B(x,y) \]

\[ 2d(D(x,y)) + D(d(x),d(y)) + 2D(d(x),D(x,y)) + 2D(d(y),D(x,y)) = B(x,y), \text{ for all } x, y \in \mathbb{R}. \quad (4.1.14) \]

We replace \( x \) by \(-x\) in (4.1.14), we get

\[ 2d(D(-x,y)) + D(d(-x),d(y)) + 2D(d(-x),D(-x,y)) + 2D(d(y),D(-x,y)) \]
\[ = B(-x,y) \]

\[ 2d(D(x,y)) + D(d(x),d(y)) - 2D(d(x),D(x,y)) - 2D(d(y),D(x,y)) = -B(x,y), \text{ for all } x, y \in \mathbb{R}. \quad (4.1.15) \]

Subtract (4.1.15) from (4.1.14), we get

\[ 4D(d(x),D(x,y)) + 4D(d(y),D(x,y)) = 2B(x,y) \]

\[ 2D(d(x),D(x,y)) + 2D(d(y),D(x,y)) = B(x,y), \text{ for all } x, y \in \mathbb{R}. \quad (4.1.16) \]

We replace \( x \) by \( 2x \) in (4.1.16), we get

\[ 2D(d(2x),D(2x,y)) + 2D(d(y),D(2x,y)) = B(2x,y) \]

\[ 16D(d(x),D(x,y)) + 4D(d(y),D(x,y)) = 2B(x,y) \]

\[ 8D(d(x),D(x,y)) + 2D(d(y),D(x,y)) = B(x,y), \text{ for all } x, y \in \mathbb{R}. \quad (4.1.17) \]
Subtract (4.1.16) from (4.1.17), we get

\[ 6D(d(x), D(x, y)) = 0 \]

Since \( R \) is 2-torison and 3-torison free ring, we get

\[ D(d(x), D(x, y)) = 0, \text{ for all } x, y \in R. \quad (4.1.18) \]

By using (4.1.18) and (4.1.16), we get

\[ B(x, y) = 0, \text{ for all } x, y \in R. \]

We replace \( y \) by \( x \) in the above equation, we get \( f(x) = 0, \text{ for all } x \in R. \) \((4.1.19)\)

By using (4.1.13) and (4.1.19), we get

\[ d(d(x)) = 0, \text{ for all } x \in R. \quad (4.1.20) \]

We replace \( y \) by \( yz \) in (4.1.18), we get

\[ D(d(x), D(x, yz)) = 0 \]
\[ D(d(x), yD(x, z) + zD(x, y)) = 0 \]
\[ D(d(x), yD(x, z)) + D(d(x), zD(x, y)) = 0 \]
\[ yD(d(x), D(x, z)) + D(x, z)D(d(x), y) + zD(d(x), D(x, y)) + D(x, y)D(d(x), z) = 0 \]

By using (4.1.18) in the above equation, we get

\[ D(x, z)D(d(x), y) + D(x, y)D(d(x), z) = 0, \text{ for all } x, y, z \in R. \quad (4.1.21) \]

We replace \( z \) by \( d(x) \) in (4.1.21), we get

\[ D(x, d(x))D(d(x), y) + D(x, y)D(d(x), d(x)) = 0 \]
By using (4.1.20) in the above equation, we get

\[ D(x, d(x))D(d(x), y) + D(x, y)d(d(x)) = 0 \]

We replace \( y \) by \( xy \) in (4.1.22), we get

\[ D(x, d(x))D(d(x), xy) = 0 \]

We replace \( y \) by \( x \) in the above equation we get

\[ D(d(x), x)xD(d(x), x) = 0 \]

which implies \( D(d(x), x) = 0 \), for all \( x \in R \) since we have assumed that \( R \) is semiprime. Now Theorem 4.1.1 completes the proof.

