Chapter 3

LEFT GENERALIZED
JORDAN DERIVATIONS
IN PRIME AND SEMIPRIME RINGS
In chapter 3, we proved some results on lie ideals with left generalized Jordan derivations and left generalized Jordan triple derivations in prime and semiprime rings. In section 3.1, we proved that let $R$ be a non-commutative prime ring with characteristic not two, $U$ is a non-central lie ideal of $R$ such that $u^2 \in U$, for all $u \in U$. If $f$ be a left generalized Jordan derivation on $U$, then $f$ is a left generalized derivation on $U$. In section 3.2, we proved that let $R$ be a 2-torsion free semiprime ring, $L \subseteq Z(R)$ a square-closed lie ideal of $R$, and $F: R \to R$ an additive mapping, then $F$ is a left generalized Jordan triple derivation on $L$ associated with a Jordan triple derivation $f$ if $F$ is a left generalized Jordan derivation on $L$ associated with a Jordan derivation $f$, and let $R$ be a 2-torsion free semiprime ring, $L \subseteq Z(R)$ a square-closed lie ideal of $R$, $F: R \to R$ an additive map such that $F(L) \subseteq L$. If $F$ is a left generalized Jordan triple derivation associated with a Jordan triple derivation $f$, then $F$ is a left generalized derivation on $L$ associated with a derivation $f$. 
3.1 Lie Ideals with Left Generalized Jordan Derivations in Prime Rings:

The notation of generalized derivation of prime ring $R$ was introduced by Hvala.B in [21]. A classical result of Herstein.I.N states that every Jordan derivation of prime ring with characteristic not two is a derivation in [17]. A brief proof of this theorem can be found in Bresar.M [13]. Latter on this result was generalized on lie ideals of $R$ such that $u^2 \in U$, for all $u \in U$ in Awtar.R [6] and generalized derivations of prime ring $R$ in Ashraf.M [4]. In [29] Oznur Gölbaşi proved some results in generalized Jordan derivation on prime rings with lie ideals. For all these concerns we proved that if $f$ be a left generalized Jordan derivation on $U$, then $f$ is a left generalized derivation on $U$.

Throughout this section, let $R$ be a prime ring with its characteristic not two. $U$ be a non-zero Lie ideal of $R$. A ring $R$ is said to be prime if $xRy = \{0\}$, for all $x,y \in R$, implies $x = 0$ or $y = 0$. While the symbol $[x,y]$ will denote the commutator $xy - yx$.

**Lemma 3.1.1:** For all $u, v, w \in U$ the following statements hold:

i) $f(uv + vu) = d(u)v + uf(v) + d(v)u + vf(u)$;

ii) $f(uvu) = d(u)vu + ud(v)u + uvf(u)$;

iii) $f(uvw + wvu) = d(u)vw + ud(v)w + uvf(w) + d(w)vu + wd(v)u + wvf(u)$.

**Proof:** i) $f(u + v)^2 = f((u + v)(u + v)) = d(u + v)(u + v) + (u + v)f(u + v)$

$= d(u)u + d(u)v + d(v)u + d(v)v + uf(u) + uf(v) + vf(u) + vf(v) + vf(v)$.
\( f(u + v)^2 = d(u)u + d(u)v + d(v)u + d(v)v + uf(u) + uf(v) + vf(u) + vf(v), \) for all \( u, v \in U. \)  

(3.1.1)

On the other hand, we have

\[ f(u + v)^2 = f((u + v)(u + v)) \]

\[ = f(u^2 + uv + vu + v^2) = f(u^2) + f(uv + vu) + f(v^2) \]

\[ = d(u)u + uf(u) + f(uv + vu) + d(v)v + vf(v). \]

\( f(u + v)^2 = d(u)u + uf(u) + f(uv + vu) + d(v)v + vf(v), \) for all \( u, v \in U. \)  

(3.1.2)

From (3.1.1) and (3.1.2), we have

\[ d(u)u + d(u)v + d(v)u + d(v)v + uf(u) + uf(v) + vf(u) + vf(v) = d(u)u + uf(u) + f(uv + vu) + d(v)v + vf(v). \]

\( f(uv + vu) = d(u)v + uf(v) + d(v)u + vf(u), \) for all \( u, v \in U. \)

ii) Let \( W = f(u(uv + vu) + (uv + vu)u). \)

