Chapter 5
SYMMETRIC SKEW 4-DERIVATIONS AND REVERSE DERIVATIONS ON SEMIPRIME RINGS
In chapter 5, we proved some results on symmetric skew 4-derivations and symmetric skew 4-reverse derivations in semiprime rings. In section 5.1, we introduce the notation of symmetric skew 4-derivation of semiprime ring and we consider $R$ be a non-commutative 2, 3-torsion free semiprime ring, $I$ be a non-zero two sided ideal of $R$, $\alpha$ be an automorphism of $R$, and $D: R^4 \to R$ be a symmetric skew 4-derivation associated with the automorphism $\alpha$. If $f$ is trace of $D$ such that $[f(x), \alpha(x)] \in Z$, for all $x \in I$, then $[f(x), \alpha(x)] = 0$, for all $x \in I$. In section 5.2, we introduce the notation of symmetric skew 4-reverse derivation of semiprime ring and we consider $R$ be a non-commutative 2,3-torsion free semiprime ring, $I$ be a non-zero two sided ideal of $R$, $\alpha$ be an anti-automorphism of $R$, and $D: R^4 \to R$ be a symmetric skew 4-reverse derivation associated with the anti-automorphism $\alpha$. Suppose that the trace function $f$ is commuting on $I$ and $[f(y), \alpha(y)] \in Z$, for all $y \in I$, then $[f(y), \alpha(y)] = 0$, for all $y \in I$. 
5.1 Symmetric Skew 4-Derivations on Semiprime Rings:

In 1957, the study of centralizing and commuting mappings on a prime rings was initiated by the result of Posner E.C. [32] which states that the existence of a non-zero centralizing derivation on a prime ring implies that the ring has to be commutative. Further Vukman [36,37] extended above result for bi derivations. Recently Jung and park [25] considered permuting 3-derivations on prime and semiprime rings and obtained the following: Let $R$ be a non-commutative 3-torsion free semiprime ring and let $I$ be a non-zero two sided ideal of $R$. Suppose that there exists a permuting 3-derivation $D: R^3 \to R$ such that $f$ is centralizing on $I$ then $f$ is commuting on $I$. Fosner.A [1] extended the above results in symmetric skew 3-derivations with prime rings and semiprime rings. Recently Faiza Shujat, Abuzaid Ansari [16] studied some results in symmetric skew 4-derivations in prime rings. In this section we proved that Symmetric skew 4-derivations in semiprime rings.

Throughout this section, $R$ will be represent a ring with a center $Z$ and $\alpha$ be an automorphism of $R$. Let $n \geq 2$ be an integer. A ring $R$ is said to be $n$-torsion free if for $x \in R$, $nx = 0$ implies $x = 0$. For all $x, y \in R$ the symbol $[x, y]$ will denote the commutator $xy - yx$. We make extensive use of basic commutator identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. Recall that a ring $R$ is semi prime if $xRx = 0$ implies that $x = 0$. An additive map $d: R \to R$ is called derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$, and it is called a skew derivation ($\alpha$-derivation) of $R$ associated with the automorphism $\alpha$ if $d(xy) = d(x)y + \alpha(x)d(y)$ for all $x, y \in R$, associated with automorphism $\alpha$ if $d(xy) = xd(y) + \alpha(y)d(x)$ for all $x, y \in R$. 

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Before starting our main theorem, let us give some basic definitions and well-known results which we will need in our further investigation.

Let $D$ be a symmetric 4-additive map of $R$, then obviously

\[ D(-p, q, r, s) = -D(p, q, r, s), \text{ for all } p, q, r, s \in R \quad (5.1.1) \]

Namely, for all $y, z \in R$, the map $D(\ldots, y, z) : R \to R$ is an endomorphism of the additive group of $R$. The map $f : R \to R$ defined by $f(x) = D(x, x, x, x)$, $x \in R$ is called trace of $D$. Note that $f$ is not additive on $R$. But for all $x, y \in R$, we have

\[ f(x + y) = [f(x) + 4D(x, x, x, y) + 6D(x, x, x, y) + 4D(x, y, y, y) + f(y)] \]

Recall that by equation (5.1.1), $f$ is even function.

More precisely, for all $p, q, r, s, u, v, w, x \in R$, we have

\[ D(pu, q, r, s) = D(p, q, r, s)u + \alpha(p)D(u, q, r, s), \]
\[ D(p, qv, r, s) = D(p, q, r, s)v + \alpha(q)D(p, v, r, s), \]
\[ D(p, q, rw, s) = D(p, q, r, s)w + \alpha(r)D(p, q, w, s), \]
\[ D(p, q, r, sx) = D(p, q, r, s)x + \alpha(s)D(p, q, r, x). \]

Of course, if $D$ is symmetric, then the above four relations are equivalent to each other.

**Lemma 5.1.1:** Let $R$ be a prime ring and $a, b \in R$. If $a[x, b] = 0$, for all $x \in R$, then either $a = 0$ or $b \in Z$.

**Proof:** Note that $0 = a[xy, b] = ax[y, b] + a[x, b]y = ax[y, b]$, for all $x, y \in R$.

Thus $aR[y, b] = 0, y \in R$, and, since $R$ is prime, either $a = 0$ or $b \in Z$. 

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Theorem 5.1.1: Let $R$ be a 2,3 -torsion free non commutative semiprime ring and $I$ be a nonzero ideal of $R$. Suppose $\alpha$ is an automorphism of $R$ and $D: R^4 \rightarrow R$ is a symmetric skew 4- derivation associated with $\alpha$. If $f$ is trace of $D$ such that $[f(x), \alpha(x)] \in Z$, for all $x \in I$, then $[f(x), \alpha(x)] = 0$, for all $x \in I$.