Assuming that \( n > 1 \), we have \( (d(d(x)) - f(x))^n = 0 \), for all \( x \in R \), that implies \( (d(d(x)) - f(x))^n \in Z(R) \), for all \( x \in R \). According to Lemma 4.1.3, we get \( d(d(x)) - f(x) = 0 \), for all \( x \in R \). Now by using the same technique of first part of the theorem we can complete the proof.

4.2 Lie Ideals with Symmetric left Bi-Derivations in Prime Rings:

The concept of a symmetric bi-derivation has been introduced by Maksa.Gy in [26,27]. A classical result in the theory of centralizing mappings is a theorem first proved by E. Posner[32] which stated that the existence of a nonzero centralizing derivation on a prime ring \( R \) implies that \( R \) is commutative. Vukman.J [36, 37] has studied some results concerning symmetric bi-derivations on prime and semiprime
rings. In [3] Argac and Yenigul and in [28] Muthana obtained the similar type of results on lie ideals of $R$. In this section we proved some results symmetric left bi-derivations in prime rings.

Throughout this section $R$ will be associative. We shall denote by $Z(R)$ the center of a ring $R$. Recall that a ring $R$ is prime if $aRb = (0)$ implies that $a = 0$ or $b = 0$. We shall write $[x, y]$ for $xy - yx$. The symbol $x \circ y$ stands for anti commutator $xy + yx$. An additive map $d: R \rightarrow R$ is called derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. A mapping $B(., .): R \times R \rightarrow R$ is said to be symmetric if $B(x, y) = B(y, x)$, for all $x, y \in R$. A mapping $f: R \rightarrow R$ defined by $f(x) = B(x, x)$, where $B(., .): R \times R \rightarrow R$ is a symmetric mapping, is called a trace of $B$. It is obvious that, in case $B(., .): R \times R \rightarrow R$ is symmetric mapping which is also bi-additive (i.e. additive in both arguments) the trace of $B$ satisfies the relation $f(x + y) = f(x) + f(y) + 2B(x, y)$, for all $x, y \in R$. We shall use also the fact that the trace of a symmetric bi-additive mapping is an even function. A symmetric bi-additive mapping $D(., .): R \times R \rightarrow R$ is called a symmetric bi-derivation if $D(xy, z) = D(x, z)y + xD(y, z)$ is fulfilled for all $x, y, z \in R$. Obviously, in this case also the relation $D(x, yz) = D(x, y)z + yD(x, z)$ for all $x, y, z \in R$. A symmetric bi-additive mapping $D(., .): R \times R \rightarrow R$ is called a symmetric left bi-derivation if $D(xy, z) = xD(y, z) + yD(x, z)$ for all $x, y, z \in R$. Obviously, in this case also the relation $D(x, yz) = yD(x, z) + zD(x, y)$ for all $x, y, z \in R$. A mapping $f: R \rightarrow R$ is said to be commuting on $R$ if $[f(x), x] = 0$, for all $x \in R$. A mapping $f: R \rightarrow R$ is said to be centralizing on $R$ if $[f(x), x] \in$
\(Z(R)\), for all \(x \in R\). A ring \(R\) is said to be \(n\)-torsion free if whenever \(na = 0\), with \(a \in R\), then \(a = 0\), where \(n\) is nonzero integer.

We shall frequently use the following identities and several well known facts about the semiprime rings without specific mention.

\([xy, z] = x[y, z] + [x, z]y\)
\([x, yz] = y[x, z] + [x, y]z\)
\(x \circ yz = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z\)
\(xy \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]\)

**Remark 4.2.1:** Let \(U\) be a square closed lie ideal of \(R\). Notice that \(xy + yx = (x + y)^2 - x^2 - y^2\), for all \(x, y \in U\). Since \(x^2 \in U\) for all \(x \in U\), \(xy + yx \in U\) for all \(x, y \in U\). Hence we find that \(2xy \in U\) for all \(x, y \in U\). Therefore, for all \(r \in R\), we get \(2r[x, y] = 2[x, ry] - 2[x, r]y \in U\) and \(2[x, y]r = 2[x, ry] - 2[y, r]y \in U\), so that \(2R[U, U] \subseteq U\) and \(2[U, U]R \subseteq U\).

