Now assume that $D \neq 0$, and let $x \in Z \setminus \{0\}$.

Then $f(x^2) = f(x)x + xD(x)Z$.

$(f(x)x + xD(x))y = y(f(x)x + xD(x))$, for all $x, y \in N$,

and lemma 2.1.3 we see that, Since both $f(x)$ in $Z$, we have

$D(x)(xy - yx) = 0$, for all $x, y \in N$.

Provided that $D(Z) \neq \{0\}$, we can conclude that $N$ is commutative.

(ii) Assume that $D \neq 0$ and $D(Z) = \{0\}$.

In particular $D(f(x)) = 0$, for all $x \in N$.

Note that for $x \in N$ such that $f(x) = 0$.

$f(x^2) = f(x)x + xD(x) = xD(x) \in Z$

Hence by lemma 2.1.2, $D(x)D(y) \in Z$ and $D(y)D(x) \in Z$, for each $x, y \in N$. If one of these is zero, the other is a central element squaring to 0, hence is also 0. The remaining possibility is that $D(x)D(y)$ and $D(y)D(x)$ are non-zero central elements, in which case $D(x)$ is not a zero divisor. Thus $D(x)D(x)D(y) = D(x)D(y)D(x)$ yields $D(x)(D(x)D(y) - D(y)D(x)) = 0$.

$D(x)D(y) - D(y)D(x) = 0$. Consequently, $N$ is commutative.

Theorem 2.1.2: Let $N$ be a 3-prime near ring and let $f$ be a generalized Jordan derivation on $N$ with associated derivation $D$. If $f^2 = 0$, then $D^3 = 0$. Moreover, if $N$ is 2-torsion free, then $D(z) = \{0\}$.

Proof: We have $f^2(x^2) = 0$. 

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Applying again $f$ to the equation

\[ f(f(x)D(x)) = 0 \]

\[ \Rightarrow f^2(x) + f(x)D(x) + f(x)D(x) + xD^2(x) = 0. \]

\[ f(x)D(x) + f(x)D(x) + xD^2(x) = 0 \]

Applying again $f$ to the equation

\[ f(f(x)D(x) + f(x)D(x) + xD^2(x)) = 0 \]

\[ f^2(x)D(x) + f(x)D^2(x) + f^2(x)D(x) + f(x)D^2(x) + f(x)D^2(x) + xD^3(x) = 0 \]

\[ f(x)D^2(x) + f(x)D^2(x) + f(x)D^2(x) + xD^3(x) = 0 \quad (2.1.8) \]

Now we have to find $f^2(xy) = 0$

\[ f(f(x)) = 0 \]

\[ f(f(x)y + xD(y)) = 0 \]

\[ f(f(x)y) + f(xD(y)) = 0 \]

\[ f^2(x)y + f(x)D(y) + f(x)D(y) + xD^2(y) = 0 \]

\[ f(x)D(y) + f(x)D(y) + xD^2(y) = 0. \quad (2.1.9) \]

Substitute $D(x)$ for $y$ in equation (2.1.9) gives

\[ f(x)D^2(x) + f(x)D^2(x) + xD^3(x) = 0. \quad (2.1.10) \]

From (2.1.8) & (2.1.10) we get $f(x)D^2(x) = 0$, for all $x \in N$. \hfill (2.1.11)

It now follows from (2.1.10) that $xD^3(x) = 0$, for all $x \in N$ and since $N$ is 3-prime, $D^3 = 0$. 

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Suppose now that $N$ is 2-torsion free and that $D(z) \neq \{0\}$ and let $z \in Z$ be such that $D(z) \neq 0$.

Then if $x, y \in N$ and $f(N)x = \{0\}$, then

$$f(xz)x = f(x)zx + xD(z)x = 0$$

$\Rightarrow xD(z)x = 0$ and since $N$ is 3-prime and $D(z)$ is not a zero divisor, $x = 0$.

It follows from equation (2.1.10) that $D^2 = 0$ and hence by (A) that $D = 0$.

But this contradicts our assumption that $D(z) \neq \{0\}$.

Hence $D(z) = \{0\}$.

**Theorem 2.1.3:** Let $N$ be a 2, 3-torsion free prime near ring with 1. If $f$ is a generalized Jordan derivation on $N$ such that $f^2 = 0$ and $f(1) \in Z$, then $f = 0$.

**Proof:** $f(x) = f(1.x) = f(1)x + 1D(x)$.

So $f(x) = cx + D(x)$. $c \in z$. \hspace{1cm} (2.1.12)

If $c = 0$, then $f = D$ and $D^2 = 0$, so $D = 0$ by (A) and therefore $f = 0$.

