CHAPTER-III

ON A RIGHT TRUNCATED GENERALIZED GAUSSIAN DISTRIBUTION

ON A RIGHT TRUNCATED GENERALIZED GAUSSIAN DISTRIBUTION

3.1. INTRODUCTION

In the earlier Chapter 2, we have developed and analyzed a left truncated generalized Gaussian distribution. There, it is assumed that the variate on the study follows a generalized Gaussian distribution and constrained with a finite value on the left end. This distribution work well in some cases where there is a minimum threshold for the variate under study. However, in some other datasets arising at quality control, agricultural experiments, reliability study, the variable under study is having constrained on the right end. i.e., there is an upper bound for the variable. For example, in manpower modeling there is an upper bound for the complete length of service known as age of superannuation. For these sorts of situation it is needed to consider right truncated generalized Gaussian distribution. In this chapter, we develop and analyze a right truncated generalized Gaussian distribution. The various distributional properties such as the probability density function, the distribution function, the four moments, the skewness, the kurtosis, the hazard function and survival function are derived. The order statistics of the distribution are also studied. Some inferential properties related to the parameters of the distribution are discussed.

3.2. RIGHT TRUNCATED GENERALIZED GAUSSIAN DISTRIBUTION

A Continuous random variable X is said to be a three parameter generalized Gaussian distribution if its probability density function (p.d.f) is of the form

$$f(x;\mu,\alpha,\beta)=\frac{\beta}{2\alpha\Gamma\left(\frac{1}{\beta}\right)}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}} \quad ; \; -\infty<x<\infty; \;\; -\infty<\mu<\infty; \;\; \alpha>0; \;\; \beta>0$$

(3.2.1)

Consider that the range variable is finite say $(-\infty, B)$. Then the probability density function (p.d.f) of a right truncated three parameter generalized Gaussian distribution is

$$f(x)=\frac{f(x;\mu,\alpha,\beta)}{F(B)};\qquad -\infty<x<B;\qquad -\infty<\mu<B;\qquad \alpha>0;\qquad \beta>0$$

where $$F(B)=\int_{-\infty}^{B}\frac{\beta}{2\alpha\Gamma\left(\frac{1}{\beta}\right)}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}dx \qquad (3.2.2)$$

The lower and upper truncation points are $-\infty$ *and* B respectively.

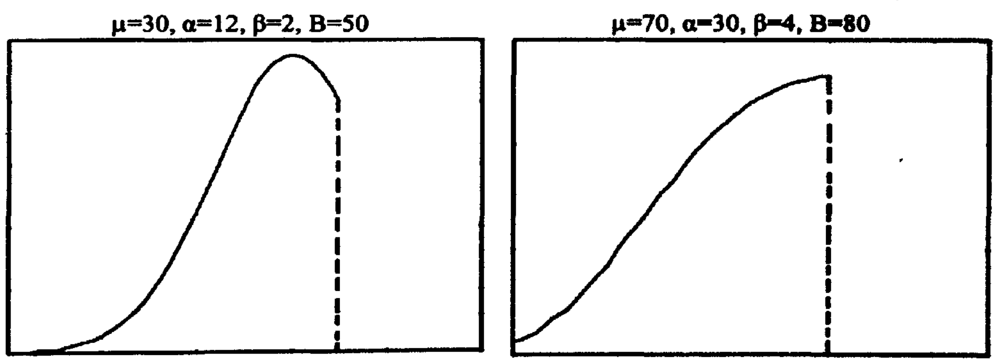
Hence, the probability density function of three parameter right truncated generalized Gaussian distribution is

$$f(x)=\frac{\beta}{\alpha}\frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\qquad for\ \ B<\mu \qquad (3.2.3)$$

$$f(x)=\frac{\beta}{\alpha}\frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\qquad for\ \ B\geq\mu \qquad (3.2.4)$$

3.3. DISTRIBUTIONAL PROPERTIES

The various distributional properties of the right truncated generalized Gaussian distribution are discussed in this section. Different shapes of the frequency curves for given values of the parameter are shown in Figure 3.1