On one hand, we have

\[ W = d(u)(uv + vu) + uf(uv + vu) + d(uv + vu)u + (uv + vu)f(u) \]

\[ W = d(u)uv + d(u)vu + u(d(u)v + uf(v) + d(v)u + vf(u)) \]

\[ + (d(u)v + ud(v) + d(v)u + vd(u))u + uvf(u) + vu(f(u)) \]

\[ W = d(u)uv + d(u)vu + ud(u)v + u^2f(v) + ud(v)u + uvf(u) + vf(u) + (d(u)v + ud(v) + d(v)u + ud(u))u + uvf(u) + vu(f(u)), \] for all \( u, v \in U. \)  

(3.1.3)
On the other hand, we have

\[ W = f(u^2v + 2uvu + vu^2). \]

\[ W = d(u^2)v + u^2f(v) + 2f(uvu) + d(v)u^2 + vf(u^2) \]

\[ W = (d(u)u + ud(u))v + u^2f(v) + 2f(uvu) + d(v)u^2 + v(d(u)u + uf(u)) \]

\[ W = d(u)uv + ud(u)v + u^2f(v) + 2f(uvu) + d(v)u^2 + vd(u)u + vuf(u), \]

for all \( u, v \in U. \) (3.1.4)

From (3.1.3) and (3.1.4), we have

\[
d(u)uv + d(u)vu + ud(u)v + u^2f(v) + ud(v)u + uvf(u) \\
+ (d(u)v + ud(v) + d(v)u + vd(u))u + uvf(u) + vuf(u) \\
= d(u)uv + ud(u)v + u^2f(v) + 2f(uvu) + d(v)u^2 + vd(u)u \\
+ vuf(u).
\]

\[ 2f(uvu) = 2d(u)vu + 2ud(v)u + 2 uvf(u) \]

Since \( R \) is 2-torsion free, we get

\[ f(uvu) = d(u)vu + ud(v)u + uvf(u), \text{ for all } u, v \in U. \]

(iii) Linearizing (ii) by replacing \( u \) by \( u + w \)

\[ f((u + w)v(u + w)) = d(u + w)v(u + w) \]

\[ +(u + w)d(v)(u + w) + (u + w)vf(u + w) \]

From L.H.S.

\[ f((u + w)v(u + w)) = f((uv + vw)(u + w)) \]
\[ f(\nu + \nu w + w \nu + w w) \]
\[ = f(\nu) + f(\nu w) + f(\nu w) + f(w w) \]
\[ = d(u)\nu + u d(v)u + \nu v f(u) + f(\nu w + w \nu) + d(w)\nu w + w d(\nu w \nu) + w v f(w) \]
\[ = d(u)\nu + u d(v)u + \nu v f(u) + f(\nu w + w \nu) + d(w)\nu w + w d(\nu w \nu) + w v f(w), \text{ for all } u, v, w \in U. \]  

From R.H.S
\[ d(u + w)\nu (u + w) + (u + w)d(v)(u + w) + (u + w)\nu f(u + w) \]
\[ = (d(u) + d(w))(\nu + \nu w) + (u d(v) + w d(v))(u + w) \]
\[ + (\nu v + w v)(f(u) + f(w)) \]
\[ = d(u)\nu + u d(v)w + d(w)\nu u + d(w)\nu w + u d(v)u + u d(v)w + w d(v)u + w d(v)w + \nu f(u) + w v f(u) + w v f(w) + w v f(w), \text{ for all } u, v, w \in U. \]  

From (3.1.5) and (3.1.6), we get
\[ d(u)\nu + u d(v)u + \nu v f(u) + f(\nu w + w \nu) + d(w)\nu w + w d(\nu w \nu) + w v f(w) \]
\[ = d(u)\nu + u d(v)w + d(w)\nu u + d(w)\nu w + u d(v)u + u d(v)w \]
\[ + w d(v)u + w d(v)w + \nu f(u) + w v f(u) + w v f(u) + w v f(w) + w v f(w). \]
\[ f(\nu w + w \nu) = d(u)\nu w + u d(v)w + \nu v f(u) + d(w)\nu u + w d(v)u + \nu v f(u), \]
for all \( u, v, w \in U. \)
Remark 3.1.1:

We introduce abbreviation

\[ u^v = f(uv) - d(u)v - uf(v), \text{ for all } u, v \in U. \]

Observe also by lemma 3.1.1(i), we have

\[ f(uv + vu) = d(u)v + uf(v) + d(v)u + vf(u) \]