If $c \neq 0$, then $c$ is not a zero divisor, hence by equation (2.1.11) $D^2 = 0$ and $D = 0$.

But then $f(x) = cx$ and $f^2(x) = c^2x = 0$, for all $x \in N$.

Since $c^2$ is not a zero divisor, we get $N = \{0\}$ is a contradiction.

Thus $c = 0$.

Let us we introduced a generalized derivation $f$ also satisfied the following property

$$f(x^2) = D(x)x + xf(x), \text{ for all } x \in N, \text{ where } D \text{ is derivation.} \hspace{1cm} (*)$$
Using the above property we prove the following theorem:

**Theorem 2.1.4:** Let $N$ be a 2, 3-torsion free prime near ring which admits a generalized Jordan derivation $f$ with non zero associated derivation $D$ such that $f$ satisfies $(*)$. Then $N$ is a commutative ring.

**Proof:** From lemma 2.1.4 says that $f(N) \subseteq Z$ or $D(f(N)) = 0$, then $f(D(N)) = 0$

Now we calculate $f(D(x)D(x))$ in two ways

Using the defining property of $f$, we obtain

$$f(D(x)D(x)) = f(D(x))D(x) + D(x)D^2(x) = D(x)D^2(x). \quad (2.1.13)$$

And using $(*)$, we obtain

$$f(D(x)D(x)) = D^2(x)D(x) + D(x)f(D(x)) = D^2(x)D(x)$$

Thus $D^2(x)D(x) = D(x)D^2(x)$, for all $x \in N$.

But from lemma 2.1.4, equation (2.1.6), we get

$$D(x)D^2(x) = 0, \text{ for all } x \in N.$$ 

Hence by lemma 2.1.1(iii) and using (A), we have $D^2 = 0$, thus $D = 0$.

Then $N$ is commutative.

### 2.2 Generalized Jordan Derivations in Near Rings:

The study of derivation of nearing was initiated by Bell H.E. and Mason.G. in [8]. The notation of generalized derivation and generalized Jordan derivation of a prime ring was introduced by Bresar M. and Hvala B. in [12] and [21]. Oznur Golbasi [30] have studied some results on near rings with generalized derivations.
Some recent results concerning commutativity in prime near rings with derivation have been generalized Jordan in several ways. Many authors have investigated these theorems for generalized Jordan derivations. In this section we proved some results on generalized Jordan derivations in near rings.

Throughout this section N stands for a near ring with multiplicative center Z.

According to [8], a near ring N is said to be prime if $xy = 0$ for all $x, y \in N$, implies $x = 0$ or $y = 0$. For $x, y \in N$ the symbol $(x, y)$ will denote the additive-group commutator $x + y - x - y$, while the symbol $[x, y]$ will denote the commutator $xy - yx$.

**Lemma 2.2.1:** [2, Lemma] Let $N$ be a near ring and $d$ be a Jordan derivation of $N$, then $a(xd(x) + d(x)x) = axd(x) + ad(x)x, \ \forall a, x \in N$.

**Lemma 2.2.2:** Let $N$ be a near ring $d$ be a non-zero Jordan derivation of $N$ and $a \in N$. If $ad(N) = 0$ and $d(N)a = 0$, then $a = 0$.

**Proof:** Suppose that $ad(N) = 0$, for every arbitrary $x \in N$, we have $0 = ad(x^2) = ad(x)x + axd(x)$

By the hypothesis $axd(x) = 0$, for all $x \in N$

Since $N$ is near ring and $d \neq 0$, we get $a = 0$

Again, if $d(N)a = 0$, for every arbitrary $x \in N$, we have

$d(x^2)a = 0$

$d(x)xa + xd(x)a = 0$
axd(x) + xd(x)a = 0

By the hypothesis, we have \( axd(x) = 0 \), for all \( x \in N \).

Since \( N \) is near ring and \( d \neq 0 \), we get \( a = 0 \).

**Lemma 2.2.3:** Let \( N \) be a 2-torsion free near ring and \( d \) be a Jordan derivation of \( N \). If \( d^2 = 0 \), then \( d = 0 \).

