CHAPTER-IV

DOUBLY TRUNCATED THREE PARAMETER GENERALIZED GAUSSIAN DISTRIBUTION

DOUBLY TRUNCATED THREE PARAMETER GENERALIZED GAUSSIAN DISTRIBUTION

4.1 INTRODUCTION

In the earlier chapters 2 and 3, we developed and analyzed left truncated generalized Gaussian distribution and right truncated generalized Gaussian distribution, assuming the variate under study is truncated either left or right of the range. But in many practical situations arising at places like industrial experiments, agricultural experiments, financial modeling, warranty studies the variate under consideration may have a finite range. For example, in inventory modeling for deteriorating item, life time of commodity is random and may have a finite range. The lower bound may be zero because there is no negative life time and the upper bound is constrained with a finite value since, the product is perishable. Approximating the finite range with the infinite range will provide the results inaccurate. So, to have an accurate analysis of the data set it is needed to consider a doubly truncated distribution. Hence, in this chapter, we develop and analyze a doubly truncated generalized Gaussian distribution with the assumption that the range of the variate under study is having both upper and lower bounds. The various distributional properties such as the probability density function, the distribution function, the four moments, the skewness, the kurtosis, the hazard function and survival function are derived. The order statistics of the distribution are also studied. Some inferential properties related to the parameters of the distribution are discussed. A numerical illustration is also presented.

4.2. DOUBLY TRUNCATED THREE PARAMETER GENERALIZED GAUSSIAN DISTRIBUTION

A Continuous random variable X is said to have a three parameter generalized Gaussian distribution if its probability density function (p.d.f) is of the form

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$$g(x;\mu,\alpha,\beta)=\frac{\beta}{2\alpha\Gamma\left(\frac{1}{\beta}\right)}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}} \quad ; \quad -\infty<x<\infty; \quad -\infty<\mu<\infty; \quad \alpha>0; \quad \beta>0$$

Consider that the range variable is finite say (A, B). Then the probability density function (p.d.f) of the doubly truncated three parameter generalized Gaussian distribution is

$$f(x)=\frac{g(x;\mu,\alpha,\beta)}{\Phi(B)-\Phi(A)}; \qquad A<x<B; \qquad A<\mu<B; \qquad \alpha>0; \qquad \beta>0 \tag{4.2.1}$$

where $$\Phi(A)=\int_{-\infty}^{A}\frac{\beta}{2\alpha\Gamma\left(\frac{1}{\beta}\right)}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}dx$$

$$\Phi(B)=\int_{-\infty}^{B}\frac{\beta}{2\alpha\Gamma\left(\frac{1}{\beta}\right)}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}dx$$

The lower and upper truncation points are A and B respectively. The degree of truncation are $\Phi\left(\frac{A-\mu}{\beta}\right)$ and $1-\Phi\left(\frac{B-\mu}{\beta}\right)$. If A is replaced by $-\infty$ or B is replaced by ∞, the distribution is singly truncated from below or above respectively.

Hence, the probability density function of doubly truncated three parameter generalized Gaussian distribution is

$$f(x)=\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \qquad for \quad A<\mu<B \tag{4.2.2}$$

$$f(x)=\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \qquad for \quad \mu<A<B \tag{4.2.3}$$

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$$f(x)=\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)} \quad \text{for } A<B<\mu \qquad (4.2.4)$$

4.3. DISTRIBUTIONAL PROPERTIES

The various distributional properties of the doubly truncated three parameter generalized Gaussian distribution are discussed in this section.

Different shapes of the frequency curves for given values of the parameter are shown in figure 4.1

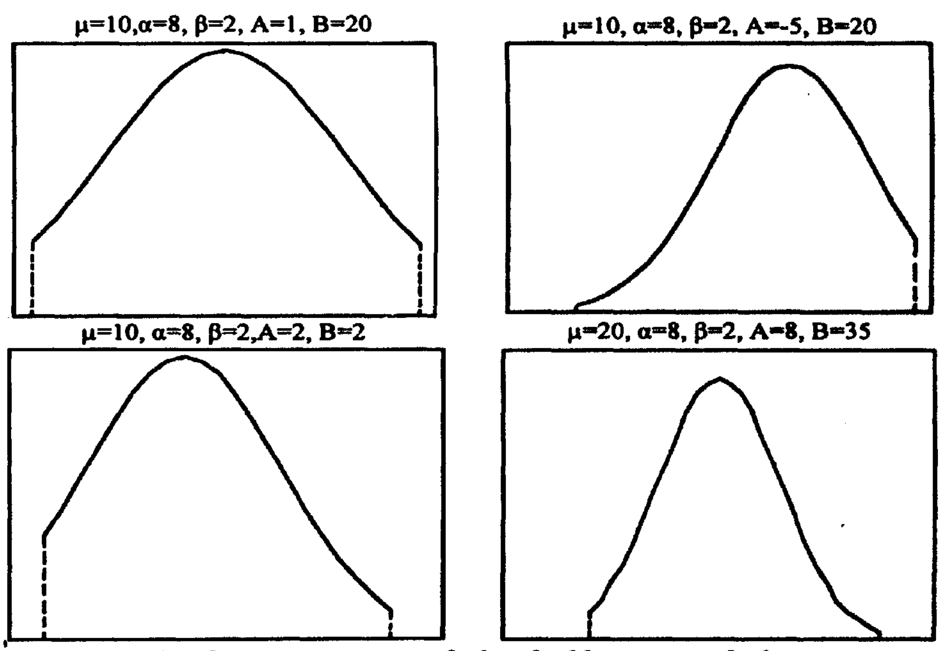

Figure 4.1: The frequency curves of the doubly truncated three parameter generalized Gaussian distribution.

The distribution function of X is given by

$$F(x)=\int_A^x f(t)dt$$

$$F(x)=\int_A^x \frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}\,dx$$

On simplification, we get

$$F(x)=\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^\beta\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)} \qquad for \quad A<\mu<B \qquad (4.3.1)$$

Similarly, we get

$$F(x)=\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^\beta\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^\beta\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^\beta\right)} \qquad for \quad \mu<A<B \qquad (4.3.2)$$

$$F(x)=\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^\beta\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)} \qquad for \quad A<B<\mu \qquad (4.3.3)$$

where $\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^\beta\right),\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right),\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)$ are incomplete gamma functions.