This remark will be freely used in the whole section without specific reference.

**Lemma 4.2.1:**[19, Corallary 2.1] Let \(R\) be a 2-torsion free semiprime ring, \(U\) be a lie ideal of \(R\) such that \(U \nsubseteq Z(R)\) and \(a, b \in U\).

(i) If \(aUa = \{0\}\), then \(a = 0\).

(ii) If \(aU = \{0\}(Ua = \{0\})\), then \(a = 0\).

(iii) If \(U\) is a square closed lie ideal and \(aUb = \{0\}\), then \(ab = 0\) and \(ba = 0\).

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Lemma 4.2.2: [3, Theorem 3] Let $R$ be 2-torsion free prime ring and $U$ be a nonzero Lie ideal of $R$. Let $B(.,.) : R \times R \rightarrow R$ be a symmetric bi-derivation and $f$ be the trace of $B$ such that

(i) $f(U) = 0$, then $U \subseteq Z(R)$ or $f = 0$.

(ii) $f(U) \subseteq Z(R)$ and $U$ be a square closed Lie ideal, then $U \subseteq Z(R)$ or $f = 0$.

Lemma 4.2.3: [18, Lemma 1] Let $R$ be a 2-torsion free semiprime ring and $U$ be a Lie ideal of $R$. Suppose that $[U, U] \subseteq Z(R)$, then $U \subseteq Z(R)$.

Lemma 4.2.4: [10, Lemma 4] Let $R$ be a 2-torsion free prime ring and $U \not\subseteq Z(R)$ be a Lie ideal of $R$ and $a, b \in R$, if $aUb = \{0\}$, then $a = 0$ and $b = 0$.

Lemma 4.2.5: Let $R$ be a 2-torsion free prime ring and $U$ be a square closed Lie ideal of $R$. Suppose that $D(.,.) : R \times R \rightarrow R$ is a symmetric left bi-derivation and $d$ the trace of $D$ such that $[d(x), y] \in Z(R)$, for all $x, y \in U$, then either $U \subseteq Z(R)$ or $d = 0$.

Proof: Suppose on the contrary that $U \not\subseteq Z(R)$.

We have $[d(x), y] \in Z(R)$, for all $x, y \in U$. \[(4.2.1)\]

We replace $y$ by $2yz$ in (4.2.1), we get

$[d(x), 2yz] \in Z(R)$

$2y[d(x), z] + 2[d(x), y]z \in Z(R)$

$\gamma[d(x), z] + [d(x), y]z \in Z(R)$, for all $x, y, z \in U$.

This implies that $[[d(x), y]z + y[d(x), z], r] = 0$, for all $x, y, z \in U$ and $r \in R$. 

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We replacing \( r \) by \( z \) in (4.2.2), we get

\[
[y, z][d(x), z] = 0, \text{ for all } x, y, z \in U.
\]  

(4.2.3)

We replacing \( y \) by \( 2yt \) in (4.2.3), we get

\[
[2yt, z][d(x), z] = 0
\]

(4.2.4)

Thus in view of lemma 4.2.4 we find that for each pair of \( x, y, z \in U \) either

\[
[y, z] = 0 \text{ or } [d(x), z] = 0.
\]

For each \( z \in U \), let \( A^1 = \{ y \in U : [y, z] = 0 \} \) and

\( B^1 = \{ x \in U : [d(x), z] = 0 \} \).

Hence \( A^1 \) and \( B^1 \) are the additive subgroups of \( U \) whose union is \( U \). By Brauer's trick, we have either \( U = A^1 \) or \( U = B^1 \). If \( A^1 = U \), then \( [y, z] = 0 \) for all \( y, z \in U \) and have \( U \subseteq Z(R) \) a contradiction. On the other hand if \( U = B^1 \), then \( [d(x), z] = 0 \) for all \( x, z \in U \) and hence \( d(U) \subseteq C_R(U) = Z(R) \), then by lemma 4.2.2, we get \( d = 0 \). This completes the proof of the lemma.

