Symmetric Skew 4-Derivations on Semi Prime Rings

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Abstract

In this paper we introduce the notation of symmetric skew 4-derivation of Semiprime ring and we consider R be a non-commutative 2, 3-torsion free semi prime ring, I be a non zero two sided ideal of R, \( \alpha \) be an automorphism of R, and \( D: R \rightarrow \) R be a symmetric skew 4-derivation associated with the automorphism \( \alpha \). If \( f \) is trace of \( D \) such that \( [f(x), \alpha(x)] \in Z \) for all \( x \in I \), then \( [f(x), \alpha(x)] = 0 \), for all \( x \in I \).

Keywords: Semiprime ring, Derivation, Bi derivation, Symmetric Skew 3-derivation, Symmetric Skew 4-derivation and Auto orphism.

Introduction

In 1957, the study of centralizing and commuting mappings on a prime rings was initiated by the result of E. C. Posner [2] which states that the existence of a non-zero centralizing derivation on a prime ring implies that the ring has to be commutative. Further Vukman [4, 5] extended above result for bi derivations. Recently Jung and park[6]considered permuting 3-derivations on prime and semi prime rings and obtained the following:Let \( R \) be a non-commutative 3-torsion free semi prime ring and let \( I \) be a non-zero two sided ideal of \( R \). Suppose that there exists a permuting 3-derivation \( D: R^3 \rightarrow R \) such that \( f \) is centralizing on \( I \) then \( f \) is commuting on \( I \). A. Fosner [1] extended the above results in symmetric skew 3-derivations with prime rings and semi prime rings. Recently Faiza Shujat, Abuzaid Ansari[3] Studied some results in symmetric skew 4-derivations in prime rings. In this Paper we proved that Symmetric skew 4-derivations in semi prime rings.
Preliminaries
Throughout this paper, $R$ will be represent a ring with a center $Z$ and a bean automorphism of $R$. Let $n \geq 2$ be an integer. A ring $R$ is said to be $n$-torsion free if for $x \in R, nx = 0$ implies $x = 0$. For all $x, y \in R$ the symbol $[x, y]$ will denote the commutator $xy - yx$. we make extensive use of basic commutator identities

$[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. Recall that a ring $R$ is semi prime if $xRx = 0$ implies that $x = 0$. An additive map $d: R \to R$ is called derivation if $d(xy) = d(x)y + d(x)d(y)$, for all $x, y \in R$, and it is called a skew derivation ($\alpha$-derivation) of $R$ associated with the automorphism $\alpha$ if $d(xy) = d(x)y + \alpha(x)d(y)$ for all $x, y \in R$, associated with automorphism $\alpha$. Before starting our main theorem, let us gives some basic definitions and well known results which we will need in our further investigation.

Let $D$ be a symmetric 4-additive map of $R$, then obviously

$$D(-p, q, r, s) = -D(p, q, r, s),$$

for all $p, q, r, s \in R$ (1)

Namely, for all $y, z \in R$, the map $D(., ., y, z): R \to R$ is an endomorphism of the additive group of $R$.

The map $f: R \to R$ defined by $f(x) = D(x, x, x, x), x \in R$ is called trace of $D$.

Note that $f$ is not additive on $R$. But for all $x, y \in R$, we have

$$f(x + y) = [f(x) + 4D(x, x, x, y) + 6D(x, x, y, y) + 4D(x, y, y, y) + f(y)]$$

Recall that by equation (1), $f$ is even function.

More precisely, for all $p, q, r, s, u, v, w, x \in R$, we have

$D(pu, q, r, s) = D(p, q, r, s)u + \alpha(p)D(u, q, r, s),$  
$D(p, qv, r, s) = D(p, q, r, s)v + \alpha(q)D(p, v, r, s),$  
$D(p, q, rw, s) = D(p, q, r, s)w + \alpha(r)D(p, q, w, s),$  
$D(p, q, r, sx) = D(p, q, r, s)x + \alpha(s)D(p, q, r, x).$

Of course, if $D$ is symmetric, then the above four relations are equivalent to each other.

**Lemma 1:**
Let $R$ be a prime ring and $a, b \in R$. If $a[x, b] = 0$, for all $x \in R$, then either $a = 0$ or $b \in Z$.

**Proof:**
Note that

$$0 = a[x, y, b] = ax[y, b] + a[x, b]y = ax[y, b],$$

for all $x, y \in R$.

Thus $aR[y, b] = 0, y \in R$, and, since $R$ is prime, either $a = 0$ or $b \in Z$.

**Theorem 1:**
Let $R$ be a $2, 3$ -torsion free non commutative semiprime ring and $I$ be a nonzero ideal of $R$. Suppose $\alpha$ is an automorphism of $R$ and $D: R^4 \to R$ is a symmetric skew 4-derivation associated with $\alpha$. If $f$ is trace of $D$ such that $[f(x), \alpha(x)] \in Z$ for all $x \in R$, then $f$ is a derivation of $R$. 


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Let \([f(x), a(x)] = 0\).