The mean of the distribution is

$$E(X)=\int_A^B x\ f(x)\,dx$$

$$E(X)=\int_A^B x\frac{\beta}{\alpha}\frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\,dx$$

On simplification, we get

$$E(X)=\mu+\alpha\left(\frac{\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right) \quad \text{for } A<\mu<B \qquad (4.3.4)$$

Similarly, we get

$$E(X)=\mu+\alpha\left(\frac{\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{2}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right) \quad \text{for } \mu<A<B \qquad (4.3.5)$$

$$E(X)=\mu+\alpha\left(\frac{\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right) \quad \text{for } A<B<\mu \qquad (4.3.6)$$

Let M be the median of the distribution, then we have

$$\int_A^M f(x)dx=\frac{1}{2}$$

$$\int_A^M \frac{\beta}{\alpha}\frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\,dx=\frac{1}{2} \qquad (4.3.7)$$

On simplification, we get

$$\frac{\gamma\left(\frac{1}{\beta},\left(\frac{M-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\left(\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)\right)}=\frac{1}{2} \qquad for\ A<\mu<B \qquad (4.3.8)$$

$$\frac{\gamma\left(\frac{1}{\beta},\left(\frac{M-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\left(\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)\right)}=\frac{1}{2} \qquad for\ \mu<A<B \qquad (4.3.9)$$

$$\frac{\gamma\left(\frac{1}{\beta},\left(\frac{M-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\left(\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}=\frac{1}{2} \qquad for\ A<B<\mu \qquad (4.3.10)$$

The median M of the distribution can be obtained by solving the equations (4.3.8), (4.3.9) and (4.3.10).

For obtaining the mode of the distribution consider the probability density function of the distribution.

$$f(x)=K_1\, e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}} \qquad for \quad A\le\mu\le B$$

Where $K_1=\dfrac{\frac{\beta}{\alpha}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}$

Taking logarithms on both sides, we get

$$\log f(x)=\log(K_1)-\left|\frac{x-\mu}{\alpha}\right|^{\beta}$$

Differentiating both sides w.r.to x, we get

$$\frac{d}{dx}\log f(x)=\frac{-\frac{\beta}{\alpha}\left|\frac{x-\mu}{\alpha}\right|^{\beta-1}\left|\frac{x-\mu}{\alpha}\right|}{\left(\frac{x-\mu}{\alpha}\right)}$$

$$\frac{d}{dx}\log f(x)=0$$

$$\Rightarrow \frac{-\frac{\beta}{\alpha}\left|\frac{x-\mu}{\alpha}\right|^{\beta-1}\left|\frac{x-\mu}{\alpha}\right|}{\left(\frac{x-\mu}{\alpha}\right)}=0 \tag{4.3.11}$$

Solving equation (4.3.11), we get x = μ.

Thus, $x=\mu$ is the unique solution which indicates this distribution is uni-model.

$$\frac{d^2}{dx^2}\log f(x)=-\frac{\beta}{\alpha^2}\left(\frac{\beta\left|\frac{x-\mu}{\alpha}\right|^{\beta}-\left(\frac{x-\mu}{\alpha}\right)}{\left(\frac{x-\mu}{\alpha}\right)^2}\right)<0$$

This distribution reaches its maximum of the point $x=\mu$

The raw moments of the distribution are

$$\mu_r'=\int_A^B x^r\ f(x)\,dx$$

$$\mu_r'=\int_A^B x^r\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}dx$$

On simplification, we get

$$\mu_r'=\sum_{j=0}^{r}\binom{r}{j}\alpha^j\mu^{r-j}\left(\frac{\gamma\left(\frac{j+1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{j+1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)\quad for\ A<\mu<B \tag{4.3.12}$$

The first four non central moments are obtained by substituting r = 1, 2, 3, 4 in equation (4.3.12)

$$\mu_1' = \mu + \alpha \left(\frac{\gamma\left(\frac{2}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{2}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right)$$

$$\mu_2' = \mu^2 + 2\alpha\mu \left(\frac{\gamma\left(\frac{2}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{2}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right) + \alpha^2 \left(\frac{\gamma\left(\frac{3}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{3}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{3}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right)$$

$$\mu_3' = \mu^3 + 3\alpha\mu^2 \left(\frac{\gamma\left(\frac{2}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{2}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right) + 3\alpha^2\mu \left(\frac{\gamma\left(\frac{3}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{3}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right)$$

$$+ \alpha^3 \left(\frac{\gamma\left(\frac{4}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{4}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right)$$

$$\mu_4' = \mu^4 + 4\alpha\mu^3 \left(\frac{\gamma\left(\frac{2}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{2}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right) + 6\alpha^2\mu^2 \left(\frac{\gamma\left(\frac{3}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{3}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right)$$

$$+ 4\alpha^3\mu \left(\frac{\gamma\left(\frac{4}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{4}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right) + \alpha^4 \left(\frac{\gamma\left(\frac{5}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{5}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right)$$

Similarly for $\mu < A < B$, the r^th^ non central moment is

$$\mu_r' = \sum_{j=0}^{r} \binom{r}{j} \alpha^j \mu^{r-j} \left(\frac{\gamma\left(\frac{j+1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{j+1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right) \qquad \text{for } \mu < A < B$$

(4.3.13)

The first four non central moments are obtained by substituting r = 1, 2, 3, 4 in equation (4.3.13)

$$\mu_1' = \mu + \alpha \left(\frac{\gamma\left(\frac{2}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right)$$

$$\mu_2' = \mu^2 + 2\alpha\mu \left(\frac{\gamma\left(\frac{2}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right) + \alpha^2 \left(\frac{\gamma\left(\frac{3}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{3}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right)$$

$$\mu_3' = \mu^3 + 3\alpha\mu^2 \left(\frac{\gamma\left(\frac{2}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right) + 3\alpha^2\mu \left(\frac{\gamma\left(\frac{3}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{3}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right)$$

$$+ \alpha^3 \left(\frac{\gamma\left(\frac{4}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{4}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right)$$

$$\mu_4' = \mu^4 + 4\alpha\mu^3 \left(\frac{\gamma\left(\frac{2}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right) + 6\alpha^2\mu^2 \left(\frac{\gamma\left(\frac{3}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{3}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right)$$

$$+ 4\alpha^3\mu \left(\frac{\gamma\left(\frac{4}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{4}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right) + \alpha^4 \left(\frac{\gamma\left(\frac{5}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{5}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right)$$