**Theorem 4.2.1:** Let \( R \) be a 2-torsion free prime ring and \( U \) be a square closed lie ideal of \( R \). Suppose that \( D(\ldots) : R \times R \to R \) is a symmetric left bi-derivation and \( d \) the trace of \( D \). If \( [d(x), x] = 0 \), for all \( x \in U \), then either \( U \subseteq Z(R) \) or \( d = 0 \).

**Proof:** Suppose on the contrary that \( U \not\subseteq Z(R) \).

Since we have given that \( [d(x), x] = 0 \), for all \( x, y \in U \).  

(4.2.4)
We replacing $x$ by $x + y$ in (4.2.4), we get

$$[d(x + y), x + y] = 0$$

$$[d(x) + d(y) + 2D(x,y), x + y] = 0$$

$$[d(x), x] + [d(x), y] + [d(y), x] + [d(y), y] + 2[D(x,y), x] + 2[D(x,y), y] = 0$$

By using (4.2.4), in the above equation we get

$$[d(x), y] + [d(y), x] + 2[D(x,y), x] + 2[D(x,y), y] = 0, \text{ for all } x, y \in U$$

(4.2.5)

We replacing $x$ by $-x$ in (4.2.5), we get

$$[d(-x), y] + [d(y), -x] + 2[D(-x,y), -x] + 2[D(-x,y), y] = 0$$

$$[d(x), y] - [d(y), x] + 2[D(x,y), x] - 2[D(x,y), y] = 0, \text{ for all } x, y \in U.$$  

(4.2.6)

By adding (4.2.5) and (4.2.6), we get

$$[d(x), y] + 2[D(x,y), x] = 0, \text{ for all } x, y \in U.$$  

(4.2.7)

We replacing $y$ by $2yz$ in (4.2.7), we get

$$[d(x), 2yz] + 2[D(x, 2yz), x] = 0$$

$$2y[d(x), z] + 2[d(x), y]z + 4[yD(x,z) + zD(x,y), x] = 0$$

$$2y[d(x), z] + 2[d(x), y]z + 4[yD(x,z), x] + 4[zD(x,y), x] = 0$$

$$2y[d(x), z] + 2[d(x), y]z + 4[y, x]D(x,z) + 4[yD(x,z), x] + 4[z, x]D(x,y)$$

$$+ 4z[D(x,y), x] = 0$$
By using (4.2.4) in the above equation we get

\[ 2y[d(x), z] + 2z[d(x), y] + 4[y, x]D(x, z) + 4y[D(x, z), x] + 4[z, x]D(x, y) + 4z[D(x, y), x] = 0 \]

\[ 2y[[d(x), z] + 2[D(x, z), x]] + 2z[[d(x), y] + 2[D(x, y), x]] + 4[y, x]D(x, z) + 4[z, x]D(x, y) = 0 \]

By using (4.2.4) in the above equation we get

\[ 4[y, x]D(x, z) + 4[z, x]D(x, y) = 0 \]

\[ [y, x]D(x, z) + [z, x]D(x, y) = 0, \text{ for all } x, y, z \in U \quad (4.2.8) \]

We replace \( z \) by \( x \) in (4.2.8) we get

\[ [y, x]D(x, x) + [x, x]D(x, y) = 0 \]

\[ [y, x]D(x, x) = 0, \text{ for all } x, y \in U. \quad (4.2.9) \]

We replacing \( y \) by \( 2yz \) in (4.2.9), we get

\[ [2yz, x]D(x, x) = 0 \]

\[ 2[y, x]zD(x, x) + 2y[z, x]D(x, x) = 0 \]

By using (4.2.9) in the above equation we get

\[ 2[y, x]zD(x, x) = 0 \]

\[ [y, x]zD(x, x) = 0, \text{ for all } x, y, z \in U, \text{ this gives } [y, x]UD(x, x) = 0, \text{ for all } x, y \in U. \]