\textbf{Proof:}

Let \([f(x), a(x)] \in Z\), for all \(x \in I\).

Linearization of (2) yields that, we have

\[ [f(x + y), a(x + y)] \in Z \]

\[ [f(x + y), a(x)] + [f(x + y), a(y)] \in Z \]

By skew 4-derivation, we have

\[ f(x + y) = [f(x) + 4D(x, x, x, y) + 6D(x, x, y, y) + 4D(x, y, y, y) + f(y)] \]

\[ [f(x), a(x)] + 4[D(x, x, x, y), a(x)] + 6[D(x, x, y, y), a(x)] + \]

\[ 4[D(x, y, y, y), a(x)] + [f(y), a(x)] + [f(x), a(y)] + 4[D(x, x, x, y), a(y)] + \]

\[ 6[D(x, x, y, y), a(y)] + 4[D(x, y, y, y), a(y)] + [f(y), a(y)] \in Z, \text{ for all } x \in I. \] (3)

From (2) & (3), we get

\[ 4[D(x, x, x, y), a(x)] + 6[D(x, x, y, y), a(x)] + 4[D(x, y, y, y), a(x)] + \]

\[ [f(y), a(x)] + [f(x), a(y)] + 4[D(x, x, x, y), a(y)] + 6[D(x, x, y, y), a(y)] + \]

\[ 4[D(x, y, y, y), a(y)] \in Z, \]

for all \(x \in I\). (4)

Replacing \(y\) by \(-y\) in (4), we find

\[ -4[D(x, x, x, y), a(x)] + 6[D(x, x, y, y), a(x)] - 4[D(x, y, y, y), a(x)] + \]

\[ [f(y), a(x)] - [f(x), a(y)] + 4[D(x, x, x, y), a(y)] - \]

\[ 6[D(x, x, y, y), a(y)] + 4[D(x, y, y, y), a(y)] \in Z, \]

for all \(x \in I\). (5)

Comparing (4) and (5) and using 2-torsion freeness of \(R\), wehave

\[ 4[D(x, x, x, y), a(x)] + 4[D(x, y, y, y), a(x)] + [f(x), a(y)] + 6[D(x, x, y, y), a(y)] \]

\[ \in Z, \]

for all \(x \in I\). (6)

Substitute \(y + z\) for \(y\) in (6) and use (6), we get

\[ 4[D(x, x, x, y + z), a(x)] + 4[D(x, x, y + z, y + z), a(x)] + [f(x), a(y + z)] + \]

\[ 6[D(x, x, y + z, y + z), a(y + z)] \in Z \]

\[ 4[D(x, x, x, y), a(x)] + 4[D(x, x, x, z), a(x)] + 4[D(x, x, y, y), a(x)] + \]

\[ 4[D(x, x, y, z), a(x)] + 4[D(x, y, z, y), a(x)] + \]

\[ 4[D(x, y, z, y), a(x)] + 4[D(x, x, z, y), a(x)] + \]

\[ 4[D(x, x, z, z), a(x)] + [f(x), a(y)] + [f(x), a(z)] + \]

\[ 6[D(x, x, y, y), a(y)] + 6[D(x, x, z, z), a(y)] + \]

\[ 6[D(x, x, z, y), a(y)] + 6[D(x, x, y, z), a(y)] + \]

\[ 6[D(x, x, z, z), a(y)] \in Z \]
\[4[D(x, y, y, z), \alpha(x)] + 4[D(x, y, z, y), \alpha(x)] + 4[D(x, y, z, z), \alpha(x)]
+ 4[D(x, z, y, y), \alpha(x)] + 4[D(x, z, y, z), \alpha(x)]
+ 4[D(x, z, z, y), \alpha(x)] + 6[D(x, x, y, z), \alpha(y)]
+ 6[D(x, x, z, y), \alpha(y)] + 6[D(x, x, z, z), \alpha(y)]
+ 6[D(x, x, x, y), \alpha(z)] + 6[D(x, x, x, z), \alpha(z)]
+ 6[D(x, x, z, y), \alpha(z)] \in Z \]

\[12[D(x, y, y, z), \alpha(x)] + 12[D(x, y, z, z), \alpha(x)] + 12[D(x, x, z, z), \alpha(y)]
+ 6[D(x, x, x, z), \alpha(y)] + 6[D(x, x, y, y), \alpha(z)]
+ 12[D(x, x, y, z), \alpha(z)] \subseteq Z, \]

for all \(x, y, z \in I.\) (7)

Replacing \(z\) in \(-z\) in (7) and compare with (7), we obtain
\[\begin{align*}
-12[D(x, y, y, z), \alpha(x)] + 12[D(x, y, z, z), \alpha(x)] - 12[D(x, x, y, z), \alpha(y)]
+ 6[D(x, x, z, z), \alpha(y)] - 6[D(x, x, y, y), \alpha(z)]
+ 12[D(x, x, y, z), \alpha(z)] \subseteq Z
\end{align*}\]