Similarly for $A < B < \mu$, the rth non central moment is

$$\mu_r' = \sum_{j=0}^{r} \binom{r}{j} \alpha^j \mu^{r-j} \left(\frac{\gamma\left(\frac{j+1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right) - \gamma\left(\frac{j+1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{+\gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)} \right) \quad for \;\; A < B < \mu$$

(4.3.14)

The first four non central moments are obtained by substituting r = 1, 2, 3, 4 in equation (4.3.14)

$$\mu_1' = \mu + \alpha \left(\frac{\gamma\left(\frac{2}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)} \right)$$

$$\mu_2' = \mu^2 + 2\alpha\mu \left(\frac{\gamma\left(\frac{2}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right) + \alpha^2 \left(\frac{\gamma\left(\frac{3}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right) - \gamma\left(\frac{3}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{3}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right)$$

$$\mu_3' = \mu^3 + 3\alpha\mu^2 \left(\frac{\gamma\left(\frac{2}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)} \right) + 3\alpha^2\mu \left(\frac{\gamma\left(\frac{3}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right) - \gamma\left(\frac{3}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)} \right)$$

$$+ \alpha^3 \left(\frac{\gamma\left(\frac{4}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right) - \gamma\left(\frac{4}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)} \right)$$

$$\mu_4' = \mu^4 + 4\alpha\mu^3\left(\frac{\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right)+6\alpha^2\mu^2\left(\frac{\gamma\left(\frac{3}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)-\gamma\left(\frac{3}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right)$$

$$+4\alpha^3\mu\left(\frac{\gamma\left(\frac{4}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)-\gamma\left(\frac{4}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right)+\alpha^4\left(\frac{\gamma\left(\frac{5}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)-\gamma\left(\frac{5}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right)$$

The central moments of this distribution

$$\mu_r = \int_A^B (x-\mu-D)^r f(x)\,dx$$

$$= \int_A^B (x-\mu-D)^r \frac{\frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}\,dx$$

On simplification, we get

$$\mu_r = \sum_{j=0}^{r}\binom{r}{j}\alpha^j(-D)^{r-j}\left(\frac{\gamma\left(\frac{j+1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)+\gamma\left(\frac{j+1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}\right) \quad \text{for } A<\mu<B \tag{4.3.15}$$

where $D = \alpha\left(\frac{\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)+\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}\right)$.

The first four central moments are obtained by substituting r = 1, 2, 3, 4 in equation (4.3.15)

$\mu_1 = 0$

$$\mu_2=\alpha^2\left[\left(\frac{\gamma\left(\frac{3}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{3}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)-\left(\frac{\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)^2\right]$$

$$\mu_3=2\alpha^3\left(\frac{\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)^3-3\alpha^3\left(\frac{\gamma\left(\frac{3}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{3}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)$$

$$\left(\frac{\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)+\alpha^3\left(\frac{\gamma\left(\frac{4}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{4}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)$$

$$\mu_4=6\alpha^4\left(\frac{\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)^2\left(\frac{\gamma\left(\frac{3}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{3}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)$$

$$-3\alpha^4\left(\frac{\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)^4-4\alpha^4\left(\frac{\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)$$

$$\left(\frac{\gamma\left(\frac{4}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{4}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)+\alpha^4\left(\frac{\gamma\left(\frac{5}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{5}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)$$

Similarly for $\mu < A < B$, the r^{th} central moment is

$$\mu_r = \sum_{j=0}^{r} \binom{r}{j} \alpha^j (-D)^{r-j} \left(\frac{\gamma\left(\frac{j+1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{j+1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right) \quad for\, \mu < A < B \tag{4.3.16}$$

where $D = \alpha \left(\frac{\gamma\left(\frac{2}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right)$

The first four central moments are obtained by substituting r = 1, 2, 3, 4 in equation (4.3.16)

$\mu_1 = 0$

$$\mu_2 = \alpha^2 \left[\left(\frac{\gamma\left(\frac{3}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{3}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right) - \left(\frac{\gamma\left(\frac{2}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right)^2 \right]$$

$$\mu_3 = 2\alpha^3 \left(\frac{\gamma\left(\frac{2}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right)^3 - 3\alpha^3 \left(\frac{\gamma\left(\frac{3}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{3}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right)$$
$$\left(\frac{\gamma\left(\frac{2}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right) + \alpha^3 \left(\frac{\gamma\left(\frac{4}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{4}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right)$$

$$\mu_4 = 6\alpha^4\left(\frac{\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{2}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right)^2\left(\frac{\gamma\left(\frac{3}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{3}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right)$$

$$-3\alpha^4\left(\frac{\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{2}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right)^4-4\alpha^4\left(\frac{\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{2}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right)$$

$$\left(\frac{\gamma\left(\frac{4}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{4}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right)+\alpha^4\left(\frac{\gamma\left(\frac{5}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{5}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right)$$

Similarly for $A < B < \mu$, the r[th] central moment is

$$\mu_r = \sum_{j=0}^{r}\binom{r}{j}\alpha^j(-D)^{r-j}\left(\frac{\gamma\left(\frac{j+1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{j+1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right) \qquad \textit{for } A < B < \mu$$

(4.3.17)

where $D = \alpha\left(\frac{\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)$

The first four central moments are obtained by substituting r = 1, 2, 3, 4 in equation (4.3.17)

$\mu_1 = 0$

$$\mu_2=\alpha^2\left(\left(\frac{\gamma\left(\frac{3}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{3}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)-\left(\frac{\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)^2\right)$$

$$\mu_3=2\alpha^3\left(\frac{\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)^3-3\alpha^3\left(\frac{\gamma\left(\frac{3}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{3}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)$$
$$\left(\frac{\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)+\alpha^3\left(\frac{\gamma\left(\frac{4}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{4}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)$$

$$\mu_4=6\alpha^4\left(\frac{\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)^2\left(\frac{\gamma\left(\frac{3}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{3}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)$$
$$-3\alpha^4\left(\frac{\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)^4-4\alpha^4\left(\frac{\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)$$
$$\left(\frac{\gamma\left(\frac{4}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{4}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)+\alpha^4\left(\frac{\gamma\left(\frac{5}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{5}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)$$