**Proof:** For arbitrary \( x \in N \), we have

\[
0 = d^2(x) = d(d(x)) = d(x)x + xd(x)
\]

By the hypothesis, we have

\[
2d(x)d(x) = 0 \quad \forall x \in N
\]

By the hypothesis, since \( N \) is a 2-torsion free near ring, we get

\[
d(x)d(x) = 0 \quad \text{for all} x \in N
\]

\[
d(N) = 0 \quad \text{for all} x \in N, \text{then} \ d = 0.
\]

**Lemma 2.2.4:** Let \( N \) be a near ring and \( d \) a non-zero Jordan derivation of \( N \). If \( d(N) \subset Z \), then \( (N, +) \) is abelian. Moreover, if \( N \) is 2-torsion free, then \( N \) is commutative ring.

**Proof:** Suppose that \( a \in N \) such that \( d(a) \neq 0 \). So that \( d(a) \in Z \setminus \{0\} \)
and \( d(a) + d(a) \in Z \setminus \{0\} \), for all \( x, y \in N \), we have

\[
(d(a) + d(a))(x + y) = (x + y)(d(a) + d(a))
\]

That is \( d(a)x + d(a)x + d(a)y + d(a)y = xd(a) + yd(a) + xd(a) + yd(a) \)

Since \( d(a) \in Z \), we get

\[
xd(a) + yd(a) = yd(a) + xd(a)
\]

and so, \((x, y) d(a) = 0\), for all \( x, y \in N \).

Since \( d(a) \in Z \setminus \{0\} \) and \( N \) is near ring, we get \((x, y) = 0\), for all \( x, y \in N \).

Thus \((N, +)\) is abelian.

Now using the hypothesis, for any \( a, c \in N \)

\[
cd(a^2) = d(a^2)
\]

By lemma 2.2.1, we obtains

\[
c[d(a)a + ad(a)] = [d(a)a + ad(a)]c
\]

\[
 cd(a)a + cad(a) = d(a)ac + ad(a)c
\]

\[
 cad(a) + cad(a) = d(a)ac + d(a)ac
\]

If \( N \) is a 2-torsion free, then

\[
cad(a) = d(a)ac \quad \Rightarrow \quad cad(a) - d(a)ac = 0
\]

\[
(ca - ac)d(a) = 0 \Rightarrow [c, a] d(a) = 0 \quad \text{for all} \quad a, c \in N
\]
Since $N$ is a near ring, then $d \neq 0$, so we get $[c, a] = 0$

$N$ is a commutative ring.

**Lemma 2.2.5:** (i) Let $f$ be right generalized Jordan derivation of $N$ associated with $d$. Then $f(x^2) = xd(x) + f(x)x$, for all $x \in N$.

(ii) Let $f$ be left generalized Jordan derivation of $N$ associated with $d$. Then $f(x^2) = xf(x) + d(x)x$, for all $x \in N$.

**Proof:** (i) For any $x \in N$,

\[
f((x+x)x) = f(x+x)x + (x+x)d(x)
\]

\[
= f(x)x + f(x)x + xd(x) + xd(x)
\]

On the other hand,

\[
f(x^2 + x^2) = f(x)x + xd(x) + f(x)x + xd(x)
\]

Comparing equations (2.2.1) and (2.2.2), we get

\[
f(x)x + xd(x) = xd(x) + f(x)x
\]

(ii) For any $x \in N$,

\[
f((x+x)x) = d(x+x)x + (x+x)f(x)
\]

\[
= d(x)x + d(x)x + xf(x) + xf(x)
\]
On the other hand

\[ f(x^2 + x^2) = d(x)x + xf(x) + d(x)x + xf(x) \]  \hspace{1cm} (2.2.4)

Comparing equations (2.2.3) and (2.2.4), we get

\[ d(x)x + xf(x) = xf(x) + d(x)x \]

\[ f(x^2) = xf(x) + d(x)x. \]

**Lemma 2.2.6:** Let \( f \) be generalized Jordan derivation of \( N \) associative with \( d \), then

\[ u(xd(x) + f(x)x) = xd(.~) + f(x)x. \quad \forall u, x \in N. \]

**Proof:** For any \( a, x \in N \), we get

\[ f(a^2) = af(x^2) + d(a)x^2 \]

\[ = a(xd(x) + f(x)x) + d(a)x^2 \]  \hspace{1cm} (2.2.5)

On the other hand,

\[ f((ax)x) = axd(x) + f(ax)x \]

\[ = axd(x) + [af(x) + d(a)x]x \]

\[ = axd(x) + af(x)x + d(a)x^2 \]  \hspace{1cm} (2.2.6)

From the equations (2.2.5) and (2.2.6), we get

\[ a(xd(x) + f(x)x) = axd(x) + af(x)x, \quad \forall a, x \in N. \]
Lemma 2.2.7: Let $N$ be a near ring, $f$ is a non-zero generalized Jordan derivations of $N$ associated with non-zero Jordan derivation $d$, and $a \in N$.