And so that

\[
\begin{align*}
 f(uv) - d(u)v - uf(v) &= d(u)v + uf(v) + d(v)u + vf(u) \\
 f(uv) - d(u)v - uf(v) &= f(uv) + d(v)u + vf(u) \\
 f(uv) - d(u)v - uf(v) &= -(f(vu) - d(v)u - vf(v))
\end{align*}
\]

From above we get,

\[ u^v = -v^u, \text{ for all } u, v \in U. \quad (3.1.7) \]

Lemma 3.1.2: For all \( u, v \in U \), then \( v^u[u, v] = 0. \)

Proof: Replace \( w \) by \( vu \) in Lemma 3.1.1(iii), and using the fact that \( \text{char}\, R \not= 2 \), we get

\[
\begin{align*}
 f(u^2v^2u + (vu)^2) &= d(u)v^2u + ud(v^2)u + uv^2f(u) + d(vu)vu + vuf(vu) \\
 &= d(u)v^2u + u(d(v)v + vd(v))u + uv^2f(u) + (d(v)u + vd(u))vu + vuf(vu) \\
 &= d(u)v^2u + ud(v)vu + uvf(v)u + d(v)uvu + vd(u)vu + vuf(vu) \\
 f(u^2v^2u + (vu)^2) &= d(u)v^2u + ud(v)vu + uvf(v)u + uv^2f(u) + d(v)uvu + \]

\[ vd(u)vu + vuf(vu), \text{ for all } u, v \in U. \quad (3.1.8) \]
On the other hand, we get

\[
\begin{align*}
  f((uv)(vu) + (vu)(vu)) &= d(u)v^2u + ud(v)vu + uvf(vu) + d(v)vu + vud(v)u + vuvf(u) \\
  &= d(u)v^2u + ud(v)vu + uvf(vu) + (d(v)u + vdu)v + vud(v)u + vuvf(u) \\
  &= d(u)v^2u + ud(v)vu + uvf(vu) + d(v)uvu + vud(v)u + vuvf(u) \\
  f((uv)(vu) + (vu)(vu)) &= d(u)v^2u + ud(v)vu + uvf(vu) + d(v)uvu + \\
  vud(v)u + vud(v)u + vuvf(u), \text{ for all } u, v \in U. & \quad (3.1.9)
\end{align*}
\]

Comparing equations (3.1.8) and (3.1.9), we have

\[
uvd(v)u + uv^2f(u) + vuf(v)u = uvf(vu) + vud(v)u + vuvf(u), \text{ for all } u, v \in U.
\]

Therefore, we get

\[
f(vu) - d(v)u - vf(u)[u, v] = 0, \text{ for all } u, v \in U.
\]

\[
v^u[u, v] = 0, \text{ for all } u, v \in U.
\]

**Theorem 3.1.1:** Let \( R \) be a non-commutative prime ring with characteristic not two, \( U \) is a non central lie ideal of \( R \) such that \( u^2 \in U \), for all \( u \in U \). If \( f \) be a left generalized Jordan derivation on \( U \), then \( f \) is a left generalized derivation on \( U \).

**Proof:** From Lemma 3.1.1 (iii), we have

\[
f(uvw + vwu) = d(u)vw + ud(w)v + uvf(v) + d(v)wu + vdw(u)w + vwf(u),
\]

for all \( u, v, w \in U \). \quad (3.1.10)

Replacing \( u \) by \( vu \) and \( v \) by \( uv \) in equation (3.1.10), we get
\[
f((vu)w(uv) + (uv)w(vu)) = f((vu)w(uv)) + f((uv)w(vu))
\]
\[
= d(vu)wuv + vwd(w)uv + vuwf(uv) + d(uv)wvu + uvd(w)vu + uvwf(vu)
\]
\[
= (d(v)u + vd(u))wuv + vwd(w)uv + vuwf(uv) + (d(u)v + ud(v))wvu
\]
\[
+uvd(w)vu + uvwf(vu)
\]
\[
= d(v)uwuv + vd(u)wuv + vud(w)uv + vuwf(uv) + d(u)vwwu + ud(v)wvu
\]
\[
+uvd(w)vuv + uvwf(vu)
\]
\[
f((vu)w(uv) + (uv)w(vu)) = d(v)uwuv + vd(u)wuv + vud(w)uv +
\]
\[
\quad vuwf(uv) + d(u)vwwu + ud(v)vwwu + vwd(w)vuv + uvwf(vu), \text{ for all } u, v, w \in U.
\] (3.1.11)