The skewness of the distribution is

$$\beta_1=\frac{\left(2S_1^3-3S_1S_2+S_3\right)^2}{\left(S_2-S_1^2\right)^3}\qquad for\quad A<\mu<B$$

$$\beta_1=\frac{\left(2P_1^3-3P_1P_2+P_3\right)^2}{\left(P_2-P_1^2\right)^3}\qquad for\quad \mu<A<B$$

$$\beta_1 = \frac{\left(2Q_1^3 - 3Q_1Q_2 + Q_3\right)^2}{\left(Q_2 - Q_1^2\right)^3} \qquad \textit{for } A < B < \mu$$

Kurtosis of the distribution is

$$\beta_2 = \frac{3S_1^2\left(2S_2 - S_1^2\right) + S_4 - 4S_1S_3}{\left(S_2 - S_1^2\right)^2} \qquad \textit{for } A < \mu < B$$

$$\beta_2 = \frac{3P_1^2\left(2P_2 - P_1^2\right) + P_4 - 4P_1P_3}{\left(P_2 - P_1^2\right)^2} \qquad \textit{for } \mu < A < B$$

$$\beta_2 = \frac{3Q_1^2\left(2Q_2 - Q_1^2\right) + Q_4 - 4Q_1Q_3}{\left(Q_2 - Q_1^2\right)^2} \qquad \textit{for } A < \mu < B$$

where $S_1 = \left(\dfrac{\gamma\left(\frac{2}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) + \gamma\left(\frac{2}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^\beta\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^\beta\right)}\right)$; $S_2 = \left(\dfrac{\gamma\left(\frac{3}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) + \gamma\left(\frac{3}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^\beta\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^\beta\right)}\right)$;

$S_3 = \left(\dfrac{\gamma\left(\frac{4}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) + \gamma\left(\frac{4}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^\beta\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^\beta\right)}\right)$; $S_4 = \left(\dfrac{\gamma\left(\frac{5}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) + \gamma\left(\frac{5}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^\beta\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^\beta\right)}\right)$.

$P_1 = \left(\dfrac{\gamma\left(\frac{2}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) - \gamma\left(\frac{2}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^\beta\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^\beta\right)}\right)$; $P_2 = \left(\dfrac{\gamma\left(\frac{3}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) - \gamma\left(\frac{3}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^\beta\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^\beta\right)}\right)$

$P_3 = \left(\dfrac{\gamma\left(\frac{4}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) - \gamma\left(\frac{4}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^\beta\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^\beta\right)}\right)$; $P_4 = \left(\dfrac{\gamma\left(\frac{5}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) - \gamma\left(\frac{5}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^\beta\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^\beta\right)}\right)$.

and

$$Q_1=\left(\frac{\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right);\qquad Q_2=\left(\frac{\gamma\left(\frac{3}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{3}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right);$$

$$Q_3=\left(\frac{\gamma\left(\frac{4}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{4}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right);\qquad Q_4=\left(\frac{\gamma\left(\frac{5}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{5}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right).$$

The hazard rate function of the distribution is

$$h(x)=\frac{f(x)}{1-F(x)}$$

$$h(x)=\left(\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+2\,\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}\right)\qquad \textit{for } A<\mu<B$$

$$h(x)=\left(\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}\right)\qquad \textit{for } \mu<A<B$$

$$h(x)=\left(\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{2\,\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}\right)\qquad \textit{for } A<\mu<B$$

The survival rate function S(x) is

$$S(x)=1-F(x)$$

$$S(x)=1-\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\qquad \textit{for } A<\mu<B$$

$$S(x)=1-\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \qquad for\ \mu<A<B$$

$$S(x)=1-\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)} \qquad for\ A<B<\mu$$

4.4. ORDER STATISTICS OF DOUBLY TRUNCATED THREE PARAMETER GENERALIZED GAUSSIAN DISTRIBUTION

The simple explicit form of the distribution function as given in equation (4.3.1, 4.3.2 & 4.3.3) leads us to derive the order statistics connected with this doubly truncated three parameter generalized Gaussian distribution.

$$f(x)=\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \qquad A<\mu<B$$

$$f(x)=\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \qquad \mu<A<B$$

$$f(x)=\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)} \qquad A<B<\mu \qquad (4.4.1)$$

Let $X_{1n}\le X_{2n}\le\ldots\le X_{nn}$ denote the order statistics obtained from a random sample of size n from the generalized truncated Gaussian distribution having the probability density function of the form given in (4.4.1). The probability density function of s[th] order statistics is given by [David (1981)],

$$f_{s:n}(x)=D_{s:n}\left[F(x)\right]^{s-1}\left[1-F(x)\right]^{n-s}f(x)$$

where $D_{s:n}=\dfrac{n!}{(s-1)!(n-s)!}$ (4.4.2)

Substituting f(x) and F(x) values given in this equation (4.4.1) and (4.3.1) in the equation (4.4.2), we get the probability density function of the s[th] order statistics is as

Case (i): For $A<\mu<B$

For $A<x<0$

$$f_{s:n}(x)=D_{s:n}\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\sum_{q=0}^{s-1}\binom{s-1}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right]^{n+q-s}$$

For $0<x<B$

$$f_{s:n}(x)=D_{s:n}\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\sum_{q=0}^{n-s}\binom{n-s}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right]^{s+q-1}$$

(4.4.3)

Substituting f(x) and F(x) values given in this equation (4.4.1) and (4.3.2) in the equation (4.4.2), we get the probability density function of the s[th] order statistics is as

Case (ii): For $\mu<A<B$

For $A<x<0$

$$f_{s:n}(x)=D_{s:n}\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\sum_{q=0}^{s-1}\binom{s-1}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right]^{n+q-s}$$

For $0<x<B$

$$f_{s:n}(x)=D_{s:n}\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\sum_{q=0}^{n-s}\binom{n-s}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right]^{s+q-1}$$

(4.4.4)

Substituting f(x) and F(x) values given in this equation (4.4.1) and (4.3.3) in the equation (4.4.2), we get the probability density function of the s^th^ order statistics is as

Case (iii): For $A<B<\mu$

For $A<x<0$

$$f_{s:n}(x)=D_{s:n}\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\sum_{q=0}^{s-1}\binom{s-1}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{n+q-s}$$

