Chapter 4

Marks in bipartite multidigraphs

In this chapter, we extend the concept of marks to bipartite multidigraphs and multipartite multidigraphs. We obtain necessary and sufficient conditions for a pair of sequences of non-negative integers to be mark sequences of some bipartite multidigraph. These characterizations give algorithms for constructing the corresponding bipartite multidigraphs. We provide analogous characterizations for multipartite multidigraphs.

0.1 Introduction

A bipartite r-digraph is an orientation of a bipartite multigraph that is without loops and contains at most r edges between any pair of vertices from distinct parts. So bipartite 1-digraph is an oriented bipartite graph and a complete bipartite 1-digraph is a bipartite tournament. Let $D(X,Y)$ be a bipartite r-digraph with $X = \{x_1, x_2, \cdots, x_m\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$. For any vertex $v_i$ in $D(X,Y)$, let $d_{v_i}^+$ and $d_{v_i}^-$ be the outdegree and indegree, respectively, of $v_i$. Define $p_{x_i}$ (or simply $p_i$) = $rn + d_{x_i}^+ - d_{x_i}^-$ and $q_{y_j}$ (or simply $q_j$) = $rm + d_{y_j}^+ - d_{y_j}^-$ as the marks (or r-scores) of $x_i$ in $X$ and $y_j$ in $Y$ respectively. Clearly, $0 \leq p_{x_i} \leq 2rn$ and $0 \leq q_{y_j} \leq 2rm$. Then the sequences $P = [p_i]_1^m$ and $Q = [q_j]_1^n$ in non-decreasing order are called the mark sequences of $D(X,Y)$.

A bipartite r-digraph can be interpreted as the result of a competition between two teams in which each player of one team plays with every player of the other team at most r times in which ties (draws) are allowed. A player receives two points for each win, and one point for each tie. With this marking system, player $x_i$ (respectively $y_j$) receives a total of $p_{x_i}$ (respectively $q_{y_j}$) points. The sequences $P$ and $Q$ of non-negative integers in non-decreasing
order are said to be realizable if there exists a bipartite r-digraph with mark sequences \( P \) and \( Q \).

In a bipartite r-digraph \( D(X, Y) \), if there are \( a_1 \) arcs directed from a vertex \( x \in X \) to a vertex \( y \in Y \) and \( a_2 \) arcs directed from vertex \( y \) to vertex \( x \), with \( 0 \leq a_1, a_2 \leq r \) and \( 0 \leq a_1 + a_2 \leq r \), we denote it by \( x(a_1 - a_2)y \). For example, if there are exactly \( r \) arcs directed from \( x \in X \) to \( y \in Y \) and no arc directed from \( y \) to \( x \), and this is denoted by \( x(r - 0)y \), and if there is no arc directed from \( x \) to \( y \) and no arc directed from \( y \) to \( x \), this is denoted by \( x(0 - 0)y \).

An oriented tetra in a bipartite r-digraph is an induced 1-subdigraph with two vertices from each part. Define oriented tetrads of the form \( x(1 - 0)y(1 - 0)x'(1 - 0)y'(1 - 0)x \) and \( x(1 - 0)y(1 - 0)x'(1 - 0)y'(0 - 0)x \) to be of \( \alpha \)-type and all other oriented tetrads to be of \( \beta \)-type. A bipartite r-digraph is said to be of \( \alpha \)-type or \( \beta \)-type according as all of its oriented tetrads are of \( \alpha \)-type or \( \beta \)-type respectively. We assume, without loss of generality, that \( \beta \)-type bipartite r-digraphs have no pair of symmetric arcs because symmetric arcs \( x(a - a)y \), where \( 1 \leq a \leq \frac{r}{2} \), can be transformed to \( x(0 - 0)y \) with the same marks. A transmitter is a vertex with indegree zero.

### 0.2 Characterization of marks in bipartite multidigraphs

The work in this section has appeared in Chishti and Samee [?]. We start with the following observation.

**Lemma 4.2.1.** Among all bipartite r-digraphs with given mark sequences, those with the fewest arcs are of \( \beta \)-type.

**Proof.** Let \( D(X, Y) \) be a bipartite r-digraph with mark sequences \( P \) and \( Q \). Assume \( D(X, Y) \) is not of \( \beta \)-type. Then \( D(X, Y) \) has an oriented tetra of \( \alpha \)-type, that is, \( x(1 - 0)y(1 - 0)x'(1 - 0)y'(1 - 0)x \) or \( x(1 - 0)y(1 - 0)x'(1 - 0)y'(0 - 0)x \) where \( x, x' \in X \) and \( y, y' \in Y \). Since \( x(1 - 0)y(1 - 0)x'(1 - 0)y'(1 - 0)x \) can be transformed to \( x(0 - 0)y(0 - 0)x'(0 - 0)y'(0 - 0)x \) with the same mark sequences and four arcs fewer, and \( x(1 - 0)y(1 - 0)x'(1 - 0)y'(0 - 0)x \) can
be transformed to $x(0 - 0)y(0 - 0)x'(0 - 0)y'(0 - 1):x$ with the same mark sequences and two arcs fewer, therefore, in both cases we obtain a bipartite $r$-digraph having same mark sequences $P$ and $Q$ with fewer arcs. Note that if there are symmetric arcs between $x$ and $y$, that is $x(a - a)y$, where $1 \leq a \leq \frac{r}{2}$, then these can be transformed to $x(0 - 0)y$ with the same mark sequences and $a$ arcs fewer. Hence the result follows.

**Lemma 4.2.2.** Let $P = [p_i]^n$ and $Q = [q_j]^n$ be mark sequences of a $\beta$-type bipartite $r$-digraph. Then either the vertex with mark $p_m$, or the vertex with mark $q_n$, or both can act as transmitters.

We know if $P = [p_1, p_2, \cdots, p_m]$ and $Q = [q_1, q_2, \cdots, q_n]$ are mark sequences of a bipartite $r$-digraph, then $p_i \leq 2rn$ and $q_j \leq 2rm$, where $1 \leq i \leq m$ and $1 \leq j \leq n$. We have the following observation.

**Lemma 4.2.3.** If $P = [p_1, p_2, \cdots, p_{m-1}, p_m]$ and $Q = [0, 0, \cdots, 0, 0]$ with each $p_i = rn$ are mark sequences of some bipartite $r$-digraph, then $P' = [p_1, p_2, \cdots, p_{m-1}]$ and $Q' = [0, 0, \cdots, 0]$ are also mark sequences of some bipartite $r$-digraph.

We now have some observations about bipartite 2-digraphs, as these will be required in application of Theorem 4.2.11.

**Lemma 4.2.4.** If $P = [p_1, p_2, \cdots, p_{m-1}, p_m]$ and $Q = [0, 0, \cdots, 0, q_n]$ with $4n - p_m = 3$ and $q_n \geq 3$ are mark sequences of some bipartite 2-digraph, then $P' = [p_1, p_2, \cdots, p_{m-1}]$ and $Q' = [0, 0, \cdots, 0, q_n - 3]$ are also mark sequences of some bipartite 2-digraph.

**Proof.** Let $P$ and $Q$ as given above be mark sequences of bipartite 2-digraph $D$ with parts $X = \{x_1, x_2, \cdots, x_{m-1}, x_m\}$ and $Y = \{y_1, y_2, \cdots, y_{n-1}, y_n\}$. Since $4n - p_m = 3$ and $3 \leq q_n \leq 4m$, therefore in $D$ necessarily $x_m(2 - 0)y_i$, for all $1 \leq i \leq n - 1$. Also $y_n(1 - 0)x_m$, because if $y_n(0 - 0)x_m$, or $y_n(0 - 2)x_m$, or $y_n(0 - 1)x_m$, then in all these cases $p_{x_m} \geq 4(n - 1) + 2$, a contradiction to our assumption. Also $y_n(2 - 0)x_m$ is not possible because in that case $p_{x_m} = 4(n - 1) < 4n - 3$.

Now delete $x_m$, obviously this keeps marks of $y_1, y_2, \cdots, y_{n-1}$ as zeros
and reduces mark of \( y_n \) by 3, and we obtain a bipartite 2-digraph with mark sequences \( P' = [p_1, p_2, \ldots, p_{m-1}] \) and \( Q' = [0, 0, \ldots, 0, q_n - 3] \), as required.

**Lemma 4.2.5.** If \( P = [p_1, p_2, \ldots, p_{m-1}, p_m] \) and \( Q = [0, 0, \ldots, 0, q_n] \) with \( 4n - p_m = 4 \) and \( q_n \geq 4 \) are mark sequences of some bipartite 2-digraph, then \( P' = [p_1, p_2, \ldots, p_{m-1}] \) and \( Q' = [0, 0, \ldots, 0, q_n - 4] \) are also mark sequences of some bipartite 2-digraph.

**Proof.** Let \( P \) and \( Q \) as given above be mark sequences of bipartite 2-digraph \( D \) with parts \( X = \{x_1, x_2, \ldots, x_{m-1}, x_m\} \) and \( Y = \{y_1, y_2, \ldots, y_{n-1}, y_n\} \). Since \( 4n - p_m = 4 \) and \( 4 \leq q_n \leq 4m \), therefore in \( D \) necessarily \( x_m(2 - 0)y_i \), for all \( 1 \leq i \leq n - 1 \). Also \( y_n(2 - 0)x_m \), because if \( y_n(0 - 0)x_m \), or \( y_n(1 - 0)x_m \), or \( y_n(0 - 2)x_m \), then in all these cases \( p_{x_m} \geq 4(n - 1) + 1 \), a contradiction to our assumption.