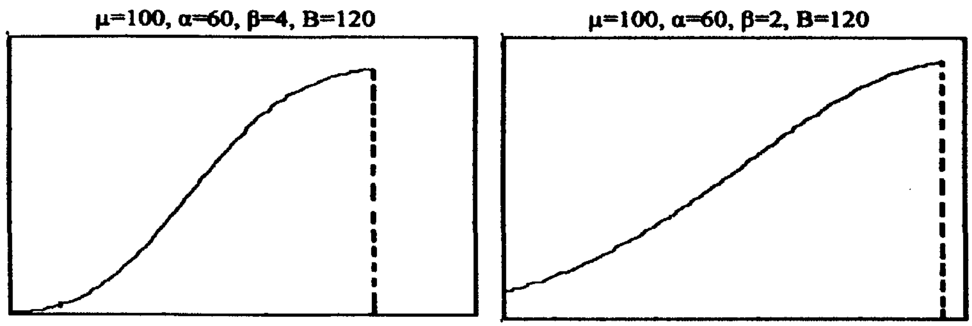


Figure 3.1: The frequency curves for different values of the right truncated generalized Gaussian distribution.

From figure 3.1 it is observed that this distribution is uni-model distribution.

The distribution function of *X* is given by

$$F(x)=\int_{-\infty}^{x} f(t)dt$$

$$F(x)=\int_{-\infty}^{x} \frac{\frac{\beta}{\alpha} e^{-\left|\frac{t-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)} dt$$

On simplification, we get

$$F(x)=\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)} \qquad for \quad B<\mu \qquad (3.3.1)$$

Similarly, we get

$$F(x)=\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)} \qquad for \quad B\geq\mu \qquad (3.3.2)$$

where, $\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)$ is an incomplete gamma function.

The mean of the distribution is

$$E(X)=\int_{-\infty}^{B} x\ f(x)\,dx$$

$$E(X)=\int_{-\infty}^{B} x\,\frac{\beta}{\alpha}\,\frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\,dx$$

On simplification, we get

$$E(X)=\mu+\alpha\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)\qquad for\quad B<\mu \qquad (3.3.3)$$

Similarly, we get

$$E(X)=\mu+\alpha\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)\qquad for\quad B\geq\mu \qquad (3.3.4)$$

Let M be the median of the distribution, then we have

$$\int_{-\infty}^{M} f(x)\,dx=\frac{1}{2}$$

$$\int_{-\infty}^{M}\frac{\beta}{\alpha}\,\frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\,dx=\frac{1}{2}$$

On simplification, we get

$$\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{M-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}=\frac{1}{2}\qquad for\quad B<\mu \qquad (3.3.5)$$

Similarly, we get

$$\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{M-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\left(\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)\right)}=\frac{1}{2} \qquad \text{for } B \geq \mu \qquad (3.3.6)$$

The median M of the distribution can be obtained by solving the equations (3.3.5) and (3.3.6).

For obtaining the mode of the distribution consider the probability density function of the distribution.

$$f(x)=K_1 e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}$$

where $K_1=\dfrac{\frac{\beta}{\alpha}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}$

Taking logarithms on both sides, we get

$$\log f(x)=\log(K_1)-\left|\frac{x-\mu}{\alpha}\right|^{\beta}$$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}\log f(x)=\frac{-\frac{\beta}{\alpha}\left|\frac{x-\mu}{\alpha}\right|^{\beta-1}\left|\frac{x-\mu}{\alpha}\right|}{\left(\frac{x-\mu}{\alpha}\right)}$$

$$\frac{d}{dx}\log f(x)=0$$

$$\Rightarrow \frac{-\frac{\beta}{\alpha}\left|\frac{x-\mu}{\alpha}\right|^{\beta-1}\left|\frac{x-\mu}{\alpha}\right|}{\left(\frac{x-\mu}{\alpha}\right)}=0 \qquad (3.3.7)$$

Solving equation (3.3.7), we get $x=\mu$.