Proof: Let $[f(x), \alpha(x)] \in Z$, for all $x \in I$.  

(5.1.2)

Linearization of (5.1.2) yields that, we have

$[f(x + y), \alpha(x + y)] \in Z \ [f(x + y), \alpha(x)] + [f(x + y), \alpha(y)] \in Z$

By skew 4- derivation, we have

$f(x + y) = [f(x) + 4D(x,x,y,y) + 6D(x,x,y,y) + 4D(x,y,y,y) + f(y)]$

$[f(x), \alpha(x)] + 4[D(x,x,y,y), \alpha(x)] + 6[D(x,x,y,y), \alpha(x)] +$

$4[D(x,y,y,y), \alpha(x)] + [f(y), \alpha(x)] + [f(x), \alpha(y)] + 4[D(x,x,y,y), \alpha(y)] +$

$6[D(x,x,y,y), \alpha(y)] + 4[D(x,y,y,y), \alpha(y)] + [f(y), \alpha(y)] \in Z$, for all $x \in I$.  

(5.1.3)

From (5.1.2) & (5.1.3), we get

$4[D(x,x,y,y), \alpha(x)] + 6[D(x,x,y,y), \alpha(x)] + 4[D(x,y,y,y), \alpha(x)] +$

$[f(y), \alpha(x)] + [f(x), \alpha(y)] + 4[D(x,x,x,y), \alpha(y)] + 6[D(x,x,y,y), \alpha(y)] +$

$4[D(x,y,y,y), \alpha(y)] \in Z$, for all $x \in I$.  

(5.1.4)

Replacing $y$ by $-y$ in (5.1.4), we find

$-4[D(x,x,y,y), \alpha(x)] + 6[D(x,x,y,y), \alpha(x)] - 4[D(x,y,y,y), \alpha(x)] +$

$[f(y), \alpha(x)] - [f(x), \alpha(y)] + 4[D(x,x,x,y), \alpha(y)] - 6[D(x,x,y,y), \alpha(y)] +$

$4[D(x,y,y,y), \alpha(y)] \in Z$, for all $x \in I$.  

(5.1.5)
Comparing (5.1.4) and (5.1.5) and using 2-torsion freeness of $R$, we have

$$4[D(x, x, x, y), \alpha(x)] + 4[D(x, y, y, y), \alpha(x)] + [f(x), \alpha(y)] +$$

$$6[D(x, x, y, y), \alpha(y)] \in Z, \text{ for all } x \in I. \tag{5.1.6}$$

Substitute $y + z$ for $y$ in (5.1.6) and use (5.1.6), we get

$$4[D(x, x, y + z), \alpha(x)] + 4[D(x, y + z, y + z, y + z), \alpha(x)] + [f(x), \alpha(y + z)]$$

$$+ 6[D(x, x, y + z, y + z), \alpha(y + z)] \in Z$$

$$4[D(x, x, y), \alpha(x)] + 4[D(x, x, z), \alpha(x)] + 4[D(x, y, y), \alpha(x)]$$

$$+ 4[D(x, y, y, z), \alpha(x)] + 4[D(x, y, z, y), \alpha(x)]$$

$$+ 4[D(x, z, y, z), \alpha(x)] + 4[D(x, z, z, y), \alpha(x)]$$

$$+ 4[D(x, z, z, z), \alpha(x)] + [f(x), \alpha(y)] + [f(x), \alpha(z)]$$

$$+ 6[D(x, x, y), \alpha(y)] + 6[D(x, x, y, z), \alpha(y)]$$

$$+ 6[D(x, x, z, y), \alpha(y)] + 6[D(x, x, z, z), \alpha(y)]$$

$$+ 6[D(x, x, y, y), \alpha(z)] + 6[D(x, x, z, z), \alpha(z)] \in Z$$

$$4[D(x, y, y, z), \alpha(x)] + 4[D(x, y, z, y), \alpha(x)] + 4[D(x, y, z, z), \alpha(x)]$$

$$+ 4[D(x, z, y, y), \alpha(y)] + 4[D(x, z, z, y), \alpha(y)]$$

$$+ 4[D(x, z, z, z), \alpha(y)] + 6[D(x, x, y, z), \alpha(y)]$$

$$+ 6[D(x, x, y, y), \alpha(z)] + 6[D(x, x, y, z), \alpha(z)]$$

$$+ 6[D(x, x, z, y), \alpha(z)] \in Z$$
Replacing $z$ in $-z$ in (5.1.7) and compare with (5.1.7), we obtain

\[-12[D(x, y, y, z), \alpha(x)] + 12[D(x, y, z, z), \alpha(x)] - 12[D(x, x, z, z), \alpha(y)]
+ 6[D(z, x, z, z), \alpha(y)] - 6[D(x, x, y, y), \alpha(z)]
+ 12[D(x, x, y, z), \alpha(z)] \in Z\]

\[2(12[D(x, z, y, y), \alpha(x)] + 12[D(x, x, y, z), \alpha(y)] + 6[D(x, x, y, y), \alpha(z)]) \in Z\]

Using of two torsion free ring, we have

\[12[D(x, z, y, y), \alpha(x)] + 12[D(x, x, y, z), \alpha(y)] + 6[D(x, x, y, y), \alpha(z)] \in Z,\]

for all $x, y, z \in I$. \hfill (5.1.8)