Now delete \( x_m \), obviously this keeps marks of \( y_1, y_2, \ldots, y_{n-1} \) as zeros and reduces mark of \( y_n \) by 4, and we obtain a bipartite 2-digraph with mark sequences \( P' = [p_1, p_2, \ldots, p_{m-1}] \) and \( Q' = [0, 0, \ldots, 0, q_n - 4] \), as required.

**Lemma 4.2.6.** If \( P = [p_1, p_2, \ldots, p_{m-1}, p_m] \) and \( Q = [0, 0, \ldots, 0, q_n] \) with \( 4n - p_m = 4 \) and \( q_n \geq 3 \) are mark sequences of some bipartite 2-digraph, then \( P' = [p_1, p_2, \ldots, p_{m-1}] \) and \( Q' = [0, 0, \ldots, 0, q_n - 3] \) are also mark sequences of some bipartite 2-digraph.

**Proof.** The proof follows by using the same argument as in Lemma 4.2.5.

**Lemma 4.2.7.** If \( P = [p_1, p_2, \ldots, p_{m-1}, p_m] \) and \( Q = [0, 0, \ldots, 0, 1, 3] \) with \( 4n - p_m = 4 \), are mark sequences of some bipartite 2-digraph, then \( P' = [p_1, p_2, \ldots, p_{m-1}] \) and \( Q' = [0, 0, \ldots, 0, 0, 0] \) are also mark sequences of some bipartite 2-digraph.

**Lemma 4.2.8.** If \( P = [p_1, p_2, \ldots, p_{m-1}, p_m] \) and \( Q = [0, 0, \ldots, 0, 1, 1, 2] \) with \( 4n - p_m = 4 \), are mark sequences of some bipartite 2-digraph, then \( P' = [p_1, p_2, \ldots, p_{m-1}] \) and \( Q' = [0, 0, \ldots, 0, 0, 0] \) are also mark sequences of some bipartite 2-digraph.

**Lemma 4.2.9.** If \( P = [p_1, p_2, \ldots, p_{m-1}, p_m] \) and \( Q = [0, 0, \ldots, 0, 1, 1, 1, 1] \)
with $4n - p_m = 4$, are mark sequences of some bipartite 2-digraph, then $P' = [p_1, p_2, \cdots, p_{m-1}]$ and $Q' = [0, 0, \cdots, 0, 0, 0]$ are also mark sequences of some bipartite 2-digraph.

Remarks 4.2.10. We note that the sequences of non-negative integers $[p_1]$ and $[q_1, q_2, \cdots, q_n]$, with $p_1 + q_1 + q_2 + \cdots + q_n = 2rn$, are always mark sequences of some bipartite $r$-digraph. We observe that the bipartite $r$-digraph $D(X, Y)$, with vertex sets $X = \{x_1\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$, where for $q_i$ even, say $2t$, we have $x_1((r - t) - t)y_i$ and for $q_i$ odd, say $2t + 1$, we have $x_1((r - t - 1) - t)y_i$, has mark sequences $[p_1]$ and $[q_1, q_2, \cdots, q_n]$. Also we note that the sequences $[0]$ and $[2r, 2r, \cdots, 2r]$ are mark sequences of some bipartite $r$-digraph.

The next result provides a useful recursive test whether or not a pair of sequences is realizable.

Theorem 4.2.11. Let $P = [p_i]^m$ and $Q = [q_j]^n$ be the sequences of non-negative integers in non-decreasing order with $p_m \geq q_n$ and $rn \leq p_m \leq 2rn$.

(A) If $q_n \leq 2r(m - 1) + 1$, let $P'$ be obtained from $P$ by deleting one entry $p_m$, and $Q'$ be obtained as follows.

For $[2r - (i - 1)]n \geq p_m \geq (2r - i)n, 1 \leq i \leq r$, reducing $[2r - (i - 1)]n - p_m$ largest entries of $Q$ by $i$ each, and reducing $p_m - (2r - i)n$ next largest entries by $i - 1$ each.

(B) In case $q_n > 2r(m - 1) + 1$, say $q_n = 2r(m - 1) + 1 + h$, where $1 \leq h \leq r - 1$, then let $P'$ be obtained from $P$ by deleting one entry $p_m$, and $Q'$ be obtained from $Q$ by reducing the entry $q_n$ by $h + 1$.

Then $P$ and $Q$ are the mark sequences of some bipartite $r$-digraph if and only if $P'$ and $Q'$ (arranged in non-decreasing order) are the mark sequences of some bipartite $r$-digraph.

Proof. Let $P'$ and $Q'$ be the mark sequences of some bipartite $r$-digraph $D'(X', Y')$. First suppose $Q'$ is obtained from $Q$ as in A. Construct a bipartite $r$-digraph $D(X, Y)$ as follows. Let $X = X' \cup x, Y = Y'$ with $X' \cap x = \phi$. Let $x((r - i) - 0)y$ for those vertices $y$ of $Y'$ whose marks are reduced by $i$ in going from $P$ to $P'$ and $Q$ to $Q'$, and $x(r - 0)y$ for those vertices $y$ of
Let $P$ will decrease the mark of $y$ mark sequences, and for constructing a corresponding bipartite sequences $r$ sequences $r_p x$ \[ x \leq 1 \] (B) Now in and $Q$ as in B, then construct a bipartite $r$-digraph $D(X,Y)$ as follows. Let $X = X' \cup x, Y = Y'$ with $X' \cap x = \phi$. Let $x((r-h-1)-0)y$ for that vertex $y$ of $Y'$ whose marks are reduced by $h$ in going from $P$ and $Q$ to $P'$ and $Q'$. Then $D(X,Y)$ is the bipartite $r$-digraph with mark sequences $P$ and $Q$.

Conversely, suppose $P$ and $Q$ be the mark sequences of a bipartite $r$-digraph $D(X,Y)$. Without loss of generality, we choose $D(X,Y)$ to be of $\beta$-type. Then by Lemma 4.2.2, any of the vertex $x \in X$ or $y \in Y$ with mark $p_m$ or $q_n$ respectively can be a transmitter. Let the vertex $x \in X$ with mark $p_m$ be a transmitter. Clearly, $p_m \geq rn$ and because if $p_m < rn$, then by deleting $p_m$ we have to reduce more than $n$ entries from $Q$, which is absurd.

(A) Now $q_n \leq 2r(m-1) + 1$ because if $q_n > 2r(m-1) + 1$, then on reduction $q'_n = q_n - 1 > 2r(m-1) + 1 - 1 = 2r(m-1)$, which is impossible.

Let $[2r - (i - 1)]n \geq p_m \geq (2r - i)n, 1 \leq i \leq r$, let $V$ be the set of $[2r - (i - 1)]n - p_m$ vertices of largest marks in $Y$, and let $W$ be the set of $p_m - (2r - i)n$ vertices of next largest marks in $Y$ and let $Z = Y - \{V,W\}$. Construct $D(X,Y)$ such that $x((r-i-1)0)v$ for all $v \in V$, $x((r-i-1)0)w$ for all $w \in W$ and $x(r-0)z$ for all $z \in Z$. Clearly, $D(X,Y) - x$ realizes $P'$ and $Q'$ (arranged in non-decreasing order).

(B) Now in $D$, let $q_n > 2r(m-1) + 1$, say $q_n = 2r(m-1) + 1 + h$, where $1 \leq h \leq r - 1$. This means $y_n(r-0)x_i$, for all $1 \leq i \leq m - 1$. Since $x_m$ is a transmitter, so there cannot be an arc from $y_n$ to $x_m$. Therefore $x_m((r-h-1)-0)y_n$, since $y_n$ needs $h+1$ more marks. Now delete $x_m$, it will decrease the mark of $y_n$ by $h+1$, and the resulting bipartite $r$-digraph will have mark sequences $P'$ and $Q'$ as desired.

Theorem 4.2.11 provides an algorithm of checking whether or not the sequences $P$ and $Q$ of non-negative integers in non-decreasing order are the mark sequences, and for constructing a corresponding bipartite $r$-digraph.

Let $P = [p_1, p_2, \cdots, p_m]$ and $Q = [q_1, q_2, \cdots, q_n]$, where $p_m \geq q_n$, $rn \leq p_m \leq 2rn$ and $q_n \leq 2r(m-1) + 1$, be the mark sequences of a bipartite $r$-digraph with parts $X = \{x_1, x_2, \cdots, x_m\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$ respec-
tively. Deleting $p_m$ and performing A of Theorem 4.2.11 if $[2r-(i-1)]n \geq p_m \geq (2r-i)n$, $1 \leq i \leq r$, we get $Q' = [q'_1, q'_2, \ldots, q'_n]$. If the marks of the vertices $y_j$ were decreased by $i$ in this process, then the construction yielded $x_m((r-i)-0)y_j$, if these were decreased by $i-1$, then the construction yielded $x_m((r-i+1)-0)y_j$. If we perform B of Theorem 4.2.11, the mark of $y_n$ was decreased by $h+1$, the construction yielded $x_m((r-h-1)-0)y_n$.