On the other hand, we have
\[
f(v(uwu)v + u(vwv)u) = f(v(uwu)v) + f(u(vwv)u)
\]
\[
= d(v)uwuv + vd(uwu)v + vuwf(v) + d(u)vwwu + ud(vwv)u + uvwf(u)
\]
\[
= d(v)uwuv + vd(u)wuv + vud(w)uv + vuwd(u)v + vuwf(v) + d(u)vwwu
\]
\[
+ud(v)vuv + uvd(w)vuv + uvwd(v)u + uvwf(u)
\]
\[
f(v(uwu)v + u(vwv)u) = d(v)uwuv + vd(u)wuv + vuwd(u)v +
\]
\[
\quad vuwd(u)v + vuwf(v) + d(u)vwwu + ud(v)vwwu + vwwu + uvwd(v)u +
\]
\[
\quad uvwf(u), \text{ for all } u, v, w \in U.
\] (3.1.12)

Comparing equations (3.1.11) and (3.1.12), we obtain
\[
\quad vuwf(uv) + uvwf(vu) = vuwd(u)v + vuwf(v) + uvwd(v)u + uvwf(u),
\]
for all \( u, v, w \in U \).

\[
uvw(f(vu) - d(v)u - vf(u)) + vuw(f(uv) - d(u)v - uf(v)) = 0, \text{ for all } u, v, w \in U.
\]

\[
uvw^u + vuw^u = 0, \text{ for all } u, v, w \in U.
\]

Using the equation (3.1.71), that is \( u^v = -v^u \), for all \( u, v \in U \). We get

\[
[v, u]Uu^v = 0, \text{ for all } u, v \in U.
\]

Since \( R \) is prime ring, we have either \( [v, u] = 0 \) or \( u^v = 0 \).

Since \( R \) is non-commutative, we have \( [v, u] \neq 0 \).

Thus we have \( u^v = 0, \text{ for all } u, v \in U \).

This completes the proof.

3.2 Lie Ideals with Left Generalized Jordan Triple Derivations in Semiprime Rings:

Every derivation is a Jordan derivation. In general, the converse is not true.

A classical result of Herstein [17] asserts that any Jordan derivation on a prime ring of characteristic different from two is a derivation. A brief proof of the Herstein theorem can be found in Bresar and Vukman [13]. Cusack [15] has generalized Herstein theorem to 2-torsion free semiprime ring. Bresar [12] had proved that any Jordan triple derivation on a 2-torsion free semiprime ring is a derivation. Zalar [39] has proved that any left(right) Jordan centralizer on a semiprime ring is a centralizer.

The concept of generalized derivation has been introduced by Bresar in [12] and also the concept of generalized Jordan derivation and generalized Jordan triple derivation.
have been introduced by Jing and Lu in [24]. Vukman [35] has shown that any generalized Jordan derivation on 2-torsion free semiprime ring is a generalized derivation, and that any generalized Jordan triple derivation on 2-torsion free semiprime ring is a generalized derivation. Hongan et al [19] proved every Jordan triple (resp. generalized Jordan triple) derivation on lie ideal L is a derivation on L (resp. generalized derivation on L). Hongan and Rehman in [20] also extended their results to generalized Jordan triple derivations on lie ideals in semiprime rings. Motivated by above work we extended our results to lie ideals with left generalized Jordan triple derivations in semiprime rings.

Throughout this section, let $R$ will denote an associative ring with centre $Z(R)$. $U$ be a non-zero lie ideal of $R$. An additive mapping $T: R \rightarrow R$ is called a left (right) centralizer in case $T(xy) = T(x)y$ ($T(xy) = xT(y)$), holds for all pairs $x, y \in R$. An additive mapping $T: R \rightarrow$ is called a two-sided centralizer if $T$ is both a left and right centralizer. An additive mapping $T: R \rightarrow R$ is called a left (right) Jordan centralizer in case $T(x^2) = T(x)x$ ($T(x^2) = xT(x)$), holds for all $x \in R$. A ring $R$ is called semiprime if $xax = 0$ implies $x = 0$, for all $x, a$ in $R$. While the symbol $[x, y]$ will denote the commutator $xy - yx$.

**Lemma 3.2.1:** Let $R$ be a 2-torsion free semiprime ring, $L \subseteq Z(R)$ a square closed Lie ideal of $R$ and 'a' a fixed element of $L$. If $[x, y]a = 0$, for all $x, y \in L$, then $a \in Z(R) \cap L(=Z(L))$.