Using of two torsion free ring, we have
\[\begin{align*}
12[D(x, y, y, z), \alpha(x)] + 12[D(x, x, y, z), \alpha(y)] + 6[D(x, x, y, y), \alpha(z)] \subseteq Z, \quad (8)
\end{align*}\]

for all \(x, y, z \in I.\)

Substitute \(y + u\) for \(y\) in (8) and use (8) we get
\[\begin{align*}
12[D(x, z, y + u, y + u), \alpha(x)] + 12[D(x, x, y + u, z), \alpha(y + u)]
+ 6[D(x, x, y + u, y + u), \alpha(z)] \subseteq Z
\end{align*}\]

\[\begin{align*}
12[D(x, z, y, y), \alpha(x)] + 12[D(x, z, y, u), \alpha(x)] + 12[D(x, z, u, y), \alpha(x)]
+ 12[D(x, z, u, u), \alpha(x)] + 12[D(x, x, y, z), \alpha(y)]
+ 12[D(x, x, u, z), \alpha(y)] + 12[D(x, x, y, z), \alpha(u)]
+ 12[D(x, x, u, z), \alpha(u)] + 6[D(x, x, y, y), \alpha(z)]
+ 6[D(x, x, y, u), \alpha(z)] + 6[D(x, x, u, y), \alpha(z)]
+ 6[D(x, x, u, u), \alpha(z)] \subseteq Z
\end{align*}\]

\[\begin{align*}
24[D(x, z, y, u), \alpha(x)] + 12[D(x, x, y, z), \alpha(u)] + 12[D(x, x, u, z), \alpha(y)] + 12[D(x, x, u, u), \alpha(z)] \subseteq Z, \quad (9)
\end{align*}\]

for all \(x, y, z \in I.\)

Since \(R\) is 2 and 3-torsion free and replacing \(y, u\) by \(x\) in (9), we have
\[\begin{align*}
24[D(x, z, x, x), \alpha(x)] + 12[D(x, x, x, z), \alpha(x)] + 12[D(x, x, x, z), \alpha(x)] + 12[D(x, x, x, x), \alpha(z)] \subseteq Z
\end{align*}\]

\[\begin{align*}
48[D(x, x, x, x), \alpha(x)] + 12[D(x, x, x, x), \alpha(z)] \subseteq Z
\end{align*}\]

\[\begin{align*}
4[D(x, x, x, z), f(x), \alpha(z)] \subseteq Z, \quad \text{for all } x, z \in I. \quad (10)
\end{align*}\]

Again replaced \(z\) by \(xz\) in (10) and using (10) we obtain
\[\begin{align*}
4[f(x), x, x, x), \alpha(x)] + 4[f(x), x, x, x), \alpha(x)] \subseteq Z, \quad \text{for all } x, z \in I.
\end{align*}\]

\[\begin{align*}
4[f(x), x, x, x), \alpha(x)] + 4[f(x), x, x, x), \alpha(x)] \subseteq Z, \quad \text{for all } x, z \in I.
\end{align*}\]

\[\begin{align*}
4[f(x), x, x, x), \alpha(x)] + 4[f(x), x, x, x), \alpha(x)] + (4x)[f(x), x, x, z), \alpha(x)] + [f(x), x, x)] \alpha(z) +
\end{align*}\]
4f(x)[z, \alpha(x)] \in Z, \text{ for all } x, z \in I.

Therefore, from (11), we get

\[ \alpha(x)[[f(x), \alpha(z)] + 4[D(x, x, z), \alpha(x)], \alpha(x)] + [(\alpha(z) + 4z)[f(x), \alpha(x)], \alpha(x)] + 4[f(x)[z, \alpha(x)], \alpha(x)] = 0, \text{ for all } x, z \in I. \quad (12) \]

Repeating (12), we get

\[ \alpha(x)[[f(x), \alpha(z)], \alpha(x)] + 4[D(x, x, z), \alpha(x), \alpha(x)] + (\alpha(z) + 4z)[f(x), \alpha(x)], \alpha(x)] + 4[f(x)[z, \alpha(x)], \alpha(x)] = 0, \text{ for all } x, z \in I. \]

Replacing z by f(x)[f(x), \alpha(x)] in (13), we get

\[ [(\alpha(f(x)[f(x), \alpha(x)] + 8f(x)[f(x), \alpha(x)], \alpha(x)] + 4[f(x)[f(x), \alpha(x)], \alpha(x)] + 4[f(x)[z, \alpha(x)], \alpha(x)] = 0, \text{ for all } x \in I. \]

\[ \quad [(\alpha(f(x)[f(x), \alpha(x)]), \alpha(x)] + 8f(x)[f(x), \alpha(x)], \alpha(x)] + 4[f(x)[f(x), \alpha(x)], \alpha(x)] + \]

\[ f(x)[f(x), \alpha(x)], \alpha(x)] + 4[f(x)[f(x), \alpha(x)], \alpha(x)] = 0, \text{ for all } x \in I. \]