For vertices $x$ of $Q$ whose marks remained unchanged, the construction yielded $x_m(r-0)y_j$. Note that if the conditions $p_m \geq rn$ does not hold, then we delete $q_n$ for which the conditions get satisfied and the same argument is used for defining arcs. If this procedure is applied recursively, then it tests whether or not $P$ and $Q$ are the mark sequences, and if $P$ and $Q$ are the mark sequences, then a bipartite $r$-digraph with mark sequences $P$ and $Q$ is constructed.

We illustrate this reduction and the resulting construction with the following examples.

**Example 4.2.12.** Consider the two sequences of non-negative integers given by $P = [14, 14, 15]$ and $Q = [6, 6, 8, 9]$. We check whether or not $P$ and $Q$ are mark sequences of some bipartite 3-digraph.

1. $P = [14, 14, 15], Q = [6, 6, 8, 9]$ 
We delete 15. Clearly $[2r-(i-1)]n = [2.3-(3-1)]4 = 16 \geq 15 \geq (2r-i)n = (2.3-3)4 = 12$. So reduce $[2r-(i-1)]n-p_m = [2.3-(3-1)4-15 = 16-15 = 1$ largest entry of $Q$ by $i = 3$ and $p_m-(2r-i)n = 15-(2.3-3)4 = 15-12 = 3$ next largest entries of $Q$ by $i - 1 = 3 - 1 = 2$ each, we get $P_1 = [14, 14], Q_1 = [4, 4, 6, 6]$, and arcs are defined as $x_3(0-0)y_4, x_3(1-0)y_3, x_3(1-0)y_2, x_3(1-0)y_1$.

2. $P_1 = [14, 14], Q_1 = [4, 4, 6, 6]$ 
We delete 14. Here $[2r-(i-1)]n = [2.3-(3-1)]4 = 16 \geq 14 \geq (2r-i)n = (2.3-3)4 = 12$. Reduce $[2r-(i-1)]n-p_m = [2.3-(3-1)4-14 = 16-14 = 2$ largest entries of $Q_1$ by $i = 3$ and $p_m-(2r-i)n = 14-(2.3-3)4 = 14-12 = 2$ next largest entries of $Q_1$ by $i - 1 = 3 - 1 = 2$ each, we get $P_2 = [14], Q_2 = [2, 2, 3, 3]$, and arcs are defined as $x_2(0-0)y_4, x_2(0-0)y_3, x_2(1-0)y_2, x_2(1-0)y_1$. 

79
3. $P_2 = [14], Q_2 = [2, 2, 3, 3]$
We delete 14. Here $[2r - (i - 1)]n = [2.3 - (3 - 1)]4 = 16 \geq 14 \geq (2r - i)n = (2.3 - 3)4 = 12$. Reduce $[2r - (i - 1)]n - p_m = [2.3 - (3 - 1)]4 - 14 = 16 - 14 = 2$
next largest entries of $Q_2$ by $i = 3$ and $p_m - (2r - i)n = 14 - (2.3 - 3)4 = 14 - 12 = 2$
next largest entries of $Q_2$ by $i - 1 = 3 - 1 = 2$ each, we get $P_3 = \phi$,$Q_3 = [0, 0, 0, 0]$, and arcs are defined as $x_1(0 - 0)y_4$, $x_1(0 - 0)y_3$, $x_1(1 - 0)y_2$, $x_1(1 - 0)y_1$.

The resulting bipartite 3-digraph has mark sequences $P = [14, 14, 15]$ and $Q = [6, 6, 8, 9]$ with vertex sets $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3, y_4\}$ and arcs as $x_3(0 - 0)y_4$, $x_3(1 - 0)y_3$, $x_3(1 - 0)y_2$, $x_3(1 - 0)y_1$, $x_2(0 - 0)y_4$, $x_2(0 - 0)y_3$, $x_2(1 - 0)y_2$, $x_2(1 - 0)y_1$, $x_1(0 - 0)y_4$, $x_1(0 - 0)y_3$, $x_1(1 - 0)y_2$, $x_1(1 - 0)y_1$.

Example 4.2.13. Consider the two sequences of non-negative integers given by $P = [13, 16, 22, 24]$ and $Q = [5, 6, 10]$. We check whether or not $P$ and $Q$ are mark sequences of some bipartite 4-digraph.

1. $P = [13, 16, 22, 24]$ and $Q = [5, 6, 10]$.
We delete 24. Here $[2r - (i - 1)]n = [2.4 - (1 - 1)]3 = 24$, so reduce $[2r - (i - 1)]n - p_m = [2.4 - (1 - 1)]3 - 24 = 24 - 24 = 0$ largest entries of $Q$ by $i = 1$, and obviously we reduce $p_m - (2r - i)n = 24 - (2.4 - 1)3 = 24 - 21 = 3$
next largest entries of $Q$ by $i - 1 = 1 - 1 = 0$ each, we get $P_1 = [13, 16, 22]$ and $Q_1 = [5, 6, 10]$, and arcs are $x_4(4 - 0)y_3$, $x_4(4 - 0)y_2$, $x_4(4 - 0)y_1$.

2. $P_1 = [13, 16, 22]$ and $Q_1 = [5, 6, 10]$.
We delete 22. Here $[2r - (i - 1)]n = [2.4 - (1 - 1)]3 = 24 \geq 22 \geq (2r - i)n = (2.4 - 1)3 = 21$. Reduce $[2r - (i - 1)]n - p_m = [2.4 - (1 - 1)]3 - 22 = 24 - 22 = 2$
next largest entries of $Q_1$ by $i = 1$ and $p_m - (2r - i)n = 22 - (2.4 - 1)3 = 22 - 21 = 1$
next largest entries of $Q_1$ by $i - 1 = 1 - 1 = 0$ each, we get $P_2 = [13, 16]$, $Q_2 = [5, 5, 9]$, and arcs are defined as $x_3(3 - 0)y_3$, $x_3(3 - 0)y_2$, $x_3(4 - 0)y_1$.

We delete 16. Here $[2r - (i - 1)]n = [2.4 - (3 - 1)]3 = 18 \geq 16 \geq (2r - i)n = (2.4 - 3)3 = 15$. Reduce $[2r - (i - 1)]n - p_m = [2.4 - (3 - 1)]3 - 16 = 18 - 16 = 2$
next largest entries of $Q_2$ by $i = 3$ and $p_m - (2r - i)n = 16 - (2.4 - 3)3 = 16 - 15 = 1$
ext largest entry of $Q_2$ by $i - 1 = 3 - 1 = 2$, we get $P_3 = [13]$, $Q_3 = [3, 2, 6]$, and arcs are defined as $x_2(3 - 0)y_3$, $x_2(3 - 0)y_2$, $x_2(2 - 0)y_1$. 80
4. $P_3 = [13], Q_3 = [3, 2, 6]$. Here $13 + 3 + 2 + 6 = 24$ which is same as $2rn = 2.4.3 = 24$. Thus by the argument as discussed in the remarks, $P_3$ and $Q_3$ are mark sequences of some bipartite 4-digraph. Here arcs are $x_1(1-3)y_3$, $x_1(3-1)y_2$, $x_1(2-1)y_1$.

The resulting bipartite 4-digraph with mark sequences $P = [13, 16, 22, 24]$ and $Q = [5, 6, 10]$ has vertex sets $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3\}$ and arcs as $x_4(4-0)y_3$, $x_4(4-0)y_2$, $x_4(4-0)y_1$, $x_3(3-0)y_3$, $x_3(3-0)y_2$, $x_3(3-0)y_1$, $x_2(3-0)y_3$, $x_2(3-0)y_2$, $x_2(2-0)y_1$, $x_1(1-3)y_3$, $x_1(3-1)y_2$, $x_1(2-1)y_1$.

Now we give a combinatorial criterion for determining whether the sequences of non-negative integers are realizable as marks. This is analogous to Landau’s theorem [?] on tournament scores and similar to the result by Beineke and Moon [?] on bipartite tournament scores.

Theorem 4.2.14. Let $P = [p_i]_1^m$ and $Q = [q_j]_1^n$ be the sequences of non-negative integers in non-decreasing order. Then $P$ and $Q$ are the mark sequences of some bipartite $r$-digraph if and only if

$$\sum_{i=1}^f p_i + \sum_{j=1}^g q_j \geq 2rfg, \quad (0.1)$$

for $1 \leq f \leq m$ and $1 \leq g \leq n$, with equality when $f = m$ and $g = n$.

Proof. The necessity of the condition follows from the fact that the sub-bipartite $r$-digraph induced by $f$ vertices from the first part and $g$ vertices from the second part has a sum of marks $2rfg$.

For sufficiency, assume that $P = [p_i]_1^m$ and $Q = [q_j]_1^n$ be the sequences of non-negative integers in non-decreasing order satisfying conditions (4.1) but are not mark sequences of any bipartite $r$-digraph. Let these sequences be chosen in such a way that $m$ and $n$ are the smallest possible and $p_1$ is the least with that choice of $m$ and $n$. We consider the following two cases.