**Proof:** Given that $[x, y]a = 0$. (3.2.1)

Substituting $ay$ for $y$ in equation (3.2.1) and using (3.2.1), we have
Chapter 3

\([x, ay]a = 0\]

Implies \(a[x, y]a + [x, a]ya = 0\)

Implies \([x, a]ya = 0\). \hspace{1cm} (3.2.2)

So, we get \([x, a]yax = 0\). \hspace{1cm} (3.2.3)

And substituting \(yx\) for \(y\) in equation (3.2.2), we get

\([x, a]yxa = 0\). \hspace{1cm} (3.2.4)

Subtracting equation (3.2.3) from equation (3.2.4), we have

\([x, a]yxa - [x, a]yax = 0\)

Implies \([x, a]y(xa - ax) = 0\)

Implies \([x, a]y[x, a] = 0\), by [19, Corollary 2.1]

We have \([x, a] = 0\), for all \(x \in L\)

i.e., \(a \in Z(L)\).

**Lemma 3.2.2:** Let \(R\) be a ring, \(L \subseteq Z(R)\) a square-closed lie ideal of \(R\), and an additive mapping \(F: R \rightarrow R\) such that \([x, y]F(x) = 0\), for all \(x, y \in L\) and \(F(L) \subseteq L\).

1. If \(R\) is 2-torsion free semiprime, then \(F(x) \in Z(L)\), for all \(x \in L\).

2. If \(R\) is prime of \(char(R) \neq 2\), then \(F(x) = 0\), for all \(x \in L\).

**Proof:** (1) Suppose that \([x, y]F(x) = 0\), for all \(x, y \in L\). \hspace{1cm} (3.2.5)

Substituting \(yz\) for \(y\) in equation (3.2.5) and using equation (3.2.5), we have

\([x, yz]F(x) = 0\)
Implies \(y[x, z]F(x) + [x, y]zF(x) = 0\)

Implies \([x, y]zF(x) = 0\) for all \(x, y, z \in L\). \((3.2.6)\)

Replacing \(x\) by \(x + u\) in equation (3.2.5) and using equation (3.2.5), we have

\([x + u, y]F(x + u) = 0\)

\([x + u, y]F(x) + [x + u, y]F(u) = 0\)

\([x, y]F(x) + [u, y]F(x) + [x, y]F(u) + [u, y]F(u) = 0\)

\([u, y]F(x) + [x, y]F(u) = 0\)

\([x, y]F(u) = 0\) for all \(x, y, u \in L\). \((3.2.7)\)

Furthermore, substituting \(F(u)z[u, y]\) for \(z\) in equation (3.2.6), we have

\([x, y]F(u)z[u, y]F(x) = 0\)

Substituting equation (3.2.7) in the above equation, we get

\([-u, y]F(x)z[u, y]F(x) = 0\)

And so, we have \([u, y]F(x)z[u, y]F(x) = 0\) for all \(z \in L\).

By [19, corollary 2.1(1)], we have

\([u, y]F(x) = 0\), for all \(u, x, y \in L\), by lemma 3.2.1

\(F(x) \in Z(L), \) for all \(x \in L\). \((3.2.8)\)

(2) Since \(F(x) \in Z(L), \) for all \(x \in L\) by equation (3.2.8),

\([x, y]LF(x) = 0\), for all \(x, y \in L\).
Now we write $\tau = F - f$.

Then we have $\tau(xy) = (F - f)(xy)$

\[
\begin{align*}
\tau(xy) &= F(xy) - f(xy) \\
&= f(x)yx + xf(y)x + xyF(x) - (f(x)yx + xf(y)x + yxf(x)) \\
&= f(x)yx + xf(y)x + xyF(x) - f(x)yx - xf(y)x - yxf(x) \\
&= xyF(x) - yxf(x) \\
&= xy(F(x) - f(x)) \\
&= xy(F - f)(x) \\
&= x\tau(x), \text{ for all } x, y \in L.
\end{align*}
\]

In other words, $\tau$ is a Jordan triple right centralizer on $L$.

Since $R$ is a 2-torsion free semiprime ring, one can conclude that $\tau$ is a Jordan right centralizer by [19, theorem 3.1].

Hence $F$ is of the form $F = \tau + f$, where $f$ is a derivation and $\tau$ is a Jordan right centralizer on $L$. Hence $F$ is a left generalized Jordan derivation on $L$.