By lemma 4.2.4 for each \( x \in U \) either \([y, x] = 0\) or \( D(x, x) = 0\), for all \( x, y \in U\). In the first case it follows that by lemma 4.2.3, \( x \in Z(R) \) for all \( x \in U \). Thus if \( x \notin Z(R) \), then \( D(x, x) = 0 \). Let \( x, z \in U \) such that \( x \in Z(R) \) and \( z \notin Z(R) \). Hence \( x + z \notin Z(R) \) and \( x - z \notin Z(R) \). Thus \( D(x + z, x + z) = 0 \) and \( D(x - z, x - z) = \)
Adding the above two relations, we get $2D(x, x) = 0$, since $R$ is 2-torsion free ring, we get $D(x, x) = 0$. Thus for all $x \in U$, $D(x, x) = 0$ and by lemma 4.2.2, $d = 0$.

**Theorem 4.2.2**: Let $R$ be a 2-torsion free prime ring and $U$ be a square closed lie ideal of $R$. Suppose that $D(., .) : R \times R \rightarrow R$ is a symmetric left bi-derivation and $d$ the trace of $D$ such that $d([x, y]) - [d(x), y] \in Z(R)$, for all $x, y \in U$. Then either $U \subseteq Z(R)$ or $d = 0$.

**Proof**: Suppose on the contrary that $U \notin Z(R)$.

We have $d([x, y]) - [d(x), y] \in Z(R)$, for all $x, y \in U$. \hspace{1cm} (4.2.10)

We replace $y$ by $y + z$ in (4.2.10), we get

$d([x, y + z]) - [d(x), y + z] \in Z(R)$

$d([x, y] + [x, z]) - [d(x), y] - [d(x), z] \in Z(R)$

$d([x, y]) + d([x, z]) + 2D([x, y], [x, z]) - [d(x), y] - [d(x), z] \in Z(R)$

By using (4.2.10) in the above equation we get

$D([x, y], [x, z]) \in Z(R)$, for all $x, y, z \in U$. \hspace{1cm} (4.2.11)

We replace $z$ by $y$ in (4.2.11), we get

$D([x, y], [x, y]) \in Z(R)$, for all $x, y \in U$. \hspace{1cm} (4.2.12)

By subtracting (4.2.10) from (4.2.12) we get

$[d(x), y] \in Z(R)$, for all $x, y \in U$. \hspace{1cm} (4.2.13)
By using lemma 4.2.5, we get the required result.

**Theorem 4.2.3:** Let $R$ be a 2-torsion free prime ring and $U$ be a square closed lie ideal of $R$. Suppose that $D(\cdot, \cdot) : R \times R \to R$ is a symmetric left bi-derivation and $d$ the trace of $D$ such that $d(x \circ y) - [d(x), y] \in Z(R)$, for all $x, y \in U$. Then either $U \subseteq Z(R)$ or $d = 0$.

**Proof:** Suppose on the contrary that $U \not\subseteq Z(R)$.

We have $d(x \circ y) - [d(x), y] \in Z(R)$, for all $x, y \in U$. \hspace{1cm} (4.2.13)

We replace $y$ by $y + z$ in (4.2.13), we get

$$d(x \circ y + z) - [d(x), y + z] \in Z(R)$$

$$d(x \circ y) + d(x \circ z) + 2D(x \circ y, x \circ z) - [d(x), y] - [d(x), z] \in Z(R)$$

By using (4.2.13) in the above equation we get

$$2D(x \circ y, x \circ z) \in Z(R)$$

$D(x \circ y, x \circ z) \in Z(R)$, for all $x, y, z \in U$. \hspace{1cm} (4.2.14)

We replace $z$ by $y$ in (4.2.14), we get

$$D(x \circ y, x \circ y) \in Z(R), \text{ for all } x, y \in U.$$ \hspace{1cm} (4.2.15)

By subtracting (4.2.13) from (4.2.15), we get

$$[d(x), y] \in Z(R), \text{ for all } x, y \in U.$$ 

By using lemma 4.2.5, we get the required result.
Theorem 4.2.4: Let $R$ be a 2-torsion free prime ring and $U$ be a square closed lie ideal of $R$. Suppose that $D(\cdot,\cdot): R \times R \to R$ is a symmetric left bi-derivation and $d$ the trace of $D$ such that $d(x) \circ y - [d(x), y] \in Z(R)$, for all $x, y \in U$. Then either $U \subseteq Z(R)$ or $d = 0$.