Case(a). Suppose the equality in (4.1) holds for some $f \leq m$ and $g \leq n$, so that

$$\sum_{i=1}^f p_i + \sum_{j=1}^g q_j = 2rfg.$$
By the minimality of $m$ and $n$, $P_1 = [p_i]_1^f$ and $Q_1 = [q_j]_1^g$ are the mark sequences of some bipartite $r$-digraph $D_1(X_1, Y_1)$. Let $P_2 = [p_{f+1} - 2rg, p_{f+2} - 2rg, \ldots, p_m - 2rg]$ and $Q_2 = [q_{g+1} - 2rf, q_{g+2} - 2rf, \ldots, q_n - 2rf]$. Consider the sum

$$\sum_{i=1}^{s}(p_{f+i} - 2rg) + \sum_{j=1}^{t}(q_{g+j} - 2rf) = \sum_{i=1}^{f+s}p_i + \sum_{j=1}^{g+t}q_j - \left(\sum_{i=1}^{f}p_i + \sum_{j=1}^{g}q_j\right)$$

$$- 2rsg - 2rft$$

$$\geq 2r(f + s)(g + t) - 2rfg - 2rsg - 2rft$$

$$= 2r(fg + ft + sg + st - fg - sg - tf)$$

$$= 2rst,$$

for $1 \leq s \leq m - f$ and $1 \leq t \leq n - g$, with equality when $s = m - f$ and $t = n - g$. Thus, by the minimality of $m$ and $n$, the sequences $P_2$ and $Q_2$ form the mark sequences of some bipartite $r$-digraph $D_2(X_2, Y_2)$. Now construct a new bipartite $r$-digraph $D(X, Y)$ as follows.

Let $X = X_1 \cup X_2$, $Y = Y_1 \cup Y_2$ with $X_1 \cap X_2 = \emptyset$, $Y_1 \cap Y_2 = \emptyset$. Let $x_2(r - 0)y_1$ and $y_2(r - 0)x_1$ for all $x_i \in X_i, y_i \in Y_i$, where $1 \leq i \leq 2$, so that we get the bipartite $r$-digraph $D(X, Y)$ with mark sequences $P$ and $Q$, which is a contradiction.

**Case (b).** Suppose the strict inequality holds in (4.1) for some $f \neq m$ and $g \neq n$. Also, assume that $p_1 > 0$. Let $P_1 = [p_1 - 1, p_2, \ldots, p_{m-1}, p_m + 1]$ and $Q_1 = [q_1, q_2, \ldots, q_n]$. Clearly, $P_1$ and $Q_1$ satisfy the conditions (2.1). Thus, by the minimality of $p_1$, the sequences $P_1$ and $Q_1$ are the mark sequences of some bipartite $r$-digraph $D_1(X_1, Y_1)$. Let $p_{x_1} = p_1 - 1$ and $p_{x_m} = p_m + 1$. Since $p_{x_m} > p_1 + 1$, therefore there exists a vertex $y \in Y_1$ such that $x_m(1 - 0)y(1 - 0)x_1$, or $x_m(0 - 0)y(1 - 0)x_1$, or $x_m(1 - 0)y(0 - 0)x_1$, or $x_m(0 - 0)y(0 - 0)x_1$, is an induced sub-bipartite 1-digraph in $D_1(X_1, Y_1)$, and if these are changed to $x_m(0 - 0)y(0 - 0)x_1$, or $x_m(0 - 1)y(0 - 0)x_1$, or $x_m(0 - 0)y(0 - 1)x_1$, or $x_m(0 - 1)y(0 - 1)x_1$ respectively, the result is a bipartite $r$-digraph with mark sequences $P$ and $Q$, which is a contradiction. Hence the result follows.
0.3 Marks in multipartite multidigraphs

A $k$-partite 2-digraph (or briefly multipartite 2-digraph (M2D)) is an orientation of a $k$-partite multigraph that is without loops and contains at most 2 edges between any pair of vertices from distinct parts. So $k$-partite 1-digraph is an oriented $k$-partite graph, and a complete $k$-partite 1-digraph is a $k$-partite tournament. Let $D = D(X_1, X_2, \ldots, X_k)$ be an M2D with parts $X_i = \{x_{i1}, x_{i2}, \ldots, x_{in_i}\}$, $1 \leq i \leq k$. Let $d^+_{x_{ij}}$ and $d^-_{x_{ij}}$, $1 \leq j \leq n_i$, be respectively the outdegree and indegree of a vertex $x_{ij} \in X_i$. Define $p_{x_{ij}}$ (or simply $p_{ij}$) as the mark (or 2-score) of $x_{ij}$.

An M2D can be interpreted as a result of a competition among $k$ teams in which each player of one team plays with every player of the other $k - 1$ teams at most 2 times in which ties (draws) are allowed. A player receives two points for each win, and one point for each tie. With this marking system, player $x_{ij}$ receives a total of $p_{x_{ij}}$ points. The $k$ sequences of non-negative integers $p_i$, $1 \leq i \leq k$, in non-decreasing order are said to be realizable if there exists an M2D with mark sequences $P_i$.

For two vertices $x_{ij}$ in $X_i$ and $x_{st}$ in $X_s$, $i \neq s$ in an M2D $D(X_1, X_2, \ldots, X_k)$, we have one of the following six possibilities. (i) exactly two arcs directed from $x_{ij}$ to $x_{st}$ and no arc directed from $x_{st}$ to $x_{ij}$, this is denoted by $x_{ij}(2 - 0)x_{st}$, (ii) exactly two arcs directed from $x_{st}$ to $x_{ij}$ and no arc directed from $x_{ij}$ to $x_{st}$, this is denoted by $x_{ij}(0 - 2)x_{st}$, (iii) exactly one arc directed from $x_{ij}$ to $x_{st}$ and exactly one arc directed from $x_{st}$ to $x_{ij}$, this is denoted by $x_{ij}(1 - 1)x_{st}$, and is called a pair of symmetric arcs between $x_{ij}$ and $x_{st}$, (iv) exactly one arc directed from $x_{ij}$ to $x_{st}$ and no arc directed from $x_{st}$ to $x_{ij}$, this is denoted by $x_{ij}(1 - 0)x_{st}$, (v) exactly one arc directed from $x_{st}$ to $x_{ij}$ and no arc directed from $x_{ij}$ to $x_{st}$, this is denoted by $x_{ij}(0 - 1)x_{st}$, (vi) no arc directed from $x_{ij}$ to $x_{st}$ and no arc directed from $x_{st}$ to $x_{ij}$, this is denoted by $x_{ij}(0 - 0)x_{st}$.
A triple in M2D ($k$-partite 2-digraph) ($k \geq 3$) is an induced 2-subdigraph of three vertices with exactly one vertex from one part, and is of the form $x_{ij}(a_1 - a_2)x_{mn}(b_1 - b_2)x_{st}(c_1 - c_2)x_{ij}$, $(i \neq m \neq s$, $1 \leq j \leq n_i$, $1 \leq n \leq n_m$, $1 \leq t \leq n_s)$, where for $1 \leq g \leq 2$, $0 \leq a_g \leq 2$, $0 \leq b_g \leq 2$, $0 \leq c_g \leq 2$ and $0 \leq \sum_{g=1}^{2} a_g \leq 2$, $0 \leq \sum_{g=1}^{2} b_g \leq 2$, $0 \leq \sum_{g=1}^{2} c_g \leq 2$. An oriented triple in M2D is transitive if every oriented triple is transitive. In particular, a $p$-partite 2-digraph is transitive if every oriented triple of a $p$-partite 2-digraph is transitive. We know that if $P = [p_1, p_2, \cdots, p_l]$ and $Q = [q_1, q_2, \cdots, q_m]$ are mark sequences of a bipartite 2-digraph, then $p_i \leq 4m$, $1 \leq i \leq l$ and $q_j \leq 4l$, $1 \leq j \leq m$. Also the sequences of non-negative integers $[p_1]$ and $[q_1, q_2, \cdots, q_m]$, with $p_1 + q_1 + q_2 + \cdots + q_m = 4m$ are always mark sequences of some bipartite 2-digraph. Obviously the sequences $[0]$ and $[4, 4, \cdots, 4]$ are the mark sequences of a bipartite 2-digraph.

We have the following observation about $k$-partite 2-digraphs, $k \geq 3$.

**Lemma 4.3.1.** Let $D$ and $D'$ be two M2D’s with the same mark sequences. Then $D$ can be transformed to $D'$ by successively transforming (i) appropriate oriented triples formed by vertices $x_{ij} \in X_i$, $x_{mn} \in X_m$ and $x_{st} \in X_s$, $i \neq m \neq s$, in one of the following ways:

- either (a) by changing an intransitive oriented triple $x_{ij}(1-0)x_{mn}(1-0)x_{st}(1-0)x_{ij}$ to a transitive oriented triple $x_{ij}(0-0)x_{mn}(0-0)x_{st}(0-0)x_{ij}$, which has same mark sequences, or vice versa,

- or (b) by changing an intransitive oriented triple $x_{ij}(1-0)x_{mn}(1-0)x_{st}(0-0)x_{ij}$ to a transitive oriented triple $x_{ij}(0-0)x_{mn}(0-0)x_{st}(0-1)x_{ij}$, which has same mark sequences, or vice versa,

- or (ii) by changing the symmetric arcs $x_{ij}(1-1)x_{mn}$ to $x_{ij}(0-0)x_{mn}$, which
has same mark sequences, or vice versa.