Substitute $y + u$ for $y$ in (5.1.8) and use (5.1.8) we get

\[12[D(x, z, y + u, y + u), \alpha(x)] + 12[D(x, x, y + u, z), \alpha(y + u)]
+ 6[D(x, x, y + u, y + u), \alpha(z)] \in Z\]

\[12[D(x, z, y, y), \alpha(x)] + 12[D(x, z, y, u), \alpha(x)] + 12[D(x, z, u, y), \alpha(x)]
+ 12[D(x, z, u, u), \alpha(x)] + 12[D(x, x, y, z), \alpha(y)]
+ 12[D(x, x, y, u), \alpha(y)] + 12[D(x, x, u, z), \alpha(u)]
+ 12[D(x, x, u, z), \alpha(u)] + 6[D(x, x, y, y), \alpha(z)]
+ 6[D(x, x, y, u), \alpha(z)] + 6[D(x, x, u, y), \alpha(z)]
+ 6[D(x, x, u, u), \alpha(z)] \in Z\]
$24[D(x, y, u, \alpha(x))] + 12[D(x, y, z, \alpha(u))] + 12[D(x, u, z, \alpha(y))] + 12[D(x, x, y, \alpha(z))] \in Z$, for all $x, y, z \in I$. (5.1.9)

Since $R$ is 2 and 3-torsion free and replacing $y, u$ by $x$ in (5.1.9), we have

$24[D(x, z, x, \alpha(x))] + 12[D(x, x, z, \alpha(z))] + 12[D(x, x, x, \alpha(z))] \in Z$

$48[D(x, x, x, z, \alpha(x))] + 12[D(x, x, x, z, \alpha(z))] \in Z$

$4[D(x, x, x, z, \alpha(x))] + [f(x), \alpha(z)] \in Z$, for all $x, z \in I$. (5.1.10)

Again replaced $z$ by $xz$ in (5.1.10) and using (5.1.10) we obtain

$4[D(x, x, x, xz, \alpha(x))] + [f(x), \alpha(xz)] \in Z$, for all $x, z \in I$.

$4[D(x, x, x, xz, \alpha(x))] + [f(x), \alpha(x)\alpha(z)] \in Z$, for all $x, z \in I$.

$4[D(x, x, x, xz + \alpha(x)D(x, x, x, z), \alpha(x)] + [f(x), \alpha(x)]\alpha(z) + \alpha(x)[f(x), \alpha(z)] \in Z$, for all $x, z \in I$.

$4f(x)[x, \alpha(x)] + 4[f(x), \alpha(x)]x + 4\alpha(x)[D(x, x, x, z), \alpha(x)] + [f(x), \alpha(x)]\alpha(z) + \alpha(x)[f(x), \alpha(z)] \in Z$, for all $x, z \in I$.

$4f(x)[x, \alpha(x)] + [f(x), \alpha(z)] + 4[D(x, x, x, z), \alpha(z)] + (\alpha(z) + 4z)[f(x), \alpha(x)] + 4f(x)[z, \alpha(x)] \in Z$, for all $x, z \in I$. (5.1.11)

Therefore, from (5.1.11), we get

$[\alpha(x)([f(x), \alpha(z)] + 4[D(x, x, x, z), \alpha(x)]) + \alpha(x)] + [(\alpha(z) + 4z)[f(x), \alpha(x)], \alpha(x)] + 4f(x)[x, \alpha(x)], \alpha(x)] = 0$, for all $x, z \in I$. (5.1.12)
\[ \alpha(x)[(f(x), \alpha(z)] + 4[D(x, x, x, z), \alpha(x)], \alpha(x)] + \\
(\alpha(z) + 4z)[(f(x), \alpha(x), \alpha(x)] + [\alpha(z) + 4z, \alpha(x)](f(x), \alpha(x)] + \\
4f(x)[z, \alpha(x)], \alpha(x)] + 4[f(x), \alpha(x)][z, \alpha(x)] = 0, \text{ for all } x, z \in I.
\]
\[ \alpha(x)[(f(x), \alpha(z)], \alpha(x)] + 4\alpha(x)[(D(x, x, x, z), \alpha(x)], \alpha(x)] + (\alpha(z) + \\
4z)[(f(x), \alpha(x), \alpha(x)] + [\alpha(z), \alpha(x)](f(x), \alpha(x)] + 4[z, \alpha(x)](f(x), \alpha(x)] + \\
4f(x)[z, \alpha(x)], \alpha(x)] + 4[f(x), \alpha(x)][z, \alpha(x)] = 0, \text{ for all } x, z \in I.
\]
\[ ((\alpha(z) + 4z), \alpha(x)](f(x), \alpha(x)] + 4f(x)[z, \alpha(x)], \alpha(x)] = 0, \text{ for all } x, z \in I.
\]

(5.1.13)