Proof: Suppose on the contrary that $U \not\subseteq Z(R)$.

We have $d(x) \circ y - [d(x), y] \in Z(R)$, for all $x, y \in U$. \hfill (4.2.16)

Thus $d(x)y + yd(x) - d(x)y + yd(x) \in Z(R)$

$2yd(x) \in Z(R)$

$yd(x) \in Z(R)$, for all $x, y \in U$.

$[yd(x), r] = 0$, for all $x, y \in U$ and $r \in R$.

$y[d(x), r] + [y, r]d(x) = 0$, for all $x, y \in U$ and $r \in R$. \hfill (4.2.17)

We replace $y$ by $2ty$ in (4.2.17), we get

$2ty[d(x), r] + [2ty, r]d(x) = 0$

$2ty[d(x), r] + 2[y, r]d(x) + 2[t, r]yd(x) = 0$

By using (4.2.17) in the above equation we get

$[t, r]yd(x) = 0$, for all $x, t \in U$ and $r \in R$.

$[t, r]Ud(x) = 0$, for all $x, t \in U$ and $r \in R$.

By using lemma 4.2.4 we get either $[t, r] = 0$ or $d(x) = 0$, for all $x, t \in U$ and $r \in R$. 

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If $[t, r] = 0$, then $U \subseteq Z(R)$ a contradiction.

Hence if $d(x) = 0$, for all $x \in U$, then by lemma 4.2.2, we get $d = 0$.

**Theorem 4.2.5**: Let $R$ be a 2-torsion free prime ring and $U$ be a square closed lie ideal of $R$. Suppose that $D(\ldots): R \times R \rightarrow R$ is a symmetric left bi-derivation and $d$ the trace of $D$ and $g: R \rightarrow R$ is any mapping such that $[d(x), y] - [x, g(x)] \in Z(R)$, for all $x, y \in U$. Then either $U \subseteq Z(R)$ or $d = 0$.

**Proof**: Suppose on the contrary that $U \not\subseteq Z(R)$.

We have $[d(x), y] - [x, g(y)] \in Z(R)$, for all $x, y \in U$. (4.2.18)

We replace $x$ by $x + z$ in (4.2.18), we get

$[d(x + z), y] - [x + z, g(y)] \in Z(R)$

$[d(x), y] + [d(z), y] + 2[D(x, z), y] - [x, g(y)] - [z, g(y)] \in Z(R)$

By using (4.2.18) in the above equation we get

$2[D(x, z), y] \in Z(R)$

$[D(x, z), y] \in Z(R)$, for all $x, y, z \in U$. (4.2.19)

We replace $z$ by $x$ in (4.2.19), we get

$[D(x, x), y] \in Z(R)$, for all $x, y \in U$.

$[d(x), y] \in Z(R)$, for all $x, y \in U$.

Hence by lemma 4.2.5, we get the required result.
Theorem 4.2.6: Let $R$ be a 2-torsion free prime ring and $U$ be a square closed lie ideal of $R$. Suppose that $D(\cdot, \cdot) : R \times R \rightarrow R$ is a symmetric left bi-derivation and $d$ the trace of $D$ and $g : R \rightarrow R$ is any mapping such that $d(x) \circ d(y) - [d(x), y] \in Z(R)$, for all $x, y \in U$. Then either $U \subseteq Z(R)$ or $d = 0$.