Thus, $x=\mu$ **is the unique solution which indicates this distribution is uni-model.**

$$\frac{d^2}{dx^2}\log f(x) = -\frac{\beta}{\alpha^2}\left(\frac{\beta\left|\frac{x-\mu}{\alpha}\right|^{\beta} - \left(\frac{x-\mu}{\alpha}\right)}{\left(\frac{x-\mu}{\alpha}\right)^2}\right) < 0$$

This distribution reaches its maximum at the point $x = \mu$

The raw moments of the distribution are

$$\mu_r' = \int_{-\infty}^{B} x^r \; f(x)\,dx$$

$$\mu_r' = \int_{-\infty}^{B} x^r \frac{\frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\,dx$$

On simplification, we get

$$\mu_r' = \sum_{j=0}^{r}\binom{r}{j}\alpha^j\mu^{r-j}\left(\frac{\Gamma\left(\frac{j+1}{\beta}\right) - \gamma\left(\frac{j+1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right) \qquad for\, B < \mu \qquad (3.3.8)$$

The first four non central moments are obtained by substituting r = 1, 2, 3, 4 in equation (3.3.8)

$$\mu_1' = \mu + \alpha\left(\frac{\Gamma\left(\frac{2}{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)$$

$$\mu_2' = \mu^2 + 2\alpha\mu\left(\frac{\Gamma\left(\frac{2}{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right) + \alpha^2\left(\frac{\Gamma\left(\frac{3}{\beta}\right) - \gamma\left(\frac{3}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)$$

$$\mu_3' = \mu^3 + 3\alpha\mu^2\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right) + 3\alpha^2\mu\left(\frac{\Gamma\left(\frac{3}{\beta}\right)-\gamma\left(\frac{3}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right) + \alpha^3\left(\frac{\Gamma\left(\frac{4}{\beta}\right)-\gamma\left(\frac{4}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)$$

$$\mu_4' = \mu^4 + 4\alpha\mu^3\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right) + 6\alpha^2\mu^2\left(\frac{\Gamma\left(\frac{3}{\beta}\right)-\gamma\left(\frac{3}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)$$

$$+ 4\alpha^3\mu\left(\frac{\Gamma\left(\frac{4}{\beta}\right)-\gamma\left(\frac{4}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right) + \alpha^4\left(\frac{\Gamma\left(\frac{5}{\beta}\right)-\gamma\left(\frac{5}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)$$

Similarly for $B \geq \mu$, the rth non central moment is

$$\mu_r' = \sum_{j=0}^{r}\binom{r}{j}\alpha^j\mu^{r-j}\left(\frac{\Gamma\left(\frac{j+1}{\beta}\right)+\gamma\left(\frac{j+1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right) \qquad for\ B \geq \mu \qquad (3.3.9)$$

The first four non central moments are obtained by substituting r = 1, 2, 3, 4 in equation (3.3.9)

$$\mu_1' = \mu + \alpha\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)$$

$$\mu_2' = \mu^2 + 2\alpha\mu\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right) + \alpha^2\left(\frac{\Gamma\left(\frac{3}{\beta}\right)+\gamma\left(\frac{3}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)$$

$$\mu_3' = \mu^3 + 3\alpha\mu^2\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)+3\alpha^2\mu\left(\frac{\Gamma\left(\frac{3}{\beta}\right)+\gamma\left(\frac{3}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)+\alpha^3\left(\frac{\Gamma\left(\frac{4}{\beta}\right)+\gamma\left(\frac{4}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)$$

$$\mu_4' = \mu^4 + 4\alpha\mu^3\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)+6\alpha^2\mu^2\left(\frac{\Gamma\left(\frac{3}{\beta}\right)+\gamma\left(\frac{3}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)$$

$$+4\alpha^3\mu\left(\frac{\Gamma\left(\frac{4}{\beta}\right)+\gamma\left(\frac{4}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)+\alpha^4\left(\frac{\Gamma\left(\frac{5}{\beta}\right)+\gamma\left(\frac{5}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)$$

The central moments of this distribution are

$$\mu_r = \int_{-\infty}^{B}(x-\mu-D)^r f(x)dx$$

$$= \int_{-\infty}^{B}(x-\mu-D)^r \frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}dx$$

where $D=\alpha\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)$

On simplification, we get

$$\mu_r = \sum_{j=0}^{r}\binom{r}{j}\alpha^j(-D)^{r-j}\left(\frac{\Gamma\left(\frac{j+1}{\beta}\right)-\gamma\left(\frac{j+1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right) \quad \textit{for } B<\mu \qquad (3.3.10)$$