Replacing \( z \) by \( f(x)[f(x), \alpha(x)] \) in (5.1.13), we get

\[ [(\alpha(f(x)[f(x), \alpha(x)])) + 8f(x)[(f(x), \alpha(x)], \alpha(x)](f(x), \alpha(x)] + \\
4f(x)[(f(x)[f(x), \alpha(x)], \alpha(x)], \alpha(x)] = 0, \text{ for all } x \in I.
\]
\[ [(\alpha(f(x))\alpha([f(x), \alpha(x)], \alpha(x)])(f(x), \alpha(x)] + \\
8[f(x)[f(x), \alpha(x)], \alpha(x)](f(x), \alpha(x)] + 4f(x)[(f(x), \alpha(x)](f(x), \alpha(x)] + \\
f(x)[(f(x), \alpha(x)], \alpha(x)](f(x), \alpha(x)] = 0, \text{ for all } x \in I.
\]
\[ \alpha(f(x))[\alpha([f(x), \alpha(x)], \alpha(x)](f(x), \alpha(x)] + \\
[\alpha(f(x)), \alpha(x)]\alpha([f(x), \alpha(x)](f(x), \alpha(x)] + \\
8f(x)[(f(x), \alpha(x)], \alpha(x)](f(x), \alpha(x)] + 8[f(x), \alpha(x)](f(x), \alpha(x)](f(x), \alpha(x)] + \\
4f(x)[(f(x), \alpha(x))(f(x), \alpha(x)], \alpha(x)] = 0, \text{ for all } x \in I.
\]
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\[\alpha(f(x), \alpha(x)) \alpha([f(x), \alpha(x)]) [f(x), \alpha(x)] + \]

\[
8[f(x), \alpha(x)][f(x), \alpha(x)][f(x), \alpha(x)] + 4f(x)[f(x), \alpha(x)][f(x), \alpha(x)] + \\
4f(x)[f(x), \alpha(x)], \alpha(x)][f(x), \alpha(x)] = 0, \text{ for all } x \in I. \\
\]

\[
\alpha(f(x), \alpha(x)][f(x), \alpha(x)][f(x), \alpha(x)] + \\
8[f(x), \alpha(x)][f(x), \alpha(x)][f(x), \alpha(x)] = 0, \text{ for all } x \in I. \\
\]

\[
\alpha(f(x), \alpha(x)][f(x), \alpha(x)][f(x), \alpha(x)] + 8[f(x), \alpha(x)]^3 = 0, \text{ for all } x \in I. \\
\]

Since \(f\) is commutative on \(I\) and we have 2, 3-torsion freeness,

\[2[f(x), \alpha(x)]^3 = 0. \]

It follows that \((2[f(x), \alpha(x)]^2) R 2([f(x), \alpha(x)]^3) = 0. \]

Since \(R\) is semiprime, we have

\[2[f(x), \alpha(x)]^2 = 0, \text{ for all } x \in I. \] \hspace{1cm} (5.1.14)

On the other hand, taking \(z = x^2\) in equation (5.1.10), we get

\[4[D(x, x, x, x^2), \alpha(x)] + [f(x), \alpha(x^2)] \in Z, \text{ for all } x \in I. \]

\[4[D(x, x, x, x) x + \alpha(x)D(x, x, x, x), \alpha(x)] + [f(x), \alpha(x)\alpha(x)] \in Z, \text{ for all } x \in I. \]

\[4[f(x) x + \alpha(x)f(x), \alpha(x)] + \alpha(x)[f(x), \alpha(x)] + [f(x), \alpha(x)]\alpha(x) \in Z, \text{ for all } x \in I. \]

\[4[f(x) x, \alpha(x)] + 4[\alpha(x)f(x), \alpha(x)] + 2\alpha(x)f(x), \alpha(x)] \in Z, \text{ for all } x \in I. \]

\[4f(x)[x, \alpha(x)] + 4[f(x), \alpha(x)] x + 4\alpha(x)[f(x), \alpha(x)] + 4[\alpha(x), \alpha(x)] f(x) + 2\alpha(x) f(x), \alpha(x)] \in Z, \text{ for all } x \in I. \]
\[ 6a(x)[f(x),\alpha(x)] + 4x[f(x),\alpha(x)] + 4f(x)[x,\alpha(x)] \in Z, \text{ for all } x \in I. \quad (5.1.15) \]

Therefore, from equation (5.1.15), we get

\[ [f(x), 6a(x)[f(x),\alpha(x)] + 4x[f(x),\alpha(x)] + 4f(x)[x,\alpha(x)]] = 0, \text{ for all } x \in I. \]

\[ [f(x), 6a(x)[f(x),\alpha(x)]] + [f(x), 4x[f(x),\alpha(x)]] + [f(x), 4f(x)[x,\alpha(x)]] = 0. \]

\[ 6a(x)[f(x),[f(x),\alpha(x)]] + 6[f(x),\alpha(x)][f(x),\alpha(x)] + 4x[f(x),[f(x),\alpha(x)]] + 4[f(x),x][f(x),\alpha(x)] + 4f(x)[f(x),[x,\alpha(x)]] \]
\[ + 4[f(x),f(x)][x,\alpha(x)] = 0. \]

\[ 6[f(x),\alpha(x)]^2 + 4f(x)[f(x),[x,\alpha(x)]] = 0, \text{ for all } x \in I. \]

\[ 6[f(x),\alpha(x)]^2 + 4f(x)[f(x),[x,\alpha(x)]] = 0, \text{ for all } x \in I. \quad (5.1.16) \]

Since \( f \) is commutative and using equation (5.1.16), we get

\[ 6[f(x),\alpha(x)]^2 = 0, \text{ for all } x \in I. \]

We have 2- torsion freeness, we get

\[ 3[f(x),\alpha(x)]^2 = 0, \text{ for all } x \in I. \quad (5.1.17) \]

Comparing (5.1.14) and (5.1.17) and we have 2- torsion freeness, we get

\[ [f(x),\alpha(x)]^2 = 0, \text{ for all } x \in I. \]

Note that zero is the only nilpotent element in the center of semiprime ring.