(2) $\rightarrow$ (1): Suppose that $F(x^2) = f(x)x + xF(x)$.

Replacing $x$ by $x + y (y \in L)$ in (3.2.10) and using (3.2.10), we have

\[
\begin{align*}
F((x + y)^2) &= (f(x + y))(x + y) + (x + y)(F(x + y)) \\
F((x + y)(x + y)) &= (f(x) + f(y))(x + y) + (x + y)(F(x) + F(y))
\end{align*}
\]
Replacing $y$ by $xy + yx$ in (3.2.11) and using (3.2.11), we have

\[ F(xy + yx) = f(x)y + f(y)x + xf(y) + yf(x). \]  

On the other hand, substituting $x^2$ for $x$ in equation (3.2.11) and adding $2F(xy)$ to both sides, we have
From equation (3.2.12) and equation (3.2.13), we get

$$F(x^2 y + y x^2) + 2F(xy x) = f(x^2)y + f(y)x^2 + x^2 F(y) + y F(x^2) + 2F(xy x)$$

$$= (f(x)x + xf(x))y + f(y)x^2 + x^2 F(y) + y(f(x)x + xF(x)) + 2F(xy x)$$

$$= f(x)xy + xf(x)y + f(y)x^2 + x^2 F(y) + yf(x)x + yxF(x) + 2F(xy x) \ (3.2.13)$$

From equation (3.2.12) and equation (3.2.13), we get

$$f(x)xy + f(x)yx + f(x)yy + f(y)x + f(y)x^2 + yf(x)x + xf(x)y + x^2 F(y)$$

$$+ xf(y)x + xyF(x) + xyF(x) + yxF(x)$$

$$= f(x)xy + xf(x)y + f(y)x^2 + x^2 F(y) + yf(x)x + yxF(x)$$

$$+ 2F(xy x)$$

$$f(x)xy + f(x)yx + f(x)yx + xf(y)x + f(y)x^2 + yf(x)x + xf(x)y + x^2 F(y)$$

$$+ xf(y)x + xyF(x) + xyF(x) + yxF(x)$$

$$- (f(x)xy + xf(x)y + f(y)x^2 + x^2 F(y) + yf(x)x + yxF(x)$$

$$+ 2F(xy x)) = 0$$

$$f(x)xy + f(x)yx + f(x)yx + xf(y)x + f(y)x^2 + yf(x)x + xf(x)y + x^2 F(y)$$

$$+ xf(y)x + xyF(x) + xyF(x) + yxF(x) - f(x)xy - xf(x)y$$

$$- f(y)x^2 - x^2 F(y) - yf(x)x - yxF(x) - 2F(xy x) = 0$$

$$2f(x)yx + 2xf(y)x + 2xyF(x) - 2F(xy x) = 0$$

$$2(f(x)yx + xf(y)x + xyF(x) - F(xy x)) = 0$$

Since $R$ be 2-torsion free semi prime ring, then

$$f(x)yx + xf(y)x + xyF(x) - F(xy x) = 0$$

$$f(x)yx + xf(y)x + xyF(x) = F(xy x) \ (or)$$
Since $f$ is a derivation, $f$ is a Jordan triple derivation and $F$ is a generalized left Jordan triple derivation on $L$ associated with a Jordan triple derivation $f$.

**Theorem 3.2.2**: Let $R$ be a 2-torsion free semiprime ring, $L \subseteq Z(R)$ a square-closed lie ideal of $R, F : R \to R$ an additive map such that $F(L) \subseteq L$. If $F$ is a left generalized Jordan triple derivation associated with a Jordan triple derivation $f$, then $F$ is a left generalized derivation on $L$ associated with a derivation $f$.

**Proof**: Suppose that there exists a Jordan triple derivation $f$ on $L$ such that

$$F(xyx) = f(x)yx + xf(y)x + xyF(x), \quad x \in L.$$  \hfill (3.2.14)

Since $R$ is a 2-torsion free ring, $f$ is a derivation by [19, theorem 2.1]