The first four central moments are obtained by substituting r = 1, 2, 3, 4 in equation (3.3.10)

$$\mu_1 = 0$$

$$\mu_2 = \alpha^2\left(\left(\frac{\Gamma\left(\frac{3}{\beta}\right)-\gamma\left(\frac{3}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)-\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)^2\right)$$

$$\mu_3 = 2\alpha^3\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)^3-3\alpha^3\left(\frac{\Gamma\left(\frac{3}{\beta}\right)-\gamma\left(\frac{3}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)$$

$$\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)+\alpha^3\left(\frac{\Gamma\left(\frac{4}{\beta}\right)-\gamma\left(\frac{4}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)$$

$$\mu_4 = 6\alpha^4\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)^2\left(\frac{\Gamma\left(\frac{3}{\beta}\right)-\gamma\left(\frac{3}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)$$

$$-3\alpha^4\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)^4-4\alpha^4\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)$$

$$\left(\frac{\Gamma\left(\frac{4}{\beta}\right)-\gamma\left(\frac{4}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)+\alpha^4\left(\frac{\Gamma\left(\frac{5}{\beta}\right)-\gamma\left(\frac{5}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)$$

Similarly for $B \geq \mu$, the r^{th} non central moment is

$$\mu_r = \sum_{j=0}^{r}\binom{r}{j}\alpha^{j}(-D)^{r-j}\left(\frac{\Gamma\left(\frac{j+1}{\beta}\right)+\gamma\left(\frac{j+1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right) \qquad for\,B \geq \mu \qquad (3.3.11)$$

where $D = \alpha\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)$

The first four central moments are obtained by substituting r = 1, 2, 3, 4 in equation (3.3.11)

$\mu_1 = 0$

$$\mu_2 = \alpha^2\left(\left(\frac{\Gamma\left(\frac{3}{\beta}\right)+\gamma\left(\frac{3}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)-\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)^{2}\right)$$

$$\mu_3 = 2\alpha^3\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)^{3} - 3\alpha^3\left(\frac{\Gamma\left(\frac{3}{\beta}\right)+\gamma\left(\frac{3}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)$$

$$\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right) + \alpha^3\left(\frac{\Gamma\left(\frac{4}{\beta}\right)+\gamma\left(\frac{4}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)$$

$$\mu_4 = 6\alpha^4 \left(\frac{\Gamma\left(\frac{2}{\beta}\right) + \gamma\left(\frac{2}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)} \right)^2 \left(\frac{\Gamma\left(\frac{3}{\beta}\right) + \gamma\left(\frac{3}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)} \right)$$

$$-3\alpha^4 \left(\frac{\Gamma\left(\frac{2}{\beta}\right) + \gamma\left(\frac{2}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)} \right)^4 - 4\alpha^4 \left(\frac{\Gamma\left(\frac{2}{\beta}\right) + \gamma\left(\frac{2}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)} \right)$$

$$\left(\frac{\Gamma\left(\frac{4}{\beta}\right) + \gamma\left(\frac{4}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)} \right) + \alpha^4 \left(\frac{\Gamma\left(\frac{5}{\beta}\right) + \gamma\left(\frac{5}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)} \right)$$

The skewness of the distribution is

$$\beta_1 = \frac{\left(2S_1^3 - 3S_1S_2 + S_3\right)^2}{\left(S_2 - S_1^2\right)^3} \qquad \textit{for} \quad B < \mu$$

$$\beta_1 = \frac{\left(2P_1^3 - 3P_1P_2 + P_3\right)^2}{\left(P_2 - P_1^2\right)^3} \qquad \textit{for} \ B \geq \mu$$

Kurtosis of the distribution is

$$\beta_2 = \frac{3S_1^2\left(2S_2 - S_1^2\right) + S_4 - 4S_1S_3}{\left(S_2 - S_1^2\right)^2} \qquad \textit{for} \ B < \mu$$

$$\beta_2 = \frac{3P_1^2\left(2P_2 - P_1^2\right) + P_4 - 4P_1P_3}{\left(P_2 - P_1^2\right)^2} \qquad \textit{for} \ B \geq \mu$$

where $S_1 = \left(\frac{\Gamma\left(\frac{2}{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)} \right)$; $\quad S_2 = \left(\frac{\Gamma\left(\frac{3}{\beta}\right) - \gamma\left(\frac{3}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)} \right)$;