On the other hand, taking z = x^2 in equation (10), we get

\[ 4[D(x, x, x^2), \alpha(x)] + [f(x), \alpha(x^2)] \in Z, \text{ for all } x \in I. \]

\[ 4[D(x, x, x)x + \alpha(x)D(x, x, x), \alpha(x)] + [f(x), \alpha(x)\alpha(x)] \in Z, \text{ for all } x \in I. \]
Therefore, from equation (15), we get
\[ [f(x), 6\alpha(x)] + 4x[f(x), \alpha(x)] + 4f(x)[x, \alpha(x)] = 0, \text{ for all } x \in I. \] (15)

Since \( f \) is commutative and using equation (16), we get
\[ 6f(x, \alpha(x)) x + 4f(x)[f(x), \alpha(x)] = 0, \text{ for all } x \in I. \] (16)

Comparing (14) and (17) and we have 2-torsion freeness, we get
\[ [f(x), \alpha(x)] = 0, \text{ for all } x \in I. \] (17)

Note that zero is the only nilpotent element in the center of semiprime ring.

Thus, \( [f(x), \alpha(x)] = 0, \text{ for all } x \in I. \)

This completes the proof.

References

SYMMETRIC LEFT BI-DERIVATIONS ON SEMIPRIME RINGS

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ABSTRACT

Let R be a 2-torsion and 3-torsion free semiprime ring. Let \( D(.,.): R \times R \rightarrow R \) and \( B(.,.): R \times R \rightarrow R \) be a symmetric left bi-derivation and symmetric bi-additive mapping. If \( D(d(x), x)^n = 0 \) and \( (d(d(x)) - f(x))^n = 0 \), holds for all \( x \) in \( R \), where \( d \) be a trace of \( D \) and \( f \) be a trace of \( B \). In this case \( D \) is central mapping on \( R \).

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1. INTRODUCTION

The concept of a symmetric bi-derivation has been introduced by Maksa.Gy in [5,6]. A classical result in the theory of centralizing mappings is a theorem first proved by Posner,E[8]. Vukman.J [9, 10] has studied some results concerning symmetric bi-derivations in prime and semiprime rings. In [7] Ozturk and Jun have introduced the concept of a symmetric bi-derivation of near ring and studied some properties. Bresar,M [3] proved that, if \( R \) is a non-commutative 2-torsion free prime ring and \( D: R \times R \rightarrow R \) is a symmetric bi-derivation, then \( D = 0 \). Jaya Subba Reddy.C.et al [4] has studied some results on symmetric...
left bi-derivations on semiprime rings. Atteya.M.J [1] has studied some results concerning a bi-derivations on prime and semiprime rings. Motivated by the above results we proved some results in symmetric left bi-derivations on semiprime rings.

Throughout this paper $R$ will be associative. We shall denote by $Z(R)$ the center of a ring $R$. Recall that a ring $R$ is semiprime if $aRa = (0)$ implies that $a = 0$. We shall write $[x,y]$ for $xy - yx$ and use the identities $[xy, z] = [x, z]y + x[y, z], [x, yz] = [x, y]z + y[x, z]$. An additive map $d: R \rightarrow R$ is called derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. A mapping $B(., .): R \times R \rightarrow R$ is said to be symmetric if $B(x, y) = B(y, x)$ holds for all $x,y \in R$. A mapping $f: R \rightarrow R$ defined by $f(x) = B(x,x)$, where $B(., .): R \times R \rightarrow R$ is a symmetric mapping, is called a trace of $B$. It is obvious that, in case $B(., .): R \times R \rightarrow R$ is symmetric mapping which is also bi-additive (i.e. additive in both arguments) the trace of $B$ satisfies the relation $f(x + y) = f(x) + f(y) + 2B(x, y)$, for all $x,y \in R$. We shall use the fact that the trace of a symmetric bi-additive mapping is an even function. A symmetric bi-additive mapping $D(., .): R \times R \rightarrow R$ is called a symmetric bi-derivation if $D(xy, z) = D(x, z)y + xD(y, z)$, for all $x, y, z \in R$. Obviously, in this case also the relation $D(x, yz) = D(x, y)z + yD(x, z)$, for all $x, y, z \in R$. A symmetric bi-additive mapping $D(., .): R \times R \rightarrow R$ is called a symmetric left bi-derivation if $D(xy, z) = yD(x, z) + zD(x, y)$, for all $x, y, z \in R$. A mapping $f: R \rightarrow R$ is said to be commuting on $R$ if $[f(x), x] = 0$, for all $x \in R$. A mapping $f: R \rightarrow R$ is said to be centralizing on $R$ if $[f(x), x] \in Z(R)$, for all $x \in R$. A ring $R$ is said to be $n$-torsion free if whenever $na = 0$, with $a \in R$, then $a = 0$, where $n$ is nonzero integer.

**Lemma 1:** Let $d: R \rightarrow R$ be a derivation, where $R$ is a prime ring. Suppose that either

(i) $ad(x) = 0$, for all $x \in R$ or (ii) $d(x)a = 0$, for all $x \in R$ holds. In both the cases we have $a = 0$ or $d = 0$.