**Proof.** Let \( P \) be mark sequences of an M2D \( D \) whose parts are \( X_i, 1 \leq i \leq k \). Suppose \( D' \) be an M2D with parts \( X'_i, 1 \leq i \leq k \). To prove the result it is sufficient to show that \( D' \) can be obtained from \( D \) by transforming oriented triples in any one of the ways as given in i(a) or i(b) or by changing the arcs as given in (ii).

We fix \( n_i \) for \( 2 \leq i \leq k \) and use induction on \( n_1 \). For \( n_1 = 1, n_2 = 1, \ldots, n_k = 1 \) and \( k = 3 \) the result is obvious. Assume that the result is true when there are fewer than \( n_1 \) vertices in the first part. Let \( j_2, j_3, \ldots, j_k \) be such that for \( m_2, m_3, \ldots, m_k, 1 \leq j_i < m_i \leq n_i (2 \leq i \leq k) \), the corresponding arcs have same orientations in \( D \) and \( D' \). For \( j_2, j_3, \ldots, j_k \), \( 2 \leq i, p, q \leq k, p \neq q \), the oriented triples are of the form

(I) \( x_{1n_1}(1 - 0)x_{ijp}(1 - 0)x_{ijq} \) and \( x'_{1n_1}(0 - 0)x'_{ijp}(0 - 0)x'_{ijq} \)

(II) \( x_{1n_1}(0 - 0)x_{ijp}(0 - 1)x_{ijq} \) and \( x'_{1n_1}(1 - 0)x'_{ijp}(0 - 0)x'_{ijq} \)

(III) \( x_{1n_1}(1 - 0)x_{ijp}(0 - 0)x_{ijq} \) and \( x'_{1n_1}(0 - 0)x'_{ijp}(0 - 1)x'_{ijq} \)

(IV) \( x_{1n_1}(1 - 0)x_{ijp} \) and \( x'_{1n_1}(0 - 0)x'_{ijp} \)

**Case (I).** Since \( x_{1n_1} \) and \( x'_{1n_1} \) have equal marks, therefore \( x_{1n_1}(0 - 1)x_{ijp} \) and \( x'_{1n_1}(0 - 0)x_{ijp} \), or \( x_{1n_1}(0 - 0)x_{ijq} \) and \( x'_{1n_1}(1 - 0)x_{ijq} \). Thus there is an oriented triple \( x_{1n_1}(1 - 0)x_{ijp}(1 - 0)x_{ijq} \) in \( D \) and corresponding to these \( x'_{1n_1}(0 - 0)x'_{ijp}(0 - 0)x'_{ijq} \), or \( x'_{1n_1}(0 - 0)x'_{ijp}(0 - 0)x'_{ijq}(0 - 1)x'_{ijq} \) respectively is an oriented triple in \( D' \).

**Case II.** Since \( x_{1n_1} \) and \( x'_{1n_1} \) have equal marks, so \( x_{1n_1}(1 - 0)x_{ijp} \) and \( x'_{1n_1}(0 - 0)x_{ijp} \) and thus there is an oriented triple \( x_{1n_1}(0 - 0)x_{ijp}(0 - 1)x_{ijq} \) in \( D \) and corresponding to this \( x'_{1n_1}(1 - 0)x'_{ijp}(0 - 0)x'_{ijq}(0 - 0)x'_{1n_1} \) is an oriented triple in \( D' \).

**Case III.** Since \( x_{1n_1} \) and \( x'_{1n_1} \) have equal marks, so \( x_{1n_1}(1 - 0)x_{ijp} \) and \( x'_{1n_1}(1 - 0)x_{ijp} \) and thus there is an oriented triple \( x_{1n_1}(1 - 0)x_{ijp}(0 - 0)x_{ijq}(1 - 0)x_{1n_1} \) in \( D \) and corresponding to this \( x'_{1n_1}(0 - 0)x'_{ijp}(0 - 1)x'_{ijq}(0 - 0)x'_{1n_1} \) is an oriented triple in \( D' \).

**Case IV.** Since \( x_{1n_1} \) and \( x'_{1n_1} \) have equal marks, so \( x_{1n_1}(1 - 1)x_{ijq} \) and \( x'_{1n_1}(0 - 0)x'_{ijq} \).

Thus it follows from (I)-(IV) that there is an M2D that can be obtained from \( D \) by any one of the transformations i(a) or i(b) or (ii) with mark sequences remaining unchanged. Hence the result follows by induction. \( \square \)
Lemma 4.3.1 leads to the following observation.

**Corollary 4.3.2.** Among all M2D’s with given mark sequences those with the fewest arcs are transitive.

A transmitter is a vertex with indegree zero. We assume without loss of generality that transitive M2D’s have no arcs of the form \( x(1 - 1)y \), as they can be transformed to \( x(0 - 0)y \) with same marks. This implies that in a transitive M2D with mark sequences \( P_i = [p_{ij}]_{n_i} \), 1 \( \leq i \leq k \), any of the vertex with mark \( p_{in_i} \) can act as transmitter.

Let \( P_i = [p_{ij}]_{n_i} \), 1 \( \leq i \leq k \), be \( k \) sequences of non-negative integers in non-decreasing order with \( p_{1n_1} \geq \sum_{t=2}^{k} n_t \) and \( 0 \leq p_{in_i} \leq 4 \left( \sum_{t=2, t \neq i}^{k} n_t \right) - 3 \) for all \( 2 \leq i \leq k \). Let \( P'_1 \) be obtained from \( P_1 \) by deleting one entry \( p_{1n_1} \), and let \( P'_2, P'_3, \ldots, P'_k \) be obtained as follows.

(A)(i). If \( p_{1n_1} \geq 3 \sum_{t=2}^{k} n_t \), then reducing \( 4 \left( \sum_{t=2}^{k} n_t \right) - p_{1n_1} \) largest entries of \( P_2, P_3, \ldots, P_k \) by one each,
or(ii). If \( p_{1n_1} < 3 \sum_{t=2}^{k} n_t \), then reducing \( 3 \left( \sum_{t=2}^{k} n_t \right) - p_{1n_1} \) largest entries of \( P_2, P_3, \ldots, P_k \) by two each, and \( p_{1n_1} - 2 \left( \sum_{t=2}^{k} n_t \right) \) remaining entries by one each.

(B). In case any one of \( p_{in_i} = 4 \left( \sum_{t=2}^{k} n_t \right) - 2 \), 2 \( \leq i \leq k \), say for instance \( p_{jn_j} = 4 \sum_{t=2}^{k} n_t - 2 \), then also \( p_{1n_1} = 4 \left( \sum_{t=2}^{k} n_t \right) - 2 \) as \( p_{1n_1} \geq p_{in_i} \). In this case we reduce \( p_{jn_j} \) by two.

The next result provides a useful recursive test whether the sequences of non-negative integers form the mark sequences of some M2D.

**Theorem 4.3.3.** \( P_i \) are the mark sequences of some M2D if and only if \( P'_i \) (arranged in non-decreasing order) as obtained in (A) or (B) are the mark...
sequences of some M2D.

**Proof.** Let $P'_i$, $1 \leq i \leq k$, be the mark sequences of some M2D $D'(X'_1, X'_2, \cdots, X'_k)$.

First assume $P'_2, P'_3, \cdots, P'_k$ be obtained from $P_2, P_3, \cdots, P_k$ as in (A)(i). Construct an M2D $D(X_1, X_2, \cdots, X_k)$ as follows. Let $X_i = X'_i \cup \{x\}$, $X_i = X'_i$, $2 \leq i \leq k$, with $X'_i \cap \{x\} = \phi$. Let $x(1-0)y$ for those vertices $y$ of $X'_1, X'_3, \cdots, X'_k$ whose marks are reduced by one in going from $P_i$ to $P'_i$, and $x(2-0)y$ for those vertices $y$ of $X'_2, X'_3, \cdots, X'_k$ whose marks are not reduced in going from $P_i$ to $P'_i$, $1 \leq i \leq k$. Then $D(X_1, X_2, \cdots, X_k)$ is M2D with mark sequences $P'_i$, $1 \leq i \leq k$.

Now, if $P'_2, P'_3, \cdots, P'_k$ are obtained from $P_2, P_3, \cdots, P_k$ as in (A)(ii), then construct an M2D $D(X_1, X_2, \cdots, X_k)$ as follows. Let $X_i = X'_i \cup \{x\}$, $X_i = X'_i$, $2 \leq i \leq k$, with $X'_i \cap \{x\} = \phi$. Let $x(1-0)y$ for those vertices $y$ of $X'_2, X'_3, \cdots, X'_k$ whose marks are reduced by one in going from $P_i$ to $P'_i$, and $x(1-1)y$ for those vertices $y$ of $X'_2, X'_3, \cdots, X'_k$ whose marks are reduced by two in going from $P_i$ to $P'_i$, $1 \leq i \leq k$. For (B), we take $x(1-1)y$ for those vertices $y$ of $X'_2, X'_3, \cdots, X'_k$ whose marks are reduced by two in going from $P_i$ to $P'_i$, $1 \leq i \leq k$. Then $D(X_1, X_2, \cdots, X_k)$ is M2D with mark sequences $P'_i$, $1 \leq i \leq k$.