$$S_3 = \left(\frac{\Gamma\left(\frac{4}{\beta}\right) - \gamma\left(\frac{4}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)} \right); \qquad S_4 = \left(\frac{\Gamma\left(\frac{5}{\beta}\right) - \gamma\left(\frac{5}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)} \right).$$

and $$P_1 = \left(\frac{\Gamma\left(\frac{2}{\beta}\right) + \gamma\left(\frac{2}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)} \right); \qquad P_2 = \left(\frac{\Gamma\left(\frac{3}{\beta}\right) + \gamma\left(\frac{3}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)} \right);$$

$$P_3 = \left(\frac{\Gamma\left(\frac{4}{\beta}\right) + \gamma\left(\frac{4}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)} \right); \qquad P_4 = \left(\frac{\Gamma\left(\frac{5}{\beta}\right) + \gamma\left(\frac{5}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)} \right).$$

The hazard rate function of the distribution is

$$h(x) = \frac{f(x)}{1-F(x)}$$

$$h(x) = \frac{\frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{-\gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)} \qquad for\ B < \mu \qquad (3.3.12)$$

$$h(x) = \frac{\frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)} \qquad for\ B \geq \mu \qquad (3.3.13)$$

The survival rate function $S(x)$ is

$$S(x) = 1 - F(x)$$

$$S(x)=1-\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)} \qquad for\ B<\mu \qquad (3.3.14)$$

$$S(x)=1-\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)} \qquad for\ B\geq\mu \qquad (3.3.15)$$

3.4. ORDER STATISTICS OF RIGHT TRUNCATED THREE PARAMETER GENERALIZED GAUSSIAN DISTRIBUTION

The simple explicit form of the distribution function as given in equation (3.3.1 and 3.3.2) leads us to derive the order statistics connected with this right truncated three parameters generalized Gaussian distribution.

$$f(x)=\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)} \qquad B<\mu$$

$$f(x)=\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)} \qquad B\geq\mu$$

(3.4.1)

Let $X_{1:n}\leq X_{2:n}\leq\ldots\leq X_{n:n}$ denote the order statistics obtained from a random sample of size n from the generalized truncated Gaussian distribution having the probability density function of the form given in (3.4.1). The probability density function of s^{th} order statistics is given by [David (1981)],

$$f_{s:n}(x)=D_{s:n}[F(x)]^{s-1}[1-F(x)]^{n-s}f(x)$$

Where $D_{s:n}=\frac{n!}{(s-1)!(n-s)!}$ (3.4.2)

Substituting f(x) and F(x) values given in this equation (3.4.1) and (3.3.2) in the equation (3.4.2), we get the probability density function of the s[th] order statistics is given by

Case (i): For $B \geq \mu$

For $-\infty < x < 0$

$$f_{s:n}(x) = D_{s:n} \frac{\frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)} \sum_{q=0}^{s-1} \binom{s-1}{q} (-1)^q \left[\frac{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right]^{n+q-s}$$

For $0 < x < B$

$$f_{s:n}(x) = D_{s:n} \frac{\frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)} \sum_{q=0}^{n-s} \binom{n-s}{q} (-1)^q \left[\frac{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right]^{s+q-1} \quad (3.4.3)$$

Substituting f(x) and F(x) values given in this equation (3.4.1) and (3.3.1) in the equation (3.4.2), we get the probability density function of the s[th] order statistics is given by

Case (ii): For $B < \mu$

For $-\infty < x < 0$

$$f_{s:n}(x) = D_{s:n} \frac{\frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)} \sum_{q=0}^{s-1} \binom{s-1}{q} (-1)^q \left[\frac{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{n+q-s}$$

For $0 < x < B$

$$f_{s:n}(x) = D_{s:n} \frac{\frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)} \sum_{q=0}^{n-s} \binom{n-s}{q} (-1)^q \left[\frac{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{s+q-1} \quad (3.4.4)$$

The probability density function of the first order statistics is obtained by substituting $s = 1$ in the equation (3.4.3)

Hence, case (i): For $B \geq \mu$,

For $-\infty<x<0$

$$f_{1:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right]^{n-1}$$

For $0<x<B$

$$f_{1:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^{q}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right]^{q}$$