Thus, \([f(x),\alpha(x)] = 0, \text{ for all } x \in I.\]

This completes the proof.
Therefore, from equation (5.1.15), we get
\[ [f(x), 6\alpha(x)][f(x), \alpha(x)] + 4x[f(x), \alpha(x)] + 4f(x)[x, \alpha(x)] = 0, \text{ for all } x \in I. \]

Comparing (5.1.14) and (5.1.17) and we have 2- torsion freeness, we get
\[ [f(x), \alpha(x)]^2 = 0, \text{ for all } x \in I. \] (5.1.17)

Note that zero is the only nilpotent element in the center of semiprime ring.

Thus, \([f(x), \alpha(x)] = 0, \text{ for all } x \in I.\)

This completes the proof.
5.2 Symmetric Skew 4-Reverse Derivations on Semiprime Rings:

Bresar and Vukman [14] have introduced the notation of a reverse derivations and Samman and Alyamani [34] have studied some properties of semi prime rings with reverse derivations. The study of centralizing and commuting mappings on prime rings was initiated by the result of Posner [32] which states that the existence of a nonzero centralizing derivation on a prime ring implies that the ring has to be commutative. Vukman [36,37] investigated symmetric bi derivation on prime and semi prime rings in connection with centralizing mappings. Fosner,A [1] have studied some results in symmetric skew 3-derivations with prime rings and semiprime rings. Recently Faiza Shujat, Abuzaid Ansari [16] studied some results in symmetric skew 4-derivations in prime rings. Jaya Subba Reddy.C [22] have studied some results in symmetric skew 3-reverse derivations with semiprime rings. Motivated by the above work, in this section we proved that under certain conditions of a semiprime ring with a nonzero symmetric skew 4 - reverse derivations has to be commutative.

Throughout the section, R will represent a ring with a center Z and α an anti - automorphism of R. Let n ≥ 2 be an integer. A ring R is said to be n-torsion free if for x ∈ R, nx = 0 implies x = 0. For all x, y ∈ R the symbol [x, y] will denote the commutator xy − yx. we make extensive use of basic commutator identities [xy, z] = [x, z]y + x[y, z] and [x, yz] = [x, y]z + y[x, z]. Recall that a ring R is semiprime if xRx = 0 implies that x = 0. An additive map d: R → R is called derivation if d(xy) = d(x)y + xd(y), for all x, y ∈ R, and it is called a skew derivation (α - derivation) of R associated with the anti-automorphism α if d(xy) =
\[ d(x)y + \alpha(x)d(y), \text{ for all } x, y \in R. \] An additive map \( d: R \to R \) is called reverse derivation if \( d(xy) = xd(y) + yd(x), \) for all \( x, y \in R, \) and it is called a skew reverse derivation of \( R \) associated with anti-automorphism \( \alpha \) if \( d(xy) = xd(y) + \alpha(y)d(x), \) for all \( x, y \in R. \)

Before starting our main theorem, let us gives some basic definitions and well known results which we will need in our further investigation.

Let \( D \) be a symmetric 4-additive map of \( R, \) then obviously

\[ D(-p, q, r, s) = -D(p, q, r, s), \text{ for all } p, q, r, s \in R. \] (5.2.1)

Namely, for all \( y, z \in R, \) the map \( D(. , . , y, z): R \to R \) is endomorphism of the additive group of \( R. \)

The map \( f: R \to R \) defined by \( f(x) = D(x, x, x, x), x \in R \) is called trace of \( D. \)

Note that \( f \) is not additive on \( R. \) But for all \( x, y \in R, \) we have

\[ f(x + y) = [f(x) + 4D(x, x, x, y) + 6D(x, x, y, y) + 4D(x, y, y, y) + f(y)] \]

Recall that by (5.2.1), \( f \) is even function

More precisely, for all \( p, q, r, s, u, v, w, x \in R, \) we have

\[ D(pu, q, r, s) = pD(u, q, r, s) + \alpha(u)D(p, q, r, s), \]
\[ D(p, qv, r, s) = qD(p, v, r, s) + \alpha(v)D(p, q, r, s), \]
\[ D(p, q, rw, s) = rD(p, q, w, s) + \alpha(w)D(p, q, r, s), \]
\[ D(p, q, r, sx) = sD(p, q, r, x) + \alpha(x)D(p, q, r, s). \]
Of course, if \( D \) is symmetric, then the above four relations are equivalent to each other.

**Lemma 5.2.1:** Let \( R \) be a prime ring and \( a, b \in R \). If \( a[x, b] = 0 \) for all \( x \in R \), then either \( a = 0 \) or \( b \in Z \).

**Proof:** Note that

\[
0 = a[xy, b] = ax[y, b] + a[x, b]y = ax[y, b] \quad \text{for all } x, y \in R.
\]

Thus \( aR[y, b] = 0 \), \( y \in R \), and, since \( R \) is prime, either \( a = 0 \) or \( b \in Z \).

**Theorem 5.2.1:** Let \( R \) be a \( 2,3 \)-torsion free non commutative semiprime ring and \( I \) be a nonzero ideal of \( R \). Suppose \( \alpha \) is an anti automorphism of \( R \) and \( D: R^4 \to R \) is a symmetric skew 4-reverse derivation associated with \( \alpha \). Suppose that the trace function \( f \) is commuting on \( I \) and \( [f(y), \alpha(y)] \in Z \), for all \( y \in I \), then \( [f(y), \alpha(y)] = 0 \), for all \( y \in I \).