For $0<x<B$

$$f_{s:n}(x)=D_{s:n}\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\sum_{q=0}^{n-s}\binom{n-s}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{s+q-1}$$

(4.4.5)

The probability density function of the first order statistics is obtained by substituting $s=1$ in the equation (4.4.3)

Hence, Case (i): For $A<\mu<B$

For $A<x<0$

$$f_{1:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right]^{n-1}$$

For $0<x<B$

$$f_{1:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right]^{q}$$

(4.4.6)

The probability density function of the first order statistics is obtained by substituting $s=1$ in the equation (4.4.4)

Hence, Case (ii): For $\mu<A<B$

For $A<x<0$

$$f_{1:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right]^{n-1}$$

For $0<x<B$

$$f_{1:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right]^{q}$$

(4.4.7)

The probability density function of the first order statistics is obtained by substituting $s=1$ in the equation (4.4.5)

Hence, Case (iii): For $A<B<\mu$

For $A<x<0$

$$f_{1:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{n-1}$$

For $0<x<B$

$$f_{1:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{q} \tag{4.4.8}$$

The probability density function of the n[th] order statistics is obtained by substituting $s=n$ in equation (4.4.3)

Case (i): For $A<\mu<B$

For $A<x<0$

$$f_{n:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right]^{q}$$

For $0<x<B$

$$f_{n:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right]^{n-1} \tag{4.4.9}$$

The probability density function of the n[th] order statistics is obtained by substituting $s=n$ in equation (4.4.4)

Case (ii): For $\mu<A<B$

For $A<x<0$

$$f_{n:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right]^{q}$$

For $0<x<B$

$$f_{n:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right]^{n-1}$$

(4.4.10)

The probability density function of the n[th] order statistics is obtained by substituting $s=n$ in equation (4.4.5)

Case (iii): For $A<B<\mu$

For $A<x<0$

$$f_{n:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{q}$$

For $0<x<B$

$$f_{n:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{n-1}$$

(4.4.11)

The g[th] moment of sth order statistics is $\alpha^{(g)}{}_{s:n}=\int_{A}^{B}x^{g}f_{s:n}(x)$

$$=\frac{D_{s:n}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\left(\int_{0}^{B}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}\sum_{q=0}^{n-s}\binom{n-s}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right]^{s+q-1}\right)dx$$

$$-\frac{D_{s:n}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\left(\int_{0}^{A}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}\sum_{q=0}^{s-1}\binom{s-1}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right]^{n+q-s}\right)dx$$

$A<\mu<B$

$$= \frac{D_{sn}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\left(\int_0^B \frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}} \sum_{q=0}^{n-s}\binom{n-s}{q}(-1)^q\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right]^{s+q-1}\right)dx$$

$$- \frac{D_{sn}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\left(\int_0^A \frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}} \sum_{q=0}^{s-1}\binom{s-1}{q}(-1)^q\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right]^{n+q-s}\right)dx$$

$\mu < A < B$

$$= \frac{D_{sn}}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\left(\int_0^B \frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}} \sum_{q=0}^{n-s}\binom{n-s}{q}(-1)^q\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{s+q-1}\right)dx$$

$$- \frac{D_{sn}}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\left(\int_0^A \frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}} \sum_{q=0}^{s-1}\binom{s-1}{q}(-1)^q\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{n+q-s}\right)dx$$

$A < B < \mu$

Distribution of the Median

Let n be odd. The distribution of the median is obtained by substituting $s = \frac{n+1}{2}$ in equation (4.4.3), (4.4.4) and equation (4.4.5).

For $A < x < 0$

$$f_M(x) = \left(\frac{\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^2}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q}(-1)^q\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right]^{\frac{n-1}{2}+q} \quad A < \mu < B$$

$$f_M(x)=\left(\frac{\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^2}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^\beta\right)}\right)\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q}(-1)^q\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^\beta\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^\beta\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^\beta\right)}\right]^{\frac{n-1}{2}+q} \quad \mu<A<B$$

$$f_M(x)=\left(\frac{\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^2}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right)\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q}(-1)^q\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^\beta\right)-\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right]^{\frac{n-1}{2}+q} \quad A<B<\mu$$

For $0<x<B$

$$f_M(x)=\left(\frac{\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^2}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}\right)\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q}(-1)^q\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^\beta\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}\right]^{\frac{n-1}{2}+q} \quad A<\mu<B$$

$$f_M(x)=\left(\frac{\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^2}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^\beta\right)}\right)\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q}(-1)^q\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^\beta\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^\beta\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^\beta\right)}\right]^{\frac{n-1}{2}+q} \quad \mu<A<B$$

$$f_M(x)=\left(\frac{\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^2}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right)\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q}(-1)^q\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^\beta\right)-\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right]^{\frac{n-1}{2}+q} \quad A<B<\mu$$

Joint Moments of Order Statistics

The joint probability density function of the order statistics $X_{s:n}$ and $X_{s':n}$, $s<s'$ as given by [DAVID (1981)] is

$$f_{s,s':n}(x,y)=D_{s,s':n}[F(x)]^{s-1}[F(y)-F(x)]^{s'-s-1}[1-F(y)]^{n-s'}f(x)f(y)$$

where $D_{s,s':n}=\dfrac{n\,!}{(s-1)\,!(s'-s-1)\,!(n-s')\,!}$

and $F(x)$ is the cumulative density function of the left truncated three parameter generalized Gaussian distribution.

Following the heuristic aruguments of Balakrishna and Kocharlakota (1985) and considering

For A<μ<B

$$U(x)=\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}$$

For μ<A<B

$$U(x)=\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}$$

For A <B<μ

$$U(x)=\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}$$

We can express the joint probability density function of $X_{s:n}$ and $X_{s':n}$ as

$$f_{s,s':n}(x,y)=D_{s,s':n}[U(x)]^{s-1}[U(y)-U(x)]^{s'-s-1}[1-U(y)]^{n-s'}f(x)f(y)$$

The region $\{(x,y):A<x<y<\infty\}$ can be split in to three mutually exclusive regions:

$R_1=\{(x,y):A<x<y<0\}$

$R_2=\{(x,y):0<x<y<B\}$

$R_3=\{(x,y):A<x<0,0<y<B\}$

With this splitting of the region the product moments can be obtained as

$$E(X_s,X_{s'})=D_{s,s':n}\left\{\iint_{R_1}xy[U(-x)]^{s-1}[U(-y)-U(-x)]^{s'-s-1}[1-U(-y)]^{n-s'}f(-x)f(-y)dxdy\right.$$
$$+\iint_{R_2}xy[1-U(x)]^{s-1}[U(y)-U(x)]^{s'-s-1}[U(y)]^{n-s'}f(x)f(y)dxdy$$
$$\left.+\iint_{R3}(-x)y[U(-x)]^{s-1}[1-U(y)-U(-x)]^{s'-s-1}[U(y)]^{n-s'}f(-x)f(-y)dxdy\right\}$$

$$E(X_s,X_{s'})=D_{s,s':n}\left\{\iint_{R_1}xy\sum_{i=0}^{s'-s-1}\sum_{j=0}^{n-s'}\binom{s'-s-1}{i}\binom{n-s'}{j}(-1)^{s'-s-1+i+j}[U(-x)]^{s'-2-i}[U(-y)]^{i+j}f(x)f(y)dxdy\right.$$
$$+\iint_{R_2}xy\sum_{i=0}^{s-1}\sum_{j=0}^{s'-s-1}\binom{s-1}{i}\binom{s'-s-1}{j}(-1)^{i+j}[U(x)]^{s'-s-1+i-j}[U(y)]^{n-s'+j}f(x)f(y)dxdy$$
$$\left.+\iint_{R3}(-x)y\sum_{i=0}^{s'-s-1}\sum_{j=0}^{s'-s-1-i}\binom{s'-s-1}{i}\binom{s'-s-1-i}{j}(-1)^{i+j}[U(-x)]^{s-1}[U(-x)]^{i+j-1}[U(y)]^{n-s'+j}f(x)f(y)dxdy\right\}$$

(4.4.10)

$$\text{Let } \psi(a,b)=\int_0^\infty\int_0^\infty xy[U(x)]^a[U(y)]^b f(x)f(y)dxdy \qquad (4.4.11)$$

Substituting (4.4.11) in equation (4.4.10), we get

$$E(X_s,X_{s'})=D_{s,s':n}\left\{\sum_{i=0}^{s'-s-1}\sum_{j=0}^{n-s'}\binom{s'-s-1}{i}\binom{n-s'}{j}(-1)^{s'-s-1+i+j}\psi(s'-2-i,i+j)\right.$$
$$+\sum_{i=0}^{s-1}\sum_{j=0}^{s'-s-1}\binom{s-1}{i}\binom{s'-s-1}{j}(-1)^{i+j}\psi(s'-s-1+i-j,n-s'+j)$$
$$\left.+\sum_{i=0}^{s'-s-1}\sum_{j=0}^{s'-s-1-i}\binom{s'-s-1}{i}\binom{s'-s-1-i}{j}(-1)^{i+j}\psi(s-1+i,n-s'+j)\right\}$$

These distributions and moments of the order statistics are very useful in obtaining the optimal estimators for the scale and location parameters.

4.5. INFERENTIAL ASPECTS OF THE DOUBLY TRUNCATED THREE PARAMETER GENERALIZED GAUSSIAN DISTRIBUTION

Method of Moments

In this method, the theoretical moments of the population and the sample moments are equated correspondingly to deduce the estimators of the parameters.

Let $x_1, x_2, \ldots x_n$ be a sample of size n drawn from a population having the probability density function of the form given in equation (4.5.1). We have

$$f(x) = \frac{\beta}{\alpha} \frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \qquad A < \mu < B$$

$$f(x) = \frac{\beta}{\alpha} \frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \qquad \mu < A < B$$

$$f(x) = \frac{\beta}{\alpha} \frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)} \qquad A < B < \mu$$

(4.5.1)

This distribution is having three parameters μ, α and β. Hence we equate the first three moments of the population and the sample, which leads to the following equations.

$$\bar{x} = \mu + \alpha\left(\frac{\gamma\left(\frac{2}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{2}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right) \qquad \text{for } A < \mu < B$$

$$\bar{x} = \mu + \alpha\left(\frac{\gamma\left(\frac{2}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right) \qquad \text{for } \mu < A < B$$

$$\bar{x} = \mu + \alpha\left(\frac{\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right) - \gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right) - \gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right) \qquad \textit{for} \quad A < B < \mu \qquad (4.5.2)$$

and

$$s^2 = \alpha^2\left[\left(\frac{\gamma\left(\frac{3}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{3}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right) - \left(\frac{\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)^2\right] \quad \textit{for } A<\mu<B$$

$$s^2 = \alpha^2\left[\left(\frac{\gamma\left(\frac{3}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{3}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right) - \left(\frac{\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{2}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right)^2\right] \quad \textit{for } \mu<A<B$$

$$s^2 = \alpha^2\left[\left(\frac{\gamma\left(\frac{3}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right) - \gamma\left(\frac{3}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right) - \gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right) - \left(\frac{\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right) - \gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right) - \gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)^2\right] \quad \textit{for } A<\mu<B$$

$$(4.5.3)$$

$$\beta_2 = \frac{3S_1^2\left(2S_2 - S_1^2\right) + S_4 - 4S_1S_3}{\left(S_2 - S_1^2\right)^2} \qquad \textit{for } A < \mu < B$$

$$\beta_2 = \frac{3P_1^2\left(2P_2 - P_1^2\right) + P_4 - 4P_1P_3}{\left(P_2 - P_1^2\right)^2} \qquad \textit{for } \mu < A < B$$

$$\beta_2 = \frac{3Q_1^2\left(2Q_2 - Q_1^2\right) + Q_4 - 4Q_1Q_3}{\left(Q_2 - Q_1^2\right)^2} \qquad \textit{for } A < B < \mu \qquad (4.5.4)$$

Where $\bar{x}=\frac{1}{n}\sum_{i=1}^{n}x_i$ and $s^2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$ and $\beta_2=\frac{n\sum_{i=1}^{n}(x_i-\bar{x})^4}{\left(\sum_{i=1}^{n}(x_i-\bar{x})^2\right)^2}$

For given values of A and B, solving the above equations (4.5.2), (4.5.3) and (4.5.4) simultaneously by using Newtons-Raphson method, we can obtain the estimators for the parameters μ, α and β.

Sample mean $\bar{X}$ is unbiased estimator for the parameter μ.