The probability density function of the first order statistics is obtained by substituting $s=1$ in the equation (3.4.4)

Hence, case (ii): For $B<\mu$,

For $-\infty<x<0$

$$f_{1:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{n-1}$$

For $0<x<B$

$$f_{1:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^{q}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{q} \quad (3.4.5)$$

The probability density function of the n^{th} order statistics is obtained by substituting $s=n$ in equation (3.4.3)

Case (i): For $B\geq\mu$

For $-\infty<x<0$

$$f_{n:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^{q}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right]^{q}$$

For $0<x<B$

$$f_{n:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right]^{n-1} \qquad (3.4.6)$$

The probability density function of the n[th] order statistics is obtained by substituting $s=n$ in equation (3.4.4)

Case (ii): for $B<\mu$

For $-\infty<x<0$

$$f_{n:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^{q}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{q}$$

For $0<x<B$

$$f_{n:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{n-1} \qquad (3.4.7)$$

The gth moment of sth order statistics is

$$\alpha^{(g)}{}_{s:n}=\int_{-\infty}^{B}x^{g}f_{s:n}(x)$$

$$= \frac{D_{s:n}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\left(\int_0^B \frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}} \sum_{q=0}^{n-s}\binom{n-s}{q}(-1)^q\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right]^{s+q-1}\right)dx$$

$$- \frac{D_{s:n}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\left(\int_0^\infty \frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}} \sum_{q=0}^{s-1}\binom{s-1}{q}(-1)^q\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right]^{n+q-s}\right)dx$$

$$B \geq \mu$$

$$= \frac{D_{s:n}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\left(\int_0^B \frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}} \sum_{q=0}^{n-s}\binom{n-s}{q}(-1)^q\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{s+q-1}\right)dx$$

$$- \frac{D_{s:n}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\left(\int_0^\infty \frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}} \sum_{q=0}^{s-1}\binom{s-1}{q}(-1)^q\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{n+q-s}\right)dx$$

$$B < \mu$$

(3.4.8)

Distribution of the Median

Let n be odd. The distribution of the median is obtained by substituting $s = \frac{n+1}{2}$ in equation (3.4.3) and equation (3.4.4).

For $-\infty < x < 0$

$$f_M(x)=\left(\frac{\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^2}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)}\right)\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q}(-1)^q\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)}\right]^{\frac{n-1}{2}+q} \quad B\geq\mu$$

$$f_M(x)=\left(\frac{\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^2}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right)\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q}(-1)^q\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right]^{\frac{n-1}{2}+q} \quad B<\mu$$

For $0<x<B$

$$f_M(x)=\left(\frac{\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^2}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)}\right)\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q}(-1)^q\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)}\right]^{\frac{n-1}{2}+q} \quad B\geq\mu$$

$$f_M(x)=\left(\frac{\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^2}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right)\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q}(-1)^q\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{x-\mu}{\alpha}\right|^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right]^{\frac{n-1}{2}+q} \quad B<\mu$$

(3.4.9)

Joint Moments of Order Statistics

The joint probability density function of the order statistics $X_{s:n}$ and $X_{s':n}$, $s<s'$ as given by [DAVID (1981)] is

$$f_{s,s':n}(x,y) = D_{s,s':n}[F(x)]^{s-1}[F(y)-F(x)]^{s'-s-1}[1-F(y)]^{n-s'} f(x)f(y)$$

where $D_{s,s':n} = \dfrac{n!}{(s-1)!(s'-s-1)!(n-s')!}$ (3.4.10)

and $F(x)$ is the cumulative density function of the right truncated three parameter generalized Gaussian distribution.

Following the heuristic arguments of Balakrishna and Kocharlakota (1985) and considering

For $B \geq \mu$

$$U(x) = \frac{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}$$

For $B < \mu$

$$U(x) = \frac{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}$$

We can express the joint probability density function of $X_{s:n}$ and $X_{s':n}$ as

$$f_{s,s':n}(x,y) = D_{s,s':n}[U(x)]^{s-1}[U(y)-U(x)]^{s'-s-1}[1-U(y)]^{n-s'} f(x)f(y)$$

(3.4.11)

The region $\{(x,y): -\infty < x < y < B\}$ can be split in to three mutually exclusive regions:

$R_1 = \{(x,y): -\infty < x < y < 0\}$

$R_2 = \{(x,y): 0 < x < y < B\}$

$R_3 = \{(x,y): -\infty < x < 0, 0 < y < B\}$

With this splitting of the region the product moments can be obtained as

$$E(X_s, X_{s'}) = D_{s,s':n}\left\{\iint_{R_1} xy[U(-x)]^{s-1}[U(-y)-U(-x)]^{s'-s-1}[1-U(-y)]^{n-s'} f(-x)f(-y)dxdy\right.$$

$$+\iint_{R_2} xy[1-U(x)]^{s-1}[U(y)-U(x)]^{s'-s-1}[U(y)]^{n-s'} f(x)f(y)dxdy$$

$$\left.+\iint_{R3} (-x)y[U(-x)]^{s-1}[1-U(y)-U(-x)]^{s'-s-1}[U(y)]^{n-s'} f(-x)f(-y)dxdy\right\}$$

$$E(X_s, X_{s'}) = D_{s,s':n}\left\{\iint_{R_1} xy\sum_{i=0}^{s'-s-1}\sum_{j=0}^{n-s'}\binom{s'-s-1}{i}\binom{n-s'}{j}(-1)^{s'-s-1-i+j}[U(-x)]^{s'-2-i}[U(-y)]^{i+j} f(x)f(y)dxdy\right.$$

$$+\iint_{R_2} xy\sum_{i=0}^{s-1}\sum_{j=0}^{s'-s-1}\binom{s-1}{i}\binom{s'-s-1}{j}(-1)^{i+j}[U(x)]^{s'-s-1+i-j}[U(y)]^{n-s'+j} f(x)f(y)dxdy$$

$$\left.+\iint_{R3} (-x)y\sum_{i=0}^{s'-s-1}\sum_{j=0}^{s'-s-1-i}\binom{s'-s-1}{i}\binom{s'-s-1-i}{j}(-1)^{i+j}[U(-x)]^{s-1}[U(-x)]^{s+i-1}[U(y)]^{n-s'+j} f(x)f(y)dxdy\right\}$$

(3.4.12)

Let $\psi(a,b) = \int_0^\infty\int_0^\infty xy[U(x)]^a[U(y)]^b f(x)f(y)dxdy$ (3.4.13)

Substituting (3.4.13) in equation (3.4.12), we get

$$E(X_s, X_{s'}) = D_{s,s':n}\left\{\sum_{i=0}^{s'-s-1}\sum_{j=0}^{n-s'}\binom{s'-s-1}{i}\binom{n-s'}{j}(-1)^{s'-s-1-i+j}\psi(s'-2-i, i+j)\right.$$

$$+\sum_{i=0}^{s-1}\sum_{j=0}^{s'-s-1}\binom{s-1}{i}\binom{s'-s-1}{j}(-1)^{i+j}\psi(s'-s-1+i-j, n-s'+j)$$

$$\left.+\sum_{i=0}^{s'-s-1}\sum_{j=0}^{s'-s-1-i}\binom{s'-s-1}{i}\binom{s'-s-1-i}{j}(-1)^{i+j}\psi(s-1+i, n-s'+j)\right\}$$

These distributions and moments of the order statistics are very useful in obtaining the optimal estimators for the scale and location parameters.

3.5. INFERENTIAL ASPECTS OF THE RIGHT TRUNCATED THREE PARAMETER GENERALIZED GAUSSIAN DISTRIBUTION

Method of Moments

In this method, the theoretical moments of the population and the sample moments are equated correspondingly to deduce the estimators of the parameters.

Let $x_1, x_2, \ldots x_n$ be a sample of size n drawn from a population having the probability density function of the form given in equation (3.4.1), we have

$$f(x) = \frac{\beta}{\alpha} \frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)} \qquad B \geq \mu$$

$$f(x) = \frac{\beta}{\alpha} \frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)} \qquad B < \mu$$

This distribution is having three parameters μ, α and β. Hence we equate the first three moments of the population and the sample, which leads to the following equations.