**Proof:** Let \( [f(y), \alpha(y)] \in Z \), for all \( y \in I \).

Linearization of (5.2.2) yields that

\[
[f(x + y), \alpha(x + y)] \in Z, \quad \text{for all } x, y \in I.
\]

\[
[f(x + y), \alpha(x) + \alpha(y)] \in Z
\]

By skew 4-derivation, we have

\[
(f(x + y), \alpha(x) + \alpha(y)) = [f(x) + 4D(x, x, x, y) + 6D(x, x, y, y) + 4D(x, y, y, y) + f(y), \alpha(x) + \alpha(y)],
\]

for all \( x, y \in I \).
\[ f(x), a(x) \] + 4[D(x, x, y), a(x)] + 6[D(x, x, y), a(x)] + \\
4[D(x, y, y, a(x)] + [f(y), a(x)] + [f(x), a(y)] + 4[D(x, x, y), a(y)] + \\
6[D(x, x, y), a(y)] + 4[D(x, y, y), a(y)] + [f(y), a(y)] \in Z, \text{for all } x, y \in l.
(5.2.3)

From (5.2.2) & (5.2.3), we get

\[ 4[D(x, x, y), a(x)] + 6[D(x, x, y), a(x)] + 4[D(x, y, y), a(x)] + \\
[f(y), a(x)] + [f(x), a(y)] + 4[D(x, x, y), a(y)] + 6[D(x, x, y), a(y)] + \\
4[D(x, y, y), a(y)] \in Z, \text{for all } x, y \in l. \quad (5.2.4)

Replacing \( y \) by \(-y\) in (5.2.4) we have

\[ -4[D(x, x, y), a(x)] + 6[D(x, x, y), a(x)] - 4[D(x, y, y), a(x)] + \\
[f(y), a(x)] - [f(x), a(y)] + 4[D(x, x, y), a(y)] - 6[D(x, x, y), a(y)] + \\
4[D(x, y, y), a(y)] \in Z, \text{for all } x, y \in l. \quad (5.2.5)

Comparing (5.2.4) and (5.2.5) and using 2-torsion freeness of \( R \), we get

\[ 4[D(x, x, y), a(x)] + 4[D(x, y, y), a(x)] + [f(x), a(y)] + \\
6[D(x, x, y), a(y)] \in Z, \text{for all } x, y \in l. \quad (5.2.6)

Substitute \( y + z \) for \( y \) in (5.2.6) and using (5.2.6), we get

\[ 4[D(x, x, y + z), a(x)] + 4[D(x, y + z, y + z), a(x)] + [f(x), a(y + z)] + \\
6[D(x, x, y + z, y + z), a(y + z)] \in Z, \text{for all } x, y, z \in l. \]
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\[4[D(x,x,y),\alpha(x)] + 4[D(x,x,z),\alpha(x)] + 4[D(x,y,y),\alpha(x)] + 4[D(x,y,z),\alpha(x)] \]
\[+ 4[D(x,y,y),\alpha(x)] + 4[D(x,z,y),\alpha(x)]
\[+ 4[D(x,y,z),\alpha(x)] + 4[D(x,z,y),\alpha(x)]
\[+ 4[D(x,z,z),\alpha(x)] + [f(x),\alpha(y)] + [f(x),\alpha(z)]
\[+ 6[D(x,x,y),\alpha(y)] + 6[D(x,y,z),\alpha(y)]
\[+ 6[D(x,x,z),\alpha(y)] + 6[D(x,x,z),\alpha(y)]
\[+ 6[D(x,x,y),\alpha(z)] + 6[D(x,y,z),\alpha(z)]
\[+ 6[D(x,x,z),\alpha(z)] + 6[D(x,x,z),\alpha(z)] \in Z.
\]

\[12[D(x,y,z),\alpha(x)] + 12[D(x,y,z),\alpha(x)] + 12[D(x,y,z),\alpha(y)] + \\
6[D(x,z,z),\alpha(y)] + 6[D(x,x,y),\alpha(z)] + 12[D(x,x,y),\alpha(z)] \in Z, \text{ for all } x, y, z \in I. \quad (5.2.7)
\]

Replacing \( z \) by \(-z\) in (5.2.7) and compare with (5.2.7), we obtain

\[-12[D(x,y,y),\alpha(x)] + 12[D(x,y,z),\alpha(x)] - 12[D(x,x,y),\alpha(y)]
\[+ 6[D(x,z,z),\alpha(y)] - 6[D(x,x,y),\alpha(x)]
\[+ 12[D(x,x,y),\alpha(x)] \in Z
\]

\[2(12[D(x,z,y),\alpha(x)] + 12[D(x,x,y),\alpha(y)] + 6[D(x,x,y),\alpha(z)]) \in Z. \quad \mathbb{R} S
\]

for all \( x, y, z \in I. \)

\[\leq 2 \cdot 4 \quad \mathbb{V} 4 \mathbb{H}
\]

Using of 2- torsion free ring, we have

\[12[D(x,z,y),\alpha(x)] + 12[D(x,x,y),\alpha(y)] + 6[D(x,x,y),\alpha(z)] \in Z,
\]

for all \( x, y, z \in I. \quad (5.2.8)
\]