(i) If $af(N) = 0$, then $a = 0$

(ii) If $f(N)a = 0$, then $a = 0$

Proof: (i) For all $x \in N$, we get

$$0 = af(x^2) = a[xd(x) + f(x)x]$$

$$= axd(x) + af(x)x$$

By the hypothesis, we have,

$$aNd(N) = 0 \text{ for all } x \in N$$

Since $N$ is near ring and $d \neq 0$, we obtain $a = 0$.

(ii) For all $x \in N$, we get

$$0 = f(x^2)a = [xf(x) + d(x)x]a$$

$$= xf(x)a + d(x)xa$$

By the hypothesis, we have,

$$d(N)Na = 0 \text{ for all } x \in N$$

Since $N$ is near ring and $d \neq 0$, we obtain $a = 0$.

Theorem 2.2.1: Let $f$ be a generalized Jordan derivation of $N$ associated with non-zero derivation $d$. If $N$ is a 2 torsion free near ring and $f^2 = 0$, then $f = 0$. 
Proof: For arbitrary \( x \in N \), we have

\[
0 = f^2(x^2) = f(f(x^2))
\]

\[
= f(f(x)x + xd(x))
\]

\[
= f(f(x)x) + f(xd(x))
\]

\[
= f^2(x)x + f(x)d(x) + f(x)d(x) + xd^2(x)
\]

\[
= f^2(x)x + 2f(x)d(x) + xd^2(x)
\]

By the hypothesis, we have

\[
2f(x)d(x) + xd^2(x) = 0
\]

Writing \( f(x) \) by \( x \), we get

\[
f(x)d^2(f(x)) = 0, \forall x \in N, \text{ from this, we obtain that}
\]

\[
d^2(N) = 0 \quad \text{(or) } f = 0
\]

By lemma 2.2.3, we have

\[
d^2(N) = 0, \text{ then } d = 0
\]

Which is a contradiction, so we find \( f = 0 \)

**Theorem 2.2.2:** Let \( N \) be a near ring with a non-zero generalized Jordan derivation \( f \) associated with non-zero Jordan derivation \( d \). If \( f(N) \subset Z \), then \( (N, +) \) is abelian.

Moreover, if \( N \) is 2-torsion free, then \( N \) is a commutative ring.

**Proof:** Assume \( f(a) \notin Z \setminus \{0\} \) and \( f(a) + f(a) \in Z \setminus \{0\} \).
then we have $f(a)(x, y) = 0, \text{ for all } x, y \in N$

since $f(a) \in Z \backslash \{0\}$ and 'N' is near ring, it follows that $(x, y) = 0, \forall x, y \in N$.

Thus $(N, +)$ is abelian.

Using the hypothesis, for any $x, z \in N$

$$xf(z^2) = f(z^2)x$$

$$x(zd(z) + f(z)z) = (f(z)z + zd(z))x$$

$$xz(x^2) + xf(z)z = f(z)zx + zd(z)x$$

$$xzd(z) = zxd(z)$$

$$[x, z]d(z) = 0, \quad \forall x, z \in N$$

Substituting $z = f(y)$, then we get $(x, f(y))d(f(y)) = 0$

Since $f(y) \in Z$ and $d(f(y)) \in Z$, we have

$$d(f(y)) = 0, \text{ for all } y \in N \text{ (or) } N \text{ is commutative ring}$$

Let assume that $d(f(y)) = 0$

Substituting $y = xy$

$$0 = d(f(xy))$$

$$= d(d(x)y + xf(y))$$
Let \( y = y^2 \) in above equation, we get

\[
d^2(x)y^2 + d(x)d(y^2) + d(x)f(y^2) = 0
\]

\[
d^2(x)y^2 + d(x)[yd(y) + d(y)y] + d(x)[yd(y) + f(y)y] = 0, \quad \forall x, y \in N
\]

\[
d^2(x)y^2 + d(x)d(y)d(y) + d(x)d(y)f(y) + d(x)d(y)f(y) = 0
\]

\[
[d^2(x)y + d(x)d(y) + d(x)f(y)]y + 2d(x)yf(y) = 0
\]

Since \( N \) is 2-torsion free near ring, we get

\[
d(N)Nd(N) = 0.
\]

Thus, we obtain that \( d = 0 \). It contradicts \( d \neq 0 \).

So, we must have \( N \) is commutative ring.