$$\bar{x} = \mu + \alpha\left(\frac{\Gamma\left(\frac{2}{\beta}\right) + \gamma\left(\frac{2}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right) \qquad B \geq \mu$$

$$\bar{x} = \mu + \alpha\left(\frac{\Gamma\left(\frac{2}{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right) \qquad B < \mu \qquad (3.5.1)$$

and

$$s^2 = \alpha^2\left(\left(\frac{\Gamma\left(\frac{3}{\beta}\right)+\gamma\left(\frac{3}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)-\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)^2\right) \qquad \text{for } B \geq \mu$$

$$s^2 = \alpha^2\left(\left(\frac{\Gamma\left(\frac{3}{\beta}\right)-\gamma\left(\frac{3}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)-\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)^2\right) \qquad \text{for } B < \mu$$

(3.5.2)

$$\beta_2 = \frac{3S_1^2(2S_2 - S_1^2) + S_4 - 4S_1S_3}{(S_2 - S_1^2)^2} \qquad \text{for } B < \mu$$

$$\beta_2 = \frac{3P_1^2(2P_2 - P_1^2) + P_4 - 4P_1P_3}{(P_2 - P_1^2)^2} \qquad \text{for } B \geq \mu$$

(3.5.3)

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$ and $\beta_2 = \frac{n\sum_{i=1}^{n}(x_i - \bar{x})^4}{\left(\sum_{i=1}^{n}(x_i - \bar{x})^2\right)^2}$

For given values of A, solving the above equations (3.5.1), (3.5.2) and (3.5.3) simultaneously by using Newtons - Raphson method, we can obtain the estimators for the parameters μ, α and β.

Sample mean $\overline{X}$ is an unbiased estimator for the parameter μ.

Variance of $\overline{X}$ is

$$\text{var}(\overline{X}) = \text{var}\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)$$

$$= \frac{1}{n}\alpha^2\left[\left(\frac{\Gamma\left(\frac{3}{\beta}\right)+\gamma\left(\frac{3}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)-\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)^2\right] \qquad \text{for } B \geq \mu$$

$$= \frac{1}{n}\alpha^2\left[\left(\frac{\Gamma\left(\frac{3}{\beta}\right)-\gamma\left(\frac{3}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)-\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)^2\right] \qquad \text{for } B < \mu$$

(3.5.4)

Maximum Likelihood Method of Estimation

Case (i): For $B \geq \mu$

Let $x_1, x_2, \ldots x_n$ be a sample of size n drawn from a population having the probability density function of the form is given in equation (3.2.4), then the likelihood function of the sample is

$$L = \left(\frac{\beta}{\alpha}\right)^n \prod_{i=1}^{n} \frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\int_0^{\infty} e^{-x_i} x_i^{\frac{1}{\beta}-1} dx_i + \int_0^{\left(\frac{B-\mu}{\alpha}\right)^{\beta}} e^{-x_i} x_i^{\frac{1}{\beta}-1} dx_i} \tag{3.5.5}$$

Taking logarithms on both sides of (3.5.5), we get

$$LogL = n\log\beta - n\log\alpha - \sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta} - \log\sum_{i=1}^{n}\left(\int_0^{\infty} e^{-x_i} x_i^{\frac{1}{\beta}-1} dx_i + \int_0^{\left(\frac{B-\mu}{\alpha}\right)^{\beta}} e^{-x_i} x_i^{\frac{1}{\beta}-1} dx_i\right) \tag{3.5.6}$$

Since, Log L is not differentiable with respect to β for all values in the range $\beta > 0$, we obtain the estimate of β using the moment method of estimation in the equation (3.5.3).

For obtaining the maximum likelihood estimate of μ, we differentiate Log L with respect to μ and equate it to zero. But in equation (3.5.6) the function Log L is differentiable with respect to μ only when β is even. But in the case when β is odd we obtain the maximum likelihood estimator as in case of Laplace distribution (Keynes

(1911)) i.e., when β is odd, we find μ which maximizes log L. From eq. (3.5.6) Log L is maximum if $\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}$ is minimum when β is odd. The function $\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}$ is minimum only when μ is the median. Therefore the MLE of μ is the median of the distribution when β is odd. In case of β being even, we differentiate Log L with respect to μ and equate it to zero. This implies

$$\frac{\partial Log\,L}{\partial\mu}=\frac{\beta}{\alpha}\sum_{i=1}^{n}\frac{\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}}{\left(\frac{x_i-\