Conversely, suppose $P_i$ be mark sequences of some M2D $D(X_1, X_2, \cdots, X_k)$, $1 \leq i \leq k$. Now any of the vertex $x_{i_{m_i}} \in X_i$ with mark $p_{i_{m_i}}, 1 \leq i \leq k$, can act as a transmitter. Clearly for (i) and (ii) $p_{i_{m_i}} \geq 2 \sum_{t=2}^{k} n_t$ and $p_{i_{m_i}} \leq 4 \sum_{t=1, t \neq i}^{k} n_t - 3$ for all $2 \leq i \leq k$, because if $p_{i_{m_i}} \leq 2 \sum_{t=2}^{k} n_t$, then by deleting $p_{i_{m_i}}$ we have to reduce more than $\sum_{t=2}^{k} n_t$ entries from $P_2, P_3, \cdots, P_k$, which is absurd.

(i) If $p_{i_{m_i}} \geq 3 \sum_{t=2}^{k} n_t$, let $X$ be the set of $4 \left( \sum_{t=2}^{k} n_t \right) - p_{i_{m_i}}$ vertices of largest marks in $X_2, X_3, \cdots, X_k$ and let $Y = \cup_{t=2}^{k} X_t - X$. In case $X$ does not contain all $4 \left( \sum_{t=2}^{k} n_t \right) - p_{i_{m_i}}$ vertices of largest marks, we can bring them to $X$ by using Lemma 4.3.1. Construct $D(X_1, X_2, \cdots, X_k)$ such that $x_{i_{m_i}}(1-0)x$ for all $x$ in $X$ and $x_{i_{m_i}}(2-0)y$ for all $y$ in $Y$. Clearly, $D(X_1, X_2, \cdots, X_k) - \{x_{i_{m_i}}\}$ realizes $P'_1, P'_2, \cdots, P'_k$.

(ii) If $p_{i_{m_i}} < 3 \sum_{t=2}^{k} n_t$, let $X$ be the set of $3 \left( \sum_{t=2}^{k} n_t \right) - p_{i_{m_i}}$ vertices of largest marks in $X_2, X_3, \cdots, X_k$ and let $Y = \cup_{t=2}^{k} X_t - X$. Construct $D(X_1, X_2, \cdots, X_k)$ such that $x_{i_{m_i}}(1-1)x$ for all $x$ in $X$ and $x_{i_{m_i}}(1-0)y$ for all $y$ in $Y$. Then again $D(X_1, X_2, \cdots, X_k) - \{x_{i_{m_i}}\}$ realizes $P'_1, P'_2, \cdots, P'_k$. 87
(B) If for instance $p_{jn_1} = 4 \left( \sum_{t=2}^{k} n_t \right) - 2$, then necessarily $p_{in_1} = 4 \left( \sum_{t=2}^{k} n_t \right) - 2$ so that $x_{1n_1}(0 - 0)x_{jn_2}$ or $x_{1n_1}(1 - 1)x_{jn_2}$. Clearly, $D(X_1, X_2, \ldots, X_k) = \{x_{1n_1}\}$ realizes $P_1', P_2', \ldots, P_k'$. □

Theorem 4.3.3 provides an algorithm for determining whether or not the $k$ sequences $P_i$, $1 \leq i \leq k$, of non-negative integers in non-decreasing order are mark sequences, and for constructing a corresponding M2D. Let $P_i = [p_{i1}, p_{i2}, \ldots, p_{im_i}]$, $1 \leq i \leq k$, with (a) $p_{in_1} \geq 2 \sum_{t=2}^{k} n_t$, (b) $p_{in_1} \leq 4 \left( \sum_{t=1, t \neq i}^{k} n_t \right) - 2$ for all $2 \leq i \leq k$, be mark sequences of an M2D with parts $X_i = \{x_{i1}, x_{i2}, \ldots, x_{im_i}\}$, $1 \leq i \leq k$. Deleting $p_{in_1}$ and performing A(i) or A(ii), or B of Theorem 4.3.3 according as $p_{in_1} \geq 3 \sum_{t=2}^{k} n_t$ or $p_{in_1} < 3 \sum_{t=2}^{k} n_t$, or any one of $p_{im_i} = 4 \left( \sum_{t=2}^{k} n_t \right) - 2$, $2 \leq i \leq k$, we obtain $P_2', P_3', \ldots, P_k'$. If the marks of the vertices $x_{ij}$ were decreased by one in this process, then the construction yielded $x_{1n_1}(1 - 0)x_{ij}$, and if these were decreased by two, then the construction yielded $x_{1n_1}(1 - 1)x_{ij}$. For vertices $x_{st}$ whose marks remained unchanged, the construction yielded $x_{1n_1}(2 - 0)x_{st}$. Note that if any of the conditions A or B does not hold, then we delete $p_{in_1}$ for that $i$ for which the conditions get satisfied, and the same argument is used for defining arcs. If this procedure is applied recursively, then it tests whether or not $P_i$ are mark sequences, and if $P_i$ are mark sequences, then an M2D with mark sequences $P_i$, $1 \leq i \leq k$ is constructed. During the application of Theorem 4.3.3, the algorithm may reach a stage where we get just two sequences, and it is not possible to apply Theorem 4.3.3, in those cases we apply Lemma 4.2.3 to Lemma 4.2.9 by choosing $r = 2$.

We illustrate this reduction and the resulting construction with the following examples.

1. $[15, 16], [15, 18], [14, 18], [16, 17], [15, 16] x_{13}(0 - 0)x_{22}, x_{13}(0 - 0)x_{32}, x_{13}(0 - 0)x_{42}, x_{13}(1 - 0)x_{21}, x_{13}(1 - 0)x_{31}, x_{13}(1 - 0)x_{41}, x_{13}(1 - 0)x_{51}, x_{13}(1 - 0)x_{52}
2. $[15, 13, 16], [12, 16], [14, 15], [13, 14] x_{12}(0 - 0)x_{21}, x_{12}(0 - 0)x_{22}, x_{12}(0 -
Example 4.3.5. Consider the three sequences of non-negative integers as follows: $P_1 = [12, 18]$, $P_2 = [1, 2, 3]$, $P_3 = [10, 18]$.

1. $[12, 1, 2, 3, 10, 16]$

   
   $x_{12}(2 - 0)x_{21}, x_{12}(2 - 0)x_{22}, x_{12}(2 - 0)x_{23}, x_{12}(2 - 0)x_{31}, x_{12}(0 - 0)x_{32}$

2. $[12, 1, 2, 3, 10]$

   
   $x_{32}(2 - 0)x_{11}, x_{32}(2 - 0)x_{21}, x_{32}(2 - 0)x_{22}, x_{32}(2 - 0)x_{23}$

3. $\phi, [1, 2, 1], [8]$
3. \( \phi, [0, 0, 0], \phi \)
\[ x_{31}(1 - 0)x_{21}, x_{31}(0 - 0)x_{22}, x_{31}(1 - 0)x_{23} \]

The resulting 3-partite 2-digraph has mark sequences \( P_1 = [12, 18], P_2 = [1, 2, 3], P_3 = [10, 18] \) and vertex sets \( X_1 = \{x_{11}, x_{12}\}, X_2 = \{x_{21}, x_{22}, x_{23}\}, X_3 = \{x_{31}, x_{32}\} \) and arcs \( x_{12}(2 - 0)x_{21}, x_{12}(2 - 0)x_{22}, x_{12}(2 - 0)x_{23}, x_{12}(2 - 0)x_{31}, x_{12}(0 - 0)x_{32}, x_{32}(2 - 0)x_{11}, x_{32}(2 - 0)x_{21}, x_{32}(2 - 0)x_{22}, x_{32}(2 - 0)x_{23}, \]
\[ x_{11}(2 - 0)x_{21}, x_{11}(2 - 0)x_{22}, x_{11}(0 - 0)x_{23}, x_{11}(0 - 0)x_{31}, x_{31}(1 - 0)x_{21}, \]
\[ x_{31}(0 - 0)x_{22}, x_{31}(1 - 0)x_{23}. \]

The next result gives a combinatorial criterion for determining whether \( k \) sequences of non-negative integers in non-decreasing order are realizable as marks.

**Theorem 4.3.6.** Let \( P_i = [p_{ij}]_{i=1}^{|n_i|}, 1 \leq i \leq k, \) be \( k \) sequences of non-negative integers in non-decreasing order. Then, \( P_i \) are the mark sequences of some M2D if and only if

\[
\sum_{i=1}^{k} \sum_{j=1}^{s_i} p_{ij} \geq 4 \sum_{i=1}^{k-1} \sum_{j=i+1}^{s_i} s_i s_j, \tag{0.2}
\]

for all sequences of \( k \) integers \( s_i, 1 \leq s_i \leq n_i \), with equality when \( s_i = n_i \) for all \( i \).

**Proof.** A sub \( k \)-partite 2-digraph induced by \( s_i \) vertices for \( 1 \leq i \leq k, 1 \leq s_i \leq n_i \), has a sum of marks \( 4 \sum_{i=1}^{k-1} \sum_{j=i+1}^{s_i} s_i s_j \). This proves the necessity.