Substituting $x + z$ ($z \in L$) for $x$ in equation (3.2.14) and using (3.2.14), we have

$$F((x + z)y(x + z)) = f(x + z)y(x + z) + (x + z)f(y)(x + z)$$

$$+ (x + z)f(y)(x + z) + (x + z)yF(x + z)$$

$$F(xyx + xyz + zyx + zyz) = (f(x) + f(z))(yx + yz)$$

$$+ (xf(y) + zf(y))(x + z) + (xy + zy)(F(x) + F(z))$$

$$F(xyx) + F(xyz + zyx) + F(zyz) = f(x)yx + f(x)yz$$

$$+ f(x)yx + f(z)yz + xf(y)x + xf(y)z + zf(y)x + zf(y)z + xyF(x) + zyF(x)$$

$$+ xyF(z) + zyF(z) = [f(x)yx + xf(y)x + xyF(x)]$$

$$+ f(x)yz + f(z)yx + zf(y)x + zf(y)z + xyF(x) + zyF(z) + [f(z)yz + zf(y)z$$

$$+ zyF(z)]$$

45
\[ F(xyx) + F(xyz + zyx) + F(zyz) \]
\[ = F(xyx) + f(x)yz + f(z)yx + xf(y)z + zf(y)x + zyF(x) + xyF(z) + F(zyz) \]
\[ F(xyz + zyx) = f(x)yz + f(z)yx + xf(y)z + zf(y)x + xyF(z) + zyF(x). \]

(3.2.15)

Now we set \( A = F(xyzx + yxzx) \) and we shall compute it in two different ways using equation (3.2.14), we have

\[ A = F((x)y)z(x) + y(xzx)y) \]
\[ = F(x(zy)yx) + F(y(xzx)y) \]
\[ = f(x)yzyx + x[f(y)yz + yf(z)y + zf(y)]x + xyzyF(x) + f(y)xzx + yf(x)xzy + yxf(z)xy + yxzF(x)y + yxzxF(y) \]
\[ = f(x)yzyx + xyf(z)yx + xyf(y)x + xyzyF(x) + f(y)xzx + yf(x)xzy + yxf(z)xy + yxzF(x) + yxzxF(y). \]

(3.2.16)

Using (3.2.15), \( A = F((xy)z(xy) + (yx)z(xy)) \)
\[ = f(xy)zyx + f(yzx)zxy + xyf(z)yx + yzf(x)xy + xzf(y)yx + yxzF(xy) \]
\[ = f(x)yxz + [f(x)y + f(y)x]zxy + [f(y)x + yf(x)]zxy + xyf(x)yx + yxf(x)xy + xzF(xy) + yxzF(xy) \]
\[ = f(x)yzyx + xyf(z)zxz + yf(x)zxz + xzf(y)yx + yxf(z)yx + yxf(z)xy + xzF(xy) + yxzF(xy). \]

(3.2.17)
From equation (3.2.16) and equation (3.2.17), we get

\[ f(x)yxz + xf(y)yx + f(y)xzx + yf(x)zxy + yxf(z)yx + yxf(z)xy + xyzF(yx) + yxzF(xy) = f(x)yxz + xf(y)yx + yf(z)x + xzF(yx) \]

\[ + f(x)xzx + yf(x)xzx + yxf(z)xz + yxz(f)(y) + xzF(xy) \]

\[ f(x)yxz + xf(y)yx + f(y)xzx + yf(x)zxy + yxf(z)yx + yzf(z)xy + xzF(yx) + xzF(xy) - f(x)yxz - xf(y)xzx - yf(z)yx \]

\[ - xzF(yx) - xzF(xy) - f(x)xzx - yf(x)zxy - yzf(z)xy \]

\[ - yzF(yx) - yzF(xy) - yzF(yx) - yzF(xy) = 0 \]

\[ xzF(yx) - xzF(xy) - yzF(xy) - yzF(xy) - xzF(xy) - yzF(xy) = 0 \]  \hspace{1cm} (3.2.18)

Now putting \( \alpha(x,y) = F(xy) - f(x)y - xF(y) \).

\[ \alpha(x,y) = F(xy) - f(x)y - xF(y) \]  \hspace{1cm} (3.2.19)

Substituting equation (3.2.19) in equation (3.2.18), we have

\[ xzF(yx) - f(y)x - yF(x) + yzF(xy) - f(x)y - xF(y) = 0 \]  \hspace{1cm} (3.2.20)