Proof:

(i). We have $ad(x) = 0$, for all $x \in R$.

We replace $x$ by $xy$ in (1), we get

$$ad(xy) = 0$$

$$ad(xy) + axd(y) = 0,$$ for all $x, y \in R$.

By using (1) in the above equation we get

$$axd(y) = 0$$

$$aRd(y) = 0,$$ for all $y \in R$.

Since $R$ is prime which implies that either $a = 0$ or $d = 0$.

(ii) We have $d(x)a = 0$, for all $x \in R$.

We replace $x$ by $xy$ in (2), we get
\[ d(xy)a = 0 \]
\[ d(x)ya + xd(y)a = 0, \text{ for all } x, y \in R. \]

By using (2) in the above equation we get
\[ d(x)y a = 0 \]
\[ d(x)R a = 0, \text{ for all } x, y \in R. \]

Since \( R \) is prime which implies that either \( d = 0 \) or \( a = 0 \).

**Lemma 2:** Let \( R \) be a prime ring of characteristic not two and let \( a, b \in R \) be a fixed elements. If \( axb + bxa = 0, \text{ for all } x \in R, \) then either \( a = 0 \) or \( b = 0 \).

**Proof:** We have \( axb + bxa = 0, \text{ for all } x \in R. \)
\[ axb = -bxa, \text{ for all } x \in R. \]  \hspace{1cm} (3)

We replace \( x \) by \( xbrax \) in (3), we get
\[ a(xbrax)b = -b(xbrax)a \]
\[ axbraxb = -(b(xbr)a)xa \]

By using (3) in the above equation, we get
\[ axbraxb = (a(xbr)b)xa \]
\[ (axb)r(axb) = (axb)r(bxa) \]

Again by using (3) in the above equation, we get
\[ (axb)r(axb) = (axb)r(-axb) \]
\[ (axb)r(axb) = -(axb)r(bxa) \]
\[ 2(axb)r(axb) = 0, \text{ for all } r \in R. \]

Since \( R \) is prime ring of characteristic not 2, we get
\[ axb = 0, \text{ for all } x \in R \text{ and hence } a = 0 \text{ or } b = 0. \]

**Lemma 3:** [2] The center of semiprime ring contains no non zero nilpotent elements.

**Theorem 1:** Let \( n \) be a positive integer and \( R \) be a 2-torsion free semiprime ring. Let \( D(.,.) : R \times R \to R \) be a symmetric left bi-derivation and \( d \) is a trace of \( D \) such that
\[ D(d(x),x)^n = 0, \text{ for all } x \in R, \text{ then } D \text{ is central mapping on } R. \]

**Proof:** We have \( D(d(x),x)^n = 0, \text{ for all } x \in R. \)
If \( n = 1, \) we have \( D(d(x),x) = 0, \text{ for all } x \in R. \) \hspace{1cm} (4)

We replace \( d(x) \) by \( d(x)y \) in (4), we get
\[ D(d(x)y, x) = 0 \]

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By using (4) in the above equation, we get
\[ d(x)D(y, x) + yD(d(x), x) = 0 \]

By using (4) in the above equation, we get
\[ d(x)D(y, x) = 0 \]
\[ d(x)D(x, y) = 0, \text{ for all } x, y \in \mathbb{R}. \]  
(5)

We replace \( x \) by \( x^2 \) in (5), we get
\[ d(x^2)D(x^2, y) = 0 \]
\[ 4x^2d(x)2xD(x, y) = 0 \]
\[ 8x^2d(x)xD(x, y) = 0 \]

If \( x = 0 \) it is trivial, if \( x \neq 0 \) then \( d(x)xD(x, y) = 0 \), for all \( x, y \in \mathbb{R}. \)  
(6)

By the linearization of (4), we get
\[ D(d(x + y), x + y) = 0 \]
\[ D(d(x) + d(y) + 2D(x, y), x + y) = 0 \]
\[ D(d(x), x) + D(d(x), y) + D(d(y), x) + D(d(y), y) + D(2D(x, y), x) + D(2D(x, y), y) = 0 \]

By using (4) in the above equation, we get
\[ D(d(x), y) + D(d(y), x) + D(2D(x, y), x) + D(2D(x, y), y) = 0 \]
\[ D(d(x), y) + D(d(y), x) + 2D(D(x, y), x) + 2D(D(x, y), y) = 0, \text{ for all } x, y \in \mathbb{R}. \]  
(7)

We replace \( x \) by \( -x \) in (7), we get
\[ D(d(-x), y) + D(d(y), -x) + 2D(D(-x, y), -x) + 2D(D(-x, y), y) = 0 \]
\[ D(d(x), y) - D(d(y), x) + 2D(D(x, y), x) - 2D(D(x, y), y) = 0, \text{ for all } x, y \in \mathbb{R}. \]  
(8)