**Proof:** Suppose on the contrary that $U \not\subseteq Z(R)$.

We have $d(x) \circ d(y) - [d(x), y] \in Z(R)$, for all $x, y \in U$. \hspace{1cm} (4.2.20)

We replace $y$ by $y + z$ in (4.2.20), we get

$$d(x) \circ d(y + z) - [d(x), y + z] \in Z(R)$$

$$d(x) \circ d(y) + d(x) \circ d(z) + 2d(x) \circ D(y, z) - [d(x), y] - [d(x), z] \in Z(R), \text{ for all } x, y, z \in U.$$  

By using (4.2.20) in the above equation we get

$$2d(x) \circ D(y, z) \in Z(R)$$

$$d(x) \circ D(y, z) \in Z(R), \text{ for all } x, y, z \in U. \hspace{1cm} (4.2.21)$$

We replace $z$ by $y$ in (4.2.21), we get

$$d(x) \circ D(y, y) \in Z(R)$$

$$d(x) \circ d(y) \in Z(R), \text{ for all } x, y \in U. \hspace{1cm} (4.2.22)$$

By subtracting (4.2.20) from (4.2.22), we get

$$[d(x), y] \in Z(R), \text{ for all } x, y \in Z(R)$$

Thus by using lemma 4.2.5, we get the required result.
**Theorem 4.2.7:** Let $R$ be a 2-torsion free prime ring and $U$ be a square closed lie ideal of $R$. Suppose $D(\cdot , \cdot ) : R \times R \to R$ is a symmetric left bi-derivation and $d$ the trace of $D$ and $g : R \to R$ be any mapping such that $d(x)y - xg(y) \in Z(R)$, for all $x, y \in U$, then either $U \subseteq Z(R)$ or $d = 0$.

**Proof:** Suppose on the contrary that $U \not\subseteq Z(R)$.

We have $d(x)y - xg(y) \in Z(R)$, for all $x, y \in U$. \hfill (4.2.23)

We replace $x$ by $x + z$ in (4.2.23), we get

$$d(x + z)y - (x + z)g(y) \in Z(R)$$

$$d(x)y + d(z)y + 2D(x, z)y - xg(y) - zg(y) \in Z(R), \text{ for all } x, y, z \in U \text{ (4.2.24)}$$

By using (4.2.23) in (4.2.24), we get

$$2D(x, z)y \in Z(R)$$

$$D(x, z)y \in Z(R), \text{ for all } x, y, z \in U. \text{ (4.2.25)}$$

We replace $z$ by $x$ in (4.2.25), we get

$$D(x, x)y \in Z(R)$$

$$d(x)y \in Z(R), \text{ for all } x, y \in U.$$  \hfill (4.2.26)

We replace $y$ by $2yt$ in (4.2.26), we get

$$[d(x)y, r] = 0, \text{ for all } x, y \in U \text{ and } r \in R.$$ \hfill (4.2.26)

We replace $y$ by $2yt$ in (4.2.26), we get

$$[d(x)2yt, r] = 0$$

$$2[d(x)y, r]t + 2d(x)y[t, r] = 0$$

By using (4.2.26) in the above equation, we get
\[ 2d(x)y[t, r] = 0 \]

\[ 2d(x)y[t, r] = 0, \text{ for all } x, y \in U \text{ and } r \in R. \]

\[ 2d(x)U[t, r] = 0, \text{ for all } x, y, t \in U \text{ and } r \in R. \]

\[ d(x)U[t, r] = 0, \text{ for all } x, t \in U \text{ and } r \in R. \]

By using lemma 4.2.4, we get either \([t, r] = 0\) or \(d(x) = 0\), for all \(x, t \in U\) and \(r \in R\).

If \([t, r] = 0\), then \(U \subseteq Z(R)\) a contradiction.

Hence if \(d(x) = 0\), for all \(x \in U\), then by lemma 4.2.2, we get \(d = 0\).

**Theorem 4.2.8:** Let \(R\) be a 2-torsion free prime ring and \(U\) be a square closed Lie ideal of \(R\). Suppose \(D(., .) : R \times R \to R\) is a symmetric left bi-derivation and \(d\) the trace of \(D\) such that \(d(xy) - d(x)y - xd(y) \in Z(R)\), for all \(x, y \in U\), then either \(U \subseteq Z(R)\) or \(d = 0\).