For sufficiency, let \( P_i = [p_{ij}]_{i=1}^{|n_i|}, 1 \leq i \leq k, \) be the sequences of non-negative integers in non-decreasing order satisfying conditions (4.2) but are not the mark sequences of any M2D. Let these sequences be chosen in such a way that \( n_i, 1 \leq i \leq k, \) be smallest possible and \( p_{11} \) is the least with that choice of \( n_i \). We consider the following two cases.

**Case (i).** Assume equality in (4.2) holds for some \( s_j \leq n_j, 1 \leq j \leq k - 1, \)
\( s_k < n_k \), so that

\[
\sum_{i=1}^{k} \sum_{j=1}^{s_i} p_{ij} = 4 \sum_{i=1}^{k-1} \sum_{j=i+1}^{s_i} s_i s_j.
\]

By the minimality of \( n_i, 1 \leq i \leq k, \) the sequences \( P_i = [P_{i1}, P_{i2}, \ldots, P_{i s_i}] \)
are mark sequences of some M2D $D'(X'_1, X'_2, \cdots, X'_k)$.

For $1 \leq i \leq k$, define

$$P''_i = \left[ \left( p_{i(s_i + 1)} - 4 \sum_{t=1, t \neq i}^{k} s_t \right), \left( p_{i(s_i + 2)} - 4 \sum_{t=1, t \neq i}^{k} s_t \right), \cdots, \left( p_{i(n_i)} - 4 \sum_{t=1, t \neq i}^{k} s_t \right) \right].$$
Now consider the sum

\[
\sum_{i=1}^{k} \sum_{j=1}^{k} \left[ p_{i(s_i+j)} - 4 \sum_{t=1, t \neq i}^{k} s_t \right] = \sum_{i=1}^{k} \sum_{j=1}^{k} p_{i(s_i+j)} - 4 \sum_{i=1}^{k} \sum_{j=1, j \neq i}^{k} s_t
\]

\[
= \sum_{i=1}^{k} \sum_{j=1}^{k} p_{i(s_i+j)} - \sum_{i=1}^{k} s_i \sum_{j=1}^{k} p_{i(s_i+j)} - 4 \sum_{i=1}^{k} \sum_{j=1}^{k} s_t + 4 \sum_{i=1}^{k} \sum_{j=1}^{k} s_i
\]

\[
\geq 4 \sum_{i=1}^{k} \sum_{j=1}^{k} (s_i f_j + s_i f_j + f_i s_j + f_i f_j) - 2k \sum_{i=1}^{k} s_i f_i - 4 \sum_{i=1}^{k} f_i s_t + 4 \sum_{i=1}^{k} f_i s_i
\]

\[
= 4 \sum_{i=1}^{k} \sum_{j=1}^{k} s_i f_j + 4 \sum_{i=1}^{k} f_i s_i
\]

\[
= 4 \sum_{i=1}^{k} \sum_{j=1}^{k} s_i f_j + 4 \sum_{i=1}^{k} s_i f_j - 4 \sum_{i=1}^{k} f_i s_t + 4 \sum_{i=1}^{k} f_i s_i
\]

\[
\geq 4 \sum_{i=1}^{k} \sum_{j=1}^{k} (s_i f_i + f_i s_i + f_i s_i + f_i s_i) + \cdots + (s_i f_{i+2} + f_i s_{i+2}) + \cdots + (s_i f_k + f_i s_k)
\]

\[
- 4 \sum_{i=1}^{k} (f_i s_1 + f_i s_2 + \cdots + f_i s_k) + 4(f_1 s_1 + f_2 s_2 + \cdots + f_k s_k)
\]

\[
= 4 \sum_{i=1}^{k} \sum_{j=1}^{k} f_i f_j
\]

\[
+ 4 \sum_{i=1}^{k} [(s_i f_{i+1}) + f_i s_{i+1}) + (s_i f_{i+2}) + f_i s_{i+2}) + \cdots + (s_i f_k + f_i s_k)]
\]

\[
- 4 \sum_{i=1}^{k} (f_i s_1 + f_i s_2 + \cdots + f_i s_k) + 4(f_1 s_1 + f_2 s_2 + \cdots + f_k s_k)
\]

\[
= 4 \sum_{i=1}^{k} \sum_{j=1}^{k} f_i f_j
\]

\[
+ 4[(s_1 f_2 + f_1 s_2) + (s_1 f_3 + f_1 s_3) + \cdots + (s_1 f_k + f_1 s_k)]
\]

\[
+ [(s_2 f_3 + f_2 s_3) + (s_2 f_4 + f_2 s_4) + \cdots + (s_2 f_k + f_2 s_k)]
\]

\[
+ \cdots + [(s_{k-1} f_k + f_{k-1} s_k)]
\]
- 4[(f_1s_1 + f_1s_2 + \cdots + f_1s_k) + (f_2s_1 + f_2s_2 + \cdots + f_2s_k) \\
+ \cdots + (f_ks_1 + f_ks_2 + \cdots + f_ks_k)] \\
+ 4(f_1s_1 + f_2s_2 + \cdots + f_ks_k) \\
= 4 \sum_{i=1}^{k-1} \sum_{j=i+1}^k f_if_j,

for 1 \leq f_i \leq n_i - s_i with equality when f_i = n_i - s_i for all i, 1 \leq i \leq k. Then by minimality of n_i, 1 \leq i \leq k, the sequences P_i'' form the mark sequences of some M2D D''(X'_1, X'_2, \cdots, X''_k).

Now construct a new M2D D(X_1, X_2, \cdots, X_k) as follows. Let

\[ X_1 = X'_1 \cup X''_1, X_2 = X'_2 \cup X''_2, \cdots, X_k = X'_k \cup X''_k \]

with \( X'_i \cap X''_i = \emptyset \).

Let

\[ x_i'(2-0)x_i', x''_i(2-0)x_j', x''_i(2-0)x_i', x''_i(2-0)x_i', \cdots, x''_i(2-0)x_k', \]

for all \( x_i' \) in \( X'_i \) and for all \( x_i'' \) in \( X''_i \), 1 \leq i \leq k. Then clearly D(X_1, X_2, \cdots, X_k) is an M2D with mark sequences \( P_i, 1 \leq i \leq k \), which is a contradiction.

**Case (ii).** Assume strict inequality in (4.2) holds for some \( s_i \neq n_i, 1 \leq i \leq k \). Let \( P'_i = [p_{i1} - 1, p_{i2}, \cdots, p_{i1n_i - 1}, p_{i1n_i} + 1] \) and \( P'_j = [p_{j1}, p_{j2}, \cdots, p_{jn_j}] \) for all \( j, 2 \leq j \leq k \). Clearly the sequences \( P'_i, 1 \leq i \leq k \), satisfy conditions (4.2). Therefore by the minimality of \( p_{i1} \), the sequences \( P'_i, 1 \leq i \leq k \), are mark sequences of some M2D \( D'(X'_1, X'_2, \cdots, X'_k) \). Let \( p_{xi1} = p_{i1} - 1 \) and \( p_{x1n_i} = p_{1n_i} + 1 \). Since \( p_{x1n_i} > p_{xi1} + 1 \), there exists a vertex \( x_{ij} \) in \( X_i, 2 \leq i \leq k, 1 \leq j \leq n_i, \) such that \( x_{1n_i} \neq 0 \) \( x_{ij}(1-0)x_{11}, \) or \( x_{1n_i}(0-0)x_{ij}(1-0)x_{11}, \) or \( x_{1n_i}(1-0)x_{ij}(0-0)x_{11}, \) or \( x_{1n_i}(0-0)x_{ij}(0-0)x_{11} \) in \( D'(X'_1, X'_2, \cdots, X'_k) \), and if these are changed to \( x_{1n_i}(0-0)x_{ij}(0-0)x_{11}, \) or \( x_{1n_i}(0-0)x_{ij}(0-0)x_{11}, \) or \( x_{1n_i}(0-0)x_{ij}(0-1)x_{11}, \) or \( x_{1n_i}(0-0)x_{ij}(0-0)x_{11} \) respectively, the result is an M2D with mark sequences \( P_i, 1 \leq i \leq k \), which is again a contradiction. Hence the result follows. \( \square \)

**Definition 4.3.7.** A k-partite r-digraph (or briefly multipartite multidi-graph(MMD)) is an orientation of a k-partite multigraph that is without loops.
and contains at most $r$ edges between any pair of vertices from distinct parts. So, a $k$-partite 1-digraph is an oriented $k$-partite graph, and a complete $k$-partite 1-digraph is a $k$-partite tournament. Let $D = D(X_1, X_2, \ldots, X_k)$ be a multipartite multidigraph with parts $X_i = \{x_{i1}, x_{i2}, \ldots, x_{in_i}\}$, $1 \leq i \leq k$. Let $d^+_{x_{ij}}$ and $d^-_{x_{ij}}$, $1 \leq j \leq n_i$, be respectively the outdegree and indegree of a vertex $x_{ij} \in X_i$. Define $p_{x_{ij}}$ (or simply $p_{ij}$) as the mark (or $r$-score) of $x_{ij}$. Clearly, $0 \leq p_{x_{ij}} \leq 2r \sum_{t=1}^{k} n_t$. Then the $k$ sequences $p_i = [p_{ij}]_{n_i}^{n_i}$, $1 \leq i \leq k$, in non-decreasing order are called the mark sequences of $D$.