By adding (7) and (8), we get
\[ 2D(d(x), y) + 4D(D(x, y), x) = 0 \]
\[ D(d(x), y) + 2D(D(x, y), x) = 0, \text{ for all } x, y \in \mathbb{R}. \]  
(9)

We replace \( y \) by \( xy \) in (9), we get
\[ D(d(x), xy) + 2D(D(x, xy), x) = 0 \]
\[ xD(d(x), y) + yD(d(x), x) + 2D(xD(x, y), yD(x, x), x) = 0 \]
\[ xD(d(x), y) + yD(d(x), x) + 2D(xD(x, y), x) + 2D(yD(x, x), x) = 0 \]
\[ xD(d(x), y) + yD(d(x), x) + 2xD(D(x, y), x) + 2D(x, y)D(x, x) + 2yD(D(x, x), x) + 2D(x, x)D(y, x, y) = 0 \]
\[ xD(d(x), y) + yD(d(x), x) + 2xD(D(x, y), x) + 2D(x, y)d(x) + 2yD(d(x), x) + 2d(x)D(y, x) = 0 \]

By using (4) and (9) in the above equation, we get
\[ 2D(x, y)d(x) + 2d(x)D(y, x) = 0 \]
\[ D(x, y)d(x) + d(x)D(x, y) = 0, \text{ for all } x, y \in R. \tag{10} \]

By using (5) in (10), we get
\[ D(x, y)d(x) = 0, \text{ for all } x, y \in R. \tag{11} \]

We replace \( y \) by \( x \) in (10), we get
\[ D(x, x)d(x) + d(x)D(x, x) = 0 \]
\[ d(x)d(x) + d(x)D(x, x) = 0 \]
\[ 2d(x)d(x) = 0 \]
\[ d(x)d(x) = 0, \text{ for all } x \in R. \tag{12} \]

We replace \( y \) by \( xy \) in (10), we get
\[ D(x, xy)d(x) + d(x)D(x, xy) = 0 \]
\[ yD(x, x)d(x) + xD(x, y)d(x) + d(x)yD(x, x) + d(x)xD(y, x) = 0 \]
\[ yd(x)d(x) + xD(x, y)d(x) + d(x)yd(x) + d(x)xD(x, y) = 0 \]

By using (6), (11), (12) in above equation, we get
\[ d(x)yd(x) = 0, \text{ for all } x, y \in R. \]

Which implies that \( d(x) = 0 \), for all \( x \in R \), by semiprimeness of \( R \), which means that
\[ D(x, y) = 0, \text{ for all } x, y \in R. \]

Assume that \( n > 1 \), we have \( D(d(x), x)^n = 0 \), for all \( x \in R \). According to Lemma 3 we get
\[ D(d(x), x) = 0, \text{ for all } x \in R, \text{ so by the same technique in first part of the proof we can complete the proof of the theorem.} \]

**Theorem 2:** Let \( n \) be a positive integer and \( R \) be a 2-torsion and 3-torsion free semiprime ring. Let \( D(\cdot, \cdot): R \times R \rightarrow R \) and \( B(\cdot, \cdot): R \times R \rightarrow R \) be a symmetric left bi-derivation and symmetric bi-additive mapping respectively. Suppose that \( (d(d(x)) - f(x))^n = 0 \), for all \( x \in R \), where \( d \) be a trace of \( D \) and \( f \) be a trace of \( B \), then \( D \) is central on \( R \).

**Proof:** We have \( (d(d(x)) - f(x))^n = 0 \), for all \( x \in R \).

If \( n = 1 \), we have \( d(d(x)) = f(x), \text{ for all } x \in R. \tag{13} \]

By the linearization of (13), we get
\[ d(d(x + y)) = f(x + y) \]
By using $(13)$ in the above equation, we get
\[ d(d(x) + d(y) + 2D(x,y)) = f(x) + f(y) + 2B(x,y) \]
\[ d(d(x)) + d(d(y)) + 2D(d(x), d(y)) + 2D(d(x), 2D(x,y)) + 2D(d(y), 2D(x,y)) = f(x) + f(y) + 2B(x,y) \]
\[ d(d(x)) + d(d(y)) + 4D(D(x,y)) + 2D(d(x), d(y)) + 4D(d(x), D(x,y)) + 4D(d(y), D(x,y)) = f(x) + f(y) + 2B(x,y) \]

By using $(13)$ in the above equation, we get
\[ 4d(D(x,y)) + 2D(d(x), d(y)) + 4D(d(x), D(x,y)) + 4D(d(y), D(x,y)) = 2B(x,y) \]
\[ 2d(D(x,y)) + D(d(x), d(y)) + 2D(d(x), D(x,y)) + 2D(d(y), D(x,y)) = B(x,y), \text{ for all } x, y \in R. \]  
\begin{equation} (14) \end{equation}