**Proof:** Suppose on the contrary that \(U \not\subseteq Z(R)\).

We have \(d(xy) - d(x)y - xd(y) \in Z(R)\), for all \(x, y \in U\). \hspace{1cm} (4.2.27)

We replace \(x\) by \(x + z\) in (4.2.27), we get

\[ d((x + z)y) - d(x + z)y - (x + z)d(y) \in Z(R) \]

\[ d(xy + zy) - d(x + z)y - (x + z)d(y) \in Z(R) \]

\[ d(xy) + d(zy) + 2D(xy, zy) - d(x)y - d(z)y - 2D(x, z)y - xd(y) - zd(y) \in Z(R), \text{ for all } x, y, z \in U. \]

By using (4.2.27) in the above equation, we get
\[2D(xy,zy) - 2D(x,z)y \in Z(R)\]

\[D(xy,zy) - D(x,z)y \in Z(R), \text{ for all } x,y,z \in U. \tag{4.2.28}\]

We replace \(z\) by \(x\) in (4.2.28), we get

\[D(xy,xy) - D(x,x)y \in Z(R)\]

\[d(xy) - d(x)y \in Z(R), \text{ for all } x,y \in U. \tag{4.2.29}\]

We replace \(y\) by \(y + z\) in (4.2.29), we get

\[d(x(y + z)) - d(x)(y + z) \in Z(R)\]

\[d(xy + xz) - d(x)(y + z) \in Z(R)\]

\[d(xy) + d(xz) + 2B(xy,xz) - d(x)y - d(x)z \in Z(R), \text{ for all } x,y,z \in U.\]

By using (4.2.29) in the above equation, we get

\[B(xy,xz) \in Z(R), \text{ for all } x,y,z \in U. \tag{4.2.30}\]

We replace \(z\) by \(y\) in (4.2.30), we get

\[D(xy,xy) \in Z(R)\]

\[d(xy) \in Z(R), \text{ for all } x,y \in U. \tag{4.2.31}\]

By subtracting (4.2.29), from (4.2.31), we get

\[d(x)y \in Z(R), \text{ for all } x,y \in Z(R).\]

\[[d(x)y,r] = 0, \text{ for all } x,y \in U \text{ and } r \in R.\]

\[[d(x),r]y + d(x)[y,r] = 0, \text{ for all } x,y \in U \text{ and } r \in R. \tag{4.2.32}\]
We replace $r$ by $d(x)$ in (4.2.32), we get

\[ [d(x), d(x)]y + d(x)[y, d(x)] = 0 \]

\[ d(x)[y, d(x)] = 0, \text{ for all } x, y \in U. \]  

(4.2.33)

We replace $y$ by $2yz$ in (4.2.33), we get

\[ d(x)[2yz, d(x)] = 0 \]

\[ 2d(x)[y, d(x)] + 2d(x)y[z, d(x)] = 0 \]

By using (4.2.33) in the above equation we get

\[ 2d(x)y[z, d(x)] = 0 \]

\[ d(x)[z, d(x)] = 0, \text{ for all } x, y, z \in U. \]  

(4.2.34)

Multiplying (4.2.34) left by $z$, we get

\[ zd(x)y[z, d(x)] = 0, \text{ for all } x, y, z \in U. \]  

(4.2.35)

We replace $y$ by $2zy$ in (4.2.34), we get

\[ d(x)2zy[z, d(x)] = 0 \]

\[ d(x)zy[z, d(x)] = 0, \text{ for all } x, y, z \in U. \]  

(4.2.36)

By combining (4.2.35) and (4.2.36), we get

\[ [z, d(x)]y[z, d(x)] = 0 \]

\[ [z, d(x)]U[z, d(x)] = \{0\}. \]

By using lemma 4.2.1, we get $[z, d(x)] = 0$, for all $x, z \in U$ and lemma 4.2.5, we get $d = 0$. 

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