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Substitute $y + u$ for $y$ in (5.2.8) and use (5.2.8) we get

\[ 12[D(x, z, y + u, y + u), a(x)] + 12[D(x, x, y + u, z), a(y + u)] + 6[D(x, x, y + u, y + u), a(z)] \in Z, \text{ for all } x, y, z, u \in I. \]

\[ 12[D(x, z, y + u, y + u), a(x)] + 12[D(x, z, y + u, y), a(x)] + 12[D(x, z, y, u), a(x)] +
\]

\[ 12[D(x, z, u, u), a(x)] + 12[D(x, y, z, y), a(x)] + 12[D(x, x, y, z), a(y)] +
\]

\[ 12[D(x, x, z, u), a(u)] + 12[D(x, x, x, z), a(u)] + 6[D(x, x, y, y), a(x)] +
\]

\[ 6[D(x, x, y, u), a(z)] + 6[D(x, x, u, y), a(z)] + 6[D(x, x, u, u), a(z)] \in Z, \text{ for all } x, y, z, u \in I. \]

\[ 24[D(x, z, y, u), a(x)] + 12[D(x, x, y, z), a(u)] + 12[D(x, x, u, z), a(y)] +
\]

\[ 12[D(x, y, u), a(z)] \in Z, \text{ for all } x, y, z, u \in I. \] \quad (5.2.9)

Since $R$ is a 2 and 3-torsion free and replacing $y, u$ by $x$ in (5.2.9), we have

\[ 24[D(x, z, x, x), a(x)] + 12[D(x, x, x, z), a(x)] + 12[D(x, x, x, z), a(x)] +
\]

\[ 12[D(x, x, x, x), a(z)] \in Z, \text{ for all } x, z \in I. \]

\[ 48[D(x, x, x, z), a(x)] + 12[D(x, x, x, z), a(x)] \in Z, \text{ for all } x, z \in I. \]

\[ 4[D(x, x, x, z), a(x)] + [f(x), a(x)] \in Z, \text{ for all } x, z \in I. \] \quad (5.2.10)

Again replaced $z$ by $zx$ in (5.2.10) and using (5.2.10) we obtain

\[ 4[D(x, x, x, zx), a(x)] + [f(x), a(x)z] \in Z, \text{ for all } x, z \in I. \]

\[ 4[D(x, x, x, zx), a(x)] + [f(x), a(x)a(z)] \in Z, \text{ for all } x, z \in I. \]

\[ 4[zf(x) + a(x)D(x, x, x, z), a(x)] + [f(x), a(x)a(z) + a(x)[f(x), a(x)] \in Z, \]

for all $x, z \in I$. 

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\[ 4[z, a(x)]f(x) + 4z[f(x), a(x)] + 4a(x)[D(x, x, x, z), a(x)] + \]
\[ [f(x), a(x)]a(z) + a(x)[f(x), a(z)] \in Z, \text{ for all } x, z \in I. \]

\[ a(x)([f(x), a(z)] + 4[D(x, x, x, z), a(x)]) + (a(z) + 4z)[f(x), a(x)] + \]
\[ 4[z, a(x)]f(x) \in Z, \text{ for all } x, z \in I. \]  

(5.2.11)

Therefore, from (5.2.11), we get

\[ [a(x)([f(x), a(z)] + 4[D(x, x, x, z), a(x)]) + a(x)] + \]
\[ ([a(z) + 4z][f(x), a(x)], a(x)] + 4[[z, a(x)]f(x), a(x)] = 0, \text{ for all } x, z \in I \]  

(5.2.12)

\[ a(x)([f(x), a(z)] + 4[D(x, x, x, z), a(x)]) + a(x)] + \]
\[ (a(z) + 4z)[f(x), a(x)] + [a(z) + 4z, a(x)][f(x), a(x)] + \]
\[ 4[[z, a(x)], a(x)]f(x) + 4[z, a(x)][f(x), a(x)] = 0, \text{ for all } x, z \in I. \]

\[ a(x)([f(x), a(z)], a(x)] + 4a(x)[D(x, x, x, z), a(x)], a(x)] + (a(z) + \]
\[ 4z)[f(x), a(x)], a(x)] + [a(z), a(x)][f(x), a(x)] + 4[z, a(x)][f(x), a(x)] + \]
\[ 4[[z, a(x)], a(x)]f(x) + 4[z, a(x)][f(x), a(x)] = 0, \text{ for all } x, z \in I. \]

\[ a(x)([f(x), a(z)], a(x)] + [a(z), a(x)][f(x), a(x)] + [4z, a(x)][f(x), a(x)] + \]
\[ 4[[z, a(x)], a(x)]f(x) + [4z, a(x)][f(x), a(x)] = 0, \text{ for all } x, z \in I. \]

\[ ([a(z) + 8z), a(x)][f(x), a(x)] + 4[[z, a(x)], a(x)]f(x) = 0, \text{ for all } x, z \in I. \]  

(5.2.13)

Replacing \( z \) by \( [f(x), a(x)]f(x) \) in (5.2.13), we get
\[(\alpha(f(x), \alpha(x))f(x)) + 8f(x), \alpha(x)]f(x), \alpha(x)]f(x), \alpha(x)] +

4 \left[(f(x), \alpha(x))f(x), \alpha(x)\right]f(x) = 0, \text{ for all } x \in I.

\[(\alpha(f(x)), \alpha(x))]f(x), \alpha(x)]f(x), \alpha(x)] +

8[(f(x), \alpha(x))f(x), \alpha(x)]f(x), \alpha(x)] + 4[(f(x), \alpha(x))f(x), \alpha(x)] +

4 \left[(f(x), \alpha(x))f(x), \alpha(x)\right]f(x) = 0, \text{ for all } x \in I.