We replace $x$ by $-x$ in $(14)$, we get
\[ 2d(D(-x,y)) + D(d(-x), d(y)) + 2D(d(-x), D(-x,y)) + 2D(d(y), D(-x,y)) = B(-x,y) \]
\[ 2d(D(x,y)) + D(d(x), d(y)) - 2D(d(x), D(x,y)) - 2D(d(y), D(x,y)) = -B(x,y), \text{ for all } x, y \in R. \]  
\begin{equation} (15) \end{equation}

Subtract $(15)$ from $(14)$, we get
\[ 4D(d(x), D(x,y)) + 4D(d(y), D(x,y)) = 2B(x,y) \]
\[ 2D(d(x), D(x,y)) + 2D(d(y), D(x,y)) = B(x,y), \text{ for all } x, y \in R. \]  
\begin{equation} (16) \end{equation}

We replace $x$ by $2x$ in $(16)$, we get
\[ 2D(d(2x), D(2x,y)) + 2D(d(y), D(2x,y)) = B(2x,y) \]
\[ 16D(d(x), D(x,y)) + 4D(d(y), D(x,y)) = 2B(x,y) \]
\[ 8D(d(x), D(x,y)) + 2D(d(y), D(x,y)) = B(x,y), \text{ for all } x, y \in R. \]  
\begin{equation} (17) \end{equation}

Subtract $(16)$ from $(17)$, we get
\[ 6D(d(x), D(x,y)) = 0 \]
Since $R$ is 2-torsion and 3-torsion free ring, we get
\[ D(d(x), D(x,y)) = 0, \text{ for all } x, y \in R. \]  
\begin{equation} (18) \end{equation}

By using $(18)$ and $(16)$, we get
\[ B(x,y) = 0, \text{ for all } x, y \in R. \]

We replace $y$ by $x$ in the above equation, we get $f(x) = 0, \text{ for all } x \in R.$  
\begin{equation} (19) \end{equation}
By using (13) and (19), we get
\[ d(d(x)) = 0, \text{ for all } x \in R. \]
(20)

We replace \( y \) by \( yz \) in (18), we get
\[ D(d(x), D(x, yz)) = 0 \]
\[ D(d(x), yD(x, z) + zD(x, y)) = 0 \]
\[ D(d(x), yD(x, z)) + D(d(x), zD(x, y)) = 0 \]
\[ yD(d(x), D(x, z)) + D(x, z)D(d(x), y) + zD(d(x), D(x, y)) + D(x, y)D(d(x), z) = 0 \]

By using (18) in the above equation, we get
\[ D(x, z)D(d(x), y) + D(x, y)D(d(x), z) = 0, \text{ for all } x, y, z \in R. \]
(21)

We replace \( z \) by \( d(x) \) in (21), we get
\[ D(x, d(x))D(d(x), y) + D(x, y)D(d(x), d(x)) = 0 \]
\[ D(x, d(x))D(d(x), y) + D(x, y)d(d(x)) = 0 \]

By using (20) in the above equation, we get
\[ D(x, d(x))D(d(x), y) = 0, \text{ for all } x, y \in R. \]
(22)

We replace \( y \) by \( xy \) in (22), we get
\[ D(x, d(x))D(d(x), xy) = 0 \]
\[ D(d(x), x)(xD(d(x), y) + yD(d(x), x)) = 0 \]
\[ D(d(x), x)xD(d(x), y) + D(d(x), x)yD(d(x), x) = 0 \]

We replace \( y \) by \( x \) in the above equation we get \( D(d(x), x)xD(d(x), x) = 0 \), which implies
\[ D(d(x), x) = 0, \text{ for all } x \in R \text{ since we have assumed that } R \text{ is semiprime. Now Theorem 1 completes the proof.} \]

Assuming that \( n > 1 \), we have \( (d(d(x)) - f(x))^n = 0 \), for all \( x \in R \), that implies \( (d(d(x)) - f(x))^n \in Z(R) \), for all \( x \in R \). According to Lemma 3, we get \( d(d(x)) - f(x) = 0 \), for all \( x \in R \). Now by using the same technique of first part of the theorem we can complete the proof. \( \square \)
REFERENCES


Journal homepage: http://theojal.com/ojatm/
GENERALIZED JORDAN DERIVATIONS IN NEAR-RINGS

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Abstract: Let \( N \) be a 2-torsion free near-ring with multiplicative Center \( Z \), \( f: R \rightarrow R \) is a generalized Jordan derivation associated with derivation \( d \), the following results are proved:

(i) If \( f^2(N) = 0 \), then \( f = 0 \).
(ii) If \( f(N) \subseteq Z \), then \( N \) is commutative ring.

Keywords: Derivation, Generalized derivation, Jordan derivation, Generalized Jordan derivation, Center, Prime near-ring.

INTRODUCTION AND PRELIMINARIES

The study of derivations of near-ring was initiated by H.E. Bell and C. Mason in [2]. The notation of Generalized derivation and Generalized Jordan derivation of a prime ring was introduced by M.Bresar and B.Ivala in [3] and [4]. O.Golbasi [5] have studied some results on near-rings with generalized derivations. Some recent results concerning commutativity in Prime near-rings with derivation have been Generalized Jordan in several ways. Many authors have investigated these theorems for Generalized Jordan derivations. In this paper we proved some results Generalized Jordan derivations in near-rings.