Variance of $\bar{X}$ is

$$\mathrm{var}(\bar{X})=\mathrm{var}\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)$$

$$=\frac{1}{n}\alpha^2\left[\left(\frac{\gamma\left(\frac{3}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)+\gamma\left(\frac{3}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}\right)-\left(\frac{\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)+\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}\right)^2\right] \quad \text{for } A<\mu<B$$

$$=\frac{1}{n}\alpha^2\left[\left(\frac{\gamma\left(\frac{3}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)-\gamma\left(\frac{3}{\beta},\left(\frac{A-\mu}{\alpha}\right)^\beta\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^\beta\right)}\right)-\left(\frac{\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)-\gamma\left(\frac{2}{\beta},\left(\frac{A-\mu}{\alpha}\right)^\beta\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^\beta\right)}\right)^2\right] \quad \text{for } \mu<A<B$$

$$=\frac{1}{n}\alpha^2\left[\left(\frac{\gamma\left(\frac{3}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)-\gamma\left(\frac{3}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right)-\left(\frac{\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right)^2\right] \quad \text{for } A<B<\mu$$

(4.5.5)

Maximum Likelihood Method of Estimation

Case (i): For $A<\mu<B$

Let $x_1,x_2,\ldots x_n$ be a sample of size n drawn from a population having the probability density function of the form is given in equation (4.3.1), then the likelihood function of the sample is

$$L=\left(\frac{\beta}{\alpha}\right)^n\prod_{i=1}^{n}\frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\int_0^{\left(\frac{B-\mu}{\alpha}\right)^{\beta}} e^{-x}x^{\frac{1}{\beta}-1}dx+\int_0^{\left|\frac{A-\mu}{\alpha}\right|^{\beta}} e^{-x}x^{\frac{1}{\beta}-1}dx} \tag{4.5.6}$$

Taking logarithms on both sides of (4.5.6), we get

$$LogL=n\log\beta-n\log\alpha-\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}-\log\sum_{i=1}^{n}\left(\int_0^{\left(\frac{B-\mu}{\alpha}\right)^{\beta}} e^{-x_i}x_i^{\frac{1}{\beta}-1}dx+\int_0^{\left|\frac{A-\mu}{\alpha}\right|^{\beta}} e^{-x}x^{\frac{1}{\beta}-1}dx\right) \tag{4.5.7}$$

Since, Log L is not differentiable with respect to β for all values in the range $\beta > 0$, we obtain the estimate of β using the moment method of estimation in the equation (4.5.4).

For obtaining the maximum likelihood estimate of μ, we differentiate Log L with respect to μ and equate it to zero. But in equation (4.5.7) the function Log L is differentiable with respect to μ only when β is even. But in the case when β is odd we obtain the maximum likelihood estimator as in case of Laplace distribution (Keynes (1911)) i.e., when β is odd, we find μ which maximizes Log L. From equation (4.5.7) log L is maximum if $\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}$ is minimum when β is odd. The function $\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}$ is minimum only when μ is the median. Therefore the MLE of μ is the median of the distribution when β is odd. In case of β being even, we differentiate Log L with respect to μ and equate it to zero. This implies

$$\frac{\partial LogL}{\partial\mu}=\frac{-\beta}{\alpha}\sum_{i=1}^{n}\frac{\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}}{\left(\frac{x_i-\mu}{\alpha}\right)}-\frac{\beta}{\alpha}\frac{\frac{\left|\frac{A-\mu}{\alpha}\right|}{\left(\frac{A-\mu}{\alpha}\right)}e^{-\left|\frac{A-\mu}{\alpha}\right|^{\beta}}-e^{-\left|\frac{B-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}$$

Equating $\frac{\partial LogL}{\partial\mu}$ to zero, we get

$$\frac{-\beta}{\alpha}\sum_{i=1}^{n}\frac{\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}}{\left(\frac{x_i-\mu}{\alpha}\right)}-\frac{\beta}{\alpha}\frac{\frac{\left|\frac{A-\mu}{\alpha}\right|}{\left(\frac{A-\mu}{\alpha}\right)}e^{-\left|\frac{A-\mu}{\alpha}\right|^{\beta}}-e^{-\left|\frac{B-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}=0 \tag{4.5.8}$$

To derive maximum likelihood estimator of α, consider the derivative by L with respect to α and equate it to zero. This implies

$$\frac{\partial LogL}{\partial\alpha}=-\frac{n}{\alpha}+\frac{\beta}{\alpha}\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}-\frac{\beta}{\alpha}\frac{\left|\frac{A-\mu}{\alpha}\right|e^{-\left|\frac{A-\mu}{\alpha}\right|^{\beta}}-\left(\frac{B-\mu}{\alpha}\right)^{\beta}e^{-\left|\frac{B-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}$$

Equating $\frac{\partial LogL}{\partial\alpha}$ to zero, we get

$$-\frac{n}{\alpha}+\frac{\beta}{\alpha}\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}-\frac{\beta}{\alpha}\frac{\left|\frac{A-\mu}{\alpha}\right|e^{-\left|\frac{A-\mu}{\alpha}\right|^{\beta}}-\left(\frac{B-\mu}{\alpha}\right)^{\beta}e^{-\left|\frac{B-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}=0 \tag{4.5.9}$$

Solving the equations (4.5.4), (4.5.8) and (4.5.9) simultaneously for μ, α and β. Using numerical methods like Newton Raphson's method, we can obtain the maximum likelihood estimators of the parameters μ, α and β.

Case (ii): For $\mu<A<B$

Let $x_1, x_2, \ldots x_n$ be a sample of size n drawn from a population having the probability density function of the form is given in equation (4.3.2), then the likelihood function of the sample is

$$L=\left(\frac{\beta}{\alpha}\right)^{n}\prod_{i=1}^{n}\frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\int_0^{\left(\frac{B-\mu}{\alpha}\right)^{\beta}}e^{-x}x^{\frac{1}{\beta}-1}dx-\int_0^{\left(\frac{A-\mu}{\alpha}\right)^{\beta}}e^{-x}x^{\frac{1}{\beta}-1}dx} \tag{4.5.10}$$

Taking logarithms on both sides of (4.5.10), we get

$$LogL = n\log\beta - n\log\alpha - \sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta} - \log\sum_{i=1}^{n}\left(\int_{0}^{\left(\frac{B-\mu}{\alpha}\right)^{\beta}} e^{-x_i}x_i^{\frac{1}{\beta}-1}dx + \int_{0}^{\left(\frac{A-\mu}{\alpha}\right)^{\beta}} e^{-x}x^{\frac{1}{\beta}-1}dx\right)$$

(4.5.11)

Since, Log L is not differentiable with respect to β for all values in the range $\beta > 0$, we obtain the estimate of β using the moment method of estimation in the equation (4.5.4).