By the way, \( F \) is a generalized left derivation on \( L \) associated with a derivation \( f \) by theorem 3.2.1, and so \( F(x^2) = f(x)x + xf(x) \), for all \( x \in L \).
Substituting $x + y$ for $x$ in equation (3.2.21), we have

$$F((x + y)^2) = (f(x + y))(x + y) + (x + y)(F(x + y))$$

$$F((x + y)(x + y)) = (f(x) + f(y))(x + y) + (x + y)(F(x) + F(y))$$

$$F(x^2 + xy + yx + y^2) = f(x)x + f(x)y + f(y)x + f(y)y$$

$$+ f(y)y + xF(x) + xF(y) + yF(x) + yF(y)$$

$$F(x^2) + F(xy + yx) + F(y^2) = [f(x)x + xF(x)] + f(x)y$$

$$+ f(y)x + xF(y) + yF(x) + [f(y)y + yF(y)]$$

$$F(x^2) + F(xy + yx) + F(y^2) = F(x^2) + f(x)y + f(y)x$$

$$+ xF(y) + yF(x) + F(y^2)$$

$$F(xy + yx) = f(x)y + f(y)x + xF(y) + yF(x) , \text{ for all } x, y \in L . \quad (3.2.22)$$

$$F(xy) + F(yx) = f(x)y + f(y)x + xF(y) + yF(x)$$

$$F(xy) - f(x)y - xF(y) = -F(yx) + f(y)x + yF(x)$$

$$F(xy) - f(x)y - xF(y) = -[F(yx) - f(y)x - yF(x)]$$

Now using equation (3.2.19), we have

$$\alpha(x, y) = -\alpha(y, x) , \text{ for all } x, y \in L . \quad (3.2.23)$$

Substituting equation (3.2.23) in equation (3.2.20), we have

$$-xyz\alpha(x, y) + yxz\alpha(x, y) = 0$$

$$(yx - xy)z\alpha(x, y) = 0$$
[x, y]z \alpha(x, y) = 0, for all x, y, z \in L.

By [19, lemma 2.4], we get [u, v]z \alpha(x, y) = 0, for all \ x, y, z, u, v \in L and

we have [u, v] \alpha(x, y) = 0, for all x, y, u, v \in L by [19, corollary 2.1(3)] and

so, we get \alpha(x, y) \in Z(L) by lemma 3.2.1

Now, we put \alpha(x, y) = \alpha and B = F(xy\alpha + axy), we will compute that in two different ways.

Using equation (3.2.22), we have B = F(xy\alpha + axy)

= f(xy)\alpha + f(\alpha)xy + xyF(\alpha) + \alpha F(xy)

= [f(x)y + xf(y)]\alpha + f(\alpha)xy + xyF(\alpha) + \alpha F(xy)

= f(x)y\alpha + xf(y)\alpha + f(\alpha)xy + xyF(\alpha) + \alpha F(xy). \quad (3.2.24)

Because of \alpha \in Z(L), using equation (3.2.15), we have

B = F(xy\alpha + axy)

= f(x)y\alpha + f(\alpha)xy + xf(y)\alpha + af(x)y + xyF(\alpha) + axF(y). \quad (3.2.25)

From equation (3.2.24) and equation (3.2.25), we have

f(x)y\alpha + xf(y)\alpha + f(\alpha)xy + xyF(\alpha) + \alpha F(xy)

= f(x)y\alpha + f(\alpha)xy + xf(y)\alpha + af(x)y + xyF(\alpha) + axF(y)

f(x)y\alpha + xf(y)\alpha + f(\alpha)xy + xyF(\alpha) + \alpha F(xy) - f(x)y\alpha - f(\alpha)xy - xf(y)\alpha

- \alpha f(x)y - xyF(\alpha) - axF(y) = 0

\alpha F(xy) - \alpha f(x)y - axF(y) = 0
Using equation (3.2.19), we have

\[ \alpha(F(xy) - f(x)y - xF(y)) = 0 \]

And so, we have \( \alpha a = 0 \).

Since \( R \) is semiprime, \( \alpha = 0 \).

That is, \( F(xy) = f(x)y + xF(y) \), for all \( x, y \in L \).

By Theorems 3.2.1 and 3.2.2 we obtain the following result which explains the relationships of left generalized Jordan triple derivations, left generalized derivations and left generalized Jordan derivations.

**Corollary 3.2.1:** Let \( R \) be a 2 -torsion free semiprime ring, \( L \subseteq Z(R) \) a square-closed lie ideal of \( R \) and let \( F: R \to R \) be an additive map such that \( F(L) \subseteq L \), then the following are equivalent:

1. \( F \) is a left generalized Jordan triple derivation on \( L \) associated with a Jordan triple derivation \( f \).
2. \( F \) is a left generalized derivation on \( L \) associated with a derivation \( f \).
3. \( F \) is a left generalized Jordan derivation on \( L \) associated with a derivation \( f \).