An MMD can be interpreted as a result of a competition among $k$ teams in which each player of one team plays with every player of the other $k - 1$ teams at most $r$ times in which ties (draws) are allowed. A player receives two points for each win, and one point for each tie. With this marking system, player $x_{ij}$ receives a total of $p_{x_{ij}}$ points. The $k$ sequences of non-negative integers $p_i$, $1 \leq i \leq k$, in non-decreasing order are said to be realizable if there exists an MMD with mark sequences $P_i$. All the results on multipartite 2-digraphs can be extended to MMD. The following is the combinatorial characterization for mark sequences in MMD. We prove it here in a different way.

**Theorem 4.3.8.** Let $P_i = [p_{ij}]_{n_i}^{n_i}$, $1 \leq i \leq k$, be $k$ sequences of non-negative integers in non-decreasing order. Then, $P_i$ are the mark sequences of some MMD if and only if

$$\sum_{i=1}^{k} s_i \sum_{j=1}^{n_i} p_{ij} \geq 2r \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} s_i s_j, \quad (0.3)$$

for all sequences of $k$ integers $s_i$, $1 \leq s_i \leq n_i$, with equality when $s_i = n_i$ for all $i$.

**Proof.** A sub $k$-partite $r$-digraph induced by $s_i$ vertices for $1 \leq i \leq k$, $1 \leq s_i \leq n_i$, has a sum of marks $2r \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} s_i s_j$. This proves the necessity.

For sufficiency, let $P_i = [p_{ij}]_{n_i}^{n_i}$, $1 \leq i \leq k$, be the sequences of non-negative integers in non-decreasing order satisfying conditions (4.3) but are
not the mark sequences of any MMD. Let these sequences be chosen in such a way that \( n_i, 1 \leq i \leq k \), be smallest possible and \( p_{11} \) is the least with that choice of \( n_i \). We consider the following two cases.

**Case (i).** Assume equality in (4.3) holds for some \( s_j \leq n_j, 1 \leq j \leq k - 1, s_k < n_k \), so that

\[
\sum_{i=1}^{k} \sum_{j=1}^{k} p_{ij} = 2r \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} s_is_j.
\]

By the minimality of \( n_i, 1 \leq i \leq k \), the sequences \( P_i = [P_{i1}, P_{i2}, \cdots, P_{is_i}] \) are mark sequences of some MMD \( D'(X_1', X_2', \cdots, X_k') \).

Define

\[
P''_i = \left[ p_{i(s_i+1)} - 2r \sum_{t=1, t \neq i}^{k} s_t, p_{i(s_i+2)} - 2r \sum_{t=1, t \neq i}^{k} s_t, \cdots, p_{i(n_i)} - 2r \sum_{t=1, t \neq i}^{k} s_t \right],
\]

\( 1 \leq i \leq k \).

Now consider the sum

\[
\sum_{i=1}^{k} \sum_{j=1}^{f_i} (p_{i(s_i+j)}) - 2r \sum_{t=1, t \neq i}^{k} s_t
\]

\[
= \sum_{i=1}^{k} \sum_{j=1}^{f_i} p_{i(s_i+j)} - 2r \sum_{t=1, t \neq i}^{k} s_t
\]

\[
= \sum_{i=1}^{k} \sum_{j=1}^{f_i+s_i} p_{ij} - \sum_{i=1}^{k} \sum_{j=1}^{s_i} p_{ij} - 2r \sum_{i=1}^{k} \sum_{j=1}^{k} s_t + 2r \sum_{i=1}^{k} \sum_{j=1}^{f_i} s_i
\]

\[
\geq 2r \sum_{i=1}^{k} \sum_{j=i+1}^{k} [(s_i + f_i)(s_j + f_j)] - 2r \sum_{i=1}^{k} \sum_{j=i+1}^{k} s_is_j
\]

\[
- 2r \sum_{i=1}^{k} f_i \sum_{t=1}^{k} s_t + 2r \sum_{i=1}^{k} f_is_i
\]

95
\[
\sum_{i=1}^{k} \sum_{j=1}^{f_i} (p_{i(s_i+j)}) = 2r \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} (s_is_j + s_if_j + f_is_j + f_if_j) - 2r \sum_{i=1}^{k} s_is_j
\]

\[
- 2r \sum_{i=1}^{k-1} \sum_{j=1}^{k} f_is_t + 2r \sum_{i=1}^{k} f_is_i
\]

\[
= 2r \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} s_is_j + 2r \sum_{i=1}^{k} \sum_{j=i+1}^{k} (s_if_j + f_is_j) + 2r \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} f_if_j
\]

\[
- 2r \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} s_is_j - 2r \sum_{i=1}^{k} f_is_t + 2r \sum_{i=1}^{k} f_is_i
\]

\[
= 2r \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} f_if_j
\]

\[
+ 2r \sum_{i=1}^{k} [(s_if_{i+1}) + f_is_{i+1}) + (s_if_{i+2}) + f_is_{i+2}) + \cdots + (s_if_k) + f_is_k]
\]

\[
- 2r \sum_{i=1}^{k} (f_is_1 + f_is_2 + \cdots + f_is_k) + 2r(f_is_1 + f_2s_2 + \cdots + f ks_k)
\]

\[
= 2r \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} f_if_j
\]

\[
+ 2r \left\{ [(s_if_2 + f_1s_2) + (s_if_3 + f_1s_3) + \cdots + (s_if_k + f_1s_k)]
\right.
\]

\[
+ [(s_if_3 + f_2s_3) + (s_if_4 + f_2s_4) + \cdots + (s_if_k + f_2s_k)]
\]

\[
+ \cdots + [(s_if_k + f_1s_k)]
\]

\[
- 2r \left\{ [(f_is_1 + f_is_2 + \cdots + f_is_k) + (f_2s_1 + f_2s_2 + \cdots + f_2s_k)
\right.
\]

\[
+ \cdots + (f_ks_1 + f_ks_2 + \cdots + f_ks_k)]
\]

\[
+ 2r(f_is_1 + f_2s_2 + \cdots + f_ks_k)
\]

\[
= 2r \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} f_if_j,
\]

for \(1 \leq f_i \leq n_i - s_i\) with equality when \(f_i = n_i - s_i\) for all \(i, 1 \leq i \leq k\).

Then by minimality of \(n_i, 1 \leq i \leq k\), the sequences \(P''_i\) form the mark sequences of some MMD \(D''(X''_1, X''_2, \cdots, X''_k)\).
Now construct a new MMD $D(X_1, X_2, \cdots, X_k)$ as follows. Let

$$X_1 = X'_1 \cup X''_1, X_2 = X'_2 \cup X''_2, \cdots, X_k = X'_k \cup X''_k$$

with $X'_i \cap X''_i = \phi$.

Let

$$x''_i(r-0)x'_1, x''_i(r-0)x'_2, \cdots, x''_i(r-0)x'_{i-1}, x''_i(r-0)x'_{i+1}, \cdots, x''_i(r-0)x'_k,$$

for all $x''_i$ in $X''_i$ and for all $x'_i$ in $X'_i$, $1 \leq i \leq k$. Then clearly $D(X_1, X_2, \cdots, X_k)$ is an MMD with mark sequences $P_i$, $1 \leq i \leq k$, which is a contradiction.

**Case (ii).** Assume strict inequality in (4.3) holds for some $s_i \neq n_i$, $1 \leq i \leq k$.

Let

$$P'_1 = [p_{11} - 1, p_{12}, \cdots, p_{1n_1 - 1}, p_{1n_1} + 1]$$

and

$$P'_j = [p_{j1}, p_{j2}, \cdots, p_{jn_j}]$$

for all $j$, $2 \leq j \leq k$. Clearly the sequences $P'_i$, $1 \leq i \leq k$, satisfy conditions (4.3). Therefore by the minimality of $p_{11}$, the sequences $P'_i$, $1 \leq i \leq k$, are mark sequences of some MMD $D'(X'_1, X'_2, \cdots, X'_k)$. Let

$$p_{x_{11}} = p_{11} - 1$$

and

$$p_{x_{1n_1}} = p_{1n_1} + 1.$$

Since

$$p_{x_{1n_1}} > p_{x_{11}} + 1,$$

there exists a vertex $x_{ij}$ in $X_i$, $2 \leq i \leq k$, $1 \leq j \leq n_i$, such that $x_{1n_1}(1 - 0)x_{ij}(1 - 0)x_{11}$, or $x_{1n_1}(0 - 0)x_{ij}(1 - 0)x_{11}$, or $x_{1n_1}(1 - 0)x_{ij}(0 - 0)x_{11}$, or $x_{1n_1}(0 - 0)x_{ij}(0 - 0)x_{11}$ in $D'(X'_1, X'_2, \cdots, X'_k)$, and if these are changed to $x_{1n_1}(0 - 0)x_{ij}(0 - 0)x_{11}$, or $x_{1n_1}(0 - 1)x_{ij}(0 - 0)x_{11}$, or $x_{1n_1}(0 - 0)x_{ij}(0 - 1)x_{11}$, or $x_{1n_1}(0 - 1)x_{ij}(0 - 1)x_{11}$ respectively, the result is an MMD with mark sequences $P_i$, $1 \leq i \leq k$, which is again a contradiction. Hence the result follows. □