\[(\alpha(f(x)), \alpha(x)]\alpha(f(x), \alpha(x))]f(x), \alpha(x)] +

8[(f(x), \alpha(x))f(x), \alpha(x)]f(x), \alpha(x)] + 8f(x), \alpha(x)]f(x), \alpha(x)]f(x) +

4 \left[(f(x), \alpha(x))f(x), \alpha(x)\right]f(x) = 0, \text{ for all } x \in I.

\[(\alpha(f(x)), \alpha(x)]\alpha(f(x), \alpha(x))]f(x), \alpha(x)] +

8[f(x), \alpha(x)]f(x), \alpha(x)]f(x), \alpha(x)] = 0, \text{ for all } x \in I.

\[(\alpha(f(x)), \alpha(x)]\alpha(f(x), \alpha(x))]f(x), \alpha(x)] + 8[f(x), \alpha(x)]f(x), \alpha(x)]^3 = 0, \text{ for all } x \in I.

Since \( f \) is commutes on \( I \), and we have 2, 3- torsion freeness of \( R \), we have

\[2[f(x), \alpha(x)]^3 = 0\]

It follows that \( (2[f(x), \alpha(x)]^2) R 2([f(x), \alpha(x)]^2) = 0. \)

Since \( R \) is semiprime, we have
2[f(x), α(x)]^2 = 0, for all \(x \in I\). \hspace{1cm} (5.2.14)

On the other hand, taking \(z = x^2\) in equation (5.2.10), we get

\[
4[D(x, x, x, x^2), α(x)] + [f(x), α(x^2)] \in Z, \text{ for all } x \in I.
\]

\[
4[D(x, x, x, x) + α(x)D(x, x, x, x), α(x)] + [f(x), α(x)α(x)] \in Z, \text{ for all } x \in I.
\]

\[
4[f(x) + α(x)f(x), α(x)] + α(x)[f(x), α(x)] + [f(x), α(x)]α(x) \in Z,
\]

for all \(x \in I\).

\[
4[f(x), α(x)] + 4[α(x)f(x), α(x)] + 2α(x)[f(x), α(x)] \in Z, \text{ for all } x \in I.
\]

\[
4[x, α(x)]f(x) + 4x[f(x), α(x)] + 4α(x)[f(x), α(x)] + 4[α(x), α(x)]f(x) + 2α(x)[f(x), α(x)] \in Z, \text{ for all } x \in I.
\]

\[
6α(x)[f(x), α(x)] + 4x[f(x), α(x)] + 4[x, α(x)]f(x) \in Z, \text{ for all } x \in I. \hspace{1cm} (5.2.15)
\]

Therefore, from equation (5.2.15), we get

\[
[f(x), 6α(x)[f(x), α(x)] + 4x[f(x), α(x)] + 4[x, α(x)]f(x)] = 0, \text{ for all } x \in I.
\]

\[
[f(x), 6α(x)[f(x), α(x)] + [f(x), 4x[f(x), α(x)] + [f(x), 4x, α(x)]f(x)] = 0
\]

\[
6α(x)[f(x), [f(x), α(x)] + 6[f(x), α(x)][f(x), α(x)] + 4x[f(x), [f(x), α(x)]]
\]

\[
\hspace{1cm} + 4[f(x), x][f(x), α(x)] + 4[f(x), [x, α(x)]f(x)
\]

\[
\hspace{1cm} + 4[x, α(x)][f(x), f(x)] = 0
\]

\[
6[f(x), α(x)]^2 + 4[f(x), [x, α(x)]f(x) = 0, \text{ for all } x \in I.
\]

\[
6[f(x), α(x)]^2 + 4[[f(x), x], α(x)]f(x) = 0, \text{ for all } x \in I. \hspace{1cm} (5.2.16)
\]

Since \(f\) is commuting on \(I\) and using equation (5.2.16), we get
\[6[f(x), \alpha(x)]^2 = 0, \text{ for all } x \in I.\]

We have 2-torsion freeness, we get

\[3[f(x), \alpha(x)]^2 = 0, \text{ for all } x \in I.\]  
\hspace{10em} (5.2.17)

Comparing (5.2.14) and (5.2.17), we get

\[[f(x), \alpha(x)]^2 = 0, \text{ for all } x \in I.\]

Note that zero is the only nilpotent element in the center of semiprime ring.

Thus, \([f(x), \alpha(x)] = 0, \text{ for all } x \in I.\]

\[[f(y), \alpha(y)] = 0, \text{ for all } y \in I.\]

This completes the proof.

**Corollary 5.2.1:** Let \( R \) be a 3!-torsion free prime ring, \( I \) be a non-zero ideal of \( R \) and \( \alpha \) be an anti-automorphism of \( R \). Suppose that there exists a non-zero symmetric skew 4-derivation \( D: R^4 \to R \) associated with the anti-automorphism \( \alpha \) such that the trace function \( f \) is commuting on \( I \) and \((f(x), \alpha(x)) \in Z, \text{ for all } x \in I, \text{ then } D = 0.\)

**Proof:** From theorem 5.2.1, \([f(x), \alpha(x)] \in Z, \text{ for all } x \in I, \text{ then we have } [f(x), \alpha(x)] = 0, \text{ for all } x \in I.\) From [22, Theorem1] states that \([f(x), \alpha(x)] = 0, \text{ for all } x \in I, \text{ then } D = 0.\) Observing above two relations we concludes that \([f(x), \alpha(x)] \in Z, \text{ for all } x \in I, \text{ then } D = 0.\)