Throughout this paper \( N \) stands for a near-ring with multiplicative center \( Z \). An additive map \( d: N \rightarrow N \) is a derivation (resp. Jordan derivation), if \( d(xy) = d(x)y + xd(y) \) (resp. \( d(x^2) = d(x)x + xd(x) \)) holds for all \( x, y \in N \). An additive map \( f: N \rightarrow N \) is right generalized derivation (resp. a left generalized Jordan derivation), if \( f(xy) = f(x)y + xd(y) \) (resp. \( f(x^2) = f(x)x + xd(x) \)) holds for all \( x, y \in N \). Let \( d \) be a non-zero Jordan derivation of \( N \) and \( a \in N \).

Main Results

To prove the main theorems, first we prove the following lemmas.

**Lemma 1:** [1, Lemma 1] Let \( N \) be a near-ring and \( d \) be a Jordan derivation of \( N \), then \( d(axd(x)) = axd(x) + ad(x)x, \quad \forall a, x \in N \).

**Lemma 2:** Let \( N \) be a near-ring \( d \) be a non-zero Jordan derivation of \( N \) and \( a \in N \). If \( ad(N) = 0 \) (resp. \( d(N)a = 0 \)), then \( a = 0 \).

**Proof:** Suppose that \( ad(N) = 0 \), for every arbitrary \( x \in N \), we have \( 0 = ad(x^2) = ad(x)x + axd(x) \).

By the hypothesis, \( axd(x) = 0 \), for all \( x \in N \).

Since \( N \) is near-ring and \( d \neq 0 \), we get \( a = 0 \).

Again, if \( d(N)a = 0 \), for every arbitrary \( x \in N \), we have...
$d(x^2) a = 0$

$d(x) xa + xd(x) a = 0$

$axd(x) + xd(x) a = 0$

by the hypothesis, we have $axd(x) = 0$, for all $x \in N$

Since $N$ is near-ring and $d \neq 0$, we get $a = 0$.

**Lemma 3:** Let $N$ be a 2-torsion free near-ring and $d$ be a Jordan derivation of $N$. If $d^2 = 0$, then $d = 0$.

**Proof:** For arbitrary $x \in N$, we have

$0 = d^2(x^2) = d(d(x)x + xd(x))$

$= d^2(x)x + 2d(x)xd(x) + xd^2(x)$

By the hypothesis, we have

$2d(x)xd(x) = 0 \forall x \in N$

By the hypothesis, since $N$ is a 2-torsion near-ring, we get $d(x) = 0 \forall x \in N$.

$d(N) = 0$, for all $x \in N$, then $d = 0$.

**Lemma 4:** Let $N$ be a near-ring and $d$ a non-zero Jordan derivation of $N$. If $d^2(N) \subseteq Z$, then $(N, +)$ is abelian. Moreover, if $N$ is 2-torsion free, then $N$ is commutative ring.

**Proof:** Suppose that $a \in N$ such that $d(a) \neq 0$. So that $d(a) \in Z \setminus \{0\}$ and $d(a) + d(y) \in Z \setminus \{0\}$, for all $x, y \in N$.

we have $d(a) + d(a)(x + y) = (x + y)(d(a) + d(a))$

That is $d(a)x + d(a)x + d(a)y + d(a)y = xd(a) + yd(a) + xd(a) + yd(a)$

Since $d(a) \in Z$, we get

$xd(a) + yd(a) = yd(a) + xd(a)$ and so, $(x, y)d(a) = 0, \forall x, y \in N$

Since $d(a) \in Z \setminus \{0\}$ and $N$ is near-ring, we get $(x, y)d(a) + xd(a) = 0, \forall x, y \in N$

Thus $(N, +)$ is abelian.

Now using the hypothesis, for any $a, c \in N$

$cd(a^2) = d(a^2)$

By Lemma 1, we have

$c[d(a) a + ad(a)] = [d(a) a + ad(a)] c$

$cd(a) a + cad(a) = d(a) a c + ad(a) c$

$cad(a) a + cad(a) = d(a) a c + ad(a) c$

If $N$ is a 2-torsion free, then

$cad(a) = d(a) a c \Rightarrow cad(a) - d(a) a c = 0$

$(ca - ac)d(a) = 0 \Rightarrow [c, a]d(a) = 0, \forall a, c \in N$

Since $N$ is a near-ring, then $d \neq 0$, so we get $[c, a] = 0$

$N$ is commutative ring.

**Lemma 5:** (i) Let $f$ be right generalized Jordan derivation of $N$ associated with $d$. Then $f(x^2) = xd(x) + f(x)x$, for all $x \in N$.

(ii) Let $f$ be left generalized Jordan derivation of $N$ associated with $d$. Then $f(x^2) = xf(x) + d(x)x$, for all $x \in N$.

**Proof:** (i) For any $x \in N$,