For obtaining the maximum likelihood estimate of μ, we differentiate Log L with respect to μ and equate it to zero. But in equation (4.5.7.b) the function Log L is differentiable with respect to μ only when β is even. But in the case when β is odd we obtain the maximum likelihood estimator as in case of Laplace distribution (Keynes (1911)) i.e., when β is odd, we find μ which maximizes Log L. From equation (4.5.7.b) log L is maximum if $\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}$ is minimum when β is odd. The function $\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}$ is minimum only when μ is the median. Therefore the MLE of μ is the median of the distribution when β is odd. In case of β being even, we differentiate Log L with respect to μ and equate it to zero. This implies

$$\frac{\partial LogL}{\partial\mu} = \frac{-\beta}{\alpha}\sum_{i=1}^{n}\frac{\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}}{\left(\frac{x_i-\mu}{\alpha}\right)} - \frac{\beta}{\alpha}\frac{e^{-\left(\frac{A-\mu}{\alpha}\right)^{\beta}} - e^{-\left(\frac{B-\mu}{\alpha}\right)^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}$$

Equating $\frac{\partial LogL}{\partial\mu}$ to zero, we get

$$\frac{-\beta}{\alpha}\sum_{i=1}^{n}\frac{\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}}{\left(\frac{x_i-\mu}{\alpha}\right)} - \frac{\beta}{\alpha}\frac{e^{-\left(\frac{A-\mu}{\alpha}\right)^{\beta}} - e^{-\left(\frac{B-\mu}{\alpha}\right)^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} = 0 \qquad \textbf{(4.5.12)}$$

To derive maximum likelihood estimator of α, consider the derivative of Log L with respect to α and equate it to zero. This implies

$$\frac{\partial LogL}{\partial \alpha} = -\frac{n}{\alpha} + \frac{\beta}{\alpha}\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta} - \frac{\beta}{\alpha}\frac{\left(\frac{A-\mu}{\alpha}\right)e^{-\left(\frac{A-\mu}{\alpha}\right)^{\beta}} - \left(\frac{B-\mu}{\alpha}\right)e^{-\left(\frac{B-\mu}{\alpha}\right)^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}$$

Equating $\frac{\partial LogL}{\partial \alpha}$ to zero, we get

$$-\frac{n}{\alpha} + \frac{\beta}{\alpha}\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta} - \frac{\beta}{\alpha}\frac{\left(\frac{A-\mu}{\alpha}\right)e^{-\left(\frac{A-\mu}{\alpha}\right)^{\beta}} - \left(\frac{B-\mu}{\alpha}\right)e^{-\left(\frac{B-\mu}{\alpha}\right)^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} = 0 \qquad (4.5.13)$$

Solving the equations (4.5.3), (4.5.12) & (4.5.13) simultaneously for μ, α and β. Using numerical methods like Newton Raphson's method, we can obtain the maximum likelihood estimators of the parameters μ, α and β.

Application

The proposed doubly truncated GGD is having several applications in analyzing the data sets arising at quality control, Industrial experiments, Agricultural experiments and man power modeling.

As an illustration consider the fitting of a doubly truncated GGD to the data on length of fish. The data on length of fish of catch is given in Table 1 (Rao 2014). From this data we observed that the length of fish is a non negative random variable. Hence, we consider the doubly truncation point is zero and it is empirically observed through biological consideration that this species may be having a length not more than 9cm hence, the right truncated point taken as 9. By assuming that the Variate under study follows a doubly truncated GGD of the form distribution with the p.d.f given in section (4.2) solving the equations of maximum likelihood estimators using Mathcad. We obtain the estimates of the parameters as $\hat{\mu} = 4.55$, $\hat{\alpha} = 3.42$, $\hat{\beta} = 3.0$. The estimated frequencies are also given in Table 1. Using χ^2 test for goodness of fit of the distribution, the calculated χ^2 values is 0.70312, at 5% level of significance. Comparing the calculated

χ^2 value with the critical value of $\chi^2 = 9.488$ with 4 degrees of freedom at 5% level of significance, we observed that the data gives a good fit to the doubly truncated generalized Gaussian distribution. The goodness of fit of distribution is tried with the other distribution namely Gaussian and generalized Gaussian distribution. The calculated χ^2 values for Gaussian and generalized Gaussian distribution is 0.9985 and 0.9391 respectively. It is observed that the χ^2 values for doubly truncated Gaussian is less than the other χ^2 values. Therefore, we conclude that doubly truncated generalized Gaussian gives the good fit than other two distributions.

Table 1

Observed and Expected Frequencies of Fish Length

Length of the Fish (cm)	Observed Frequencies	Expected Frequencies
1	6	4
2	13	9
3	17	15
4	19	18
5	19	19
6	18	19
7	13	16
8	7	10
9	2	4

CONCLUSION

In this chapter, we introduced a doubly truncated Generalized Gaussian Distribution (GGD). Generalized Gaussian Distribution (GGD) got lot of applications in analyzing several data sets as an alternative to the Gaussian distribution where the variable under study is lepty or platy or meso kurtic and symmetric. A doubly truncated GGD includes GGD as limiting case where the truncated point tends to infinite. The various distributional properties such as distribution function, moments, hazard function, and survival function are derived. It is observed that the hazard function is sometimes increasing and decreasing depending upon the truncation parameter. It also includes a J-type distribution when the truncation point is greater than its location parameter. The order statistics of variate under study are also derived. The joint moments of the r^{th} order statistics are obtained. The method of obtaining maximum likelihood estimators of the parameters using Mathcad code is presented. The asymptotic properties of the estimators are studied. The utility of the distribution for fitting the data on length of fish is illustrated. This distribution is useful for analyzing several data sets in management sciences, finance, quality control and agricultural experiments. The values of the cumulative distribution function of the proposed doubly truncated GGD are useful for further statistical inferences. It is also possible to obtain other inferential aspects such as testing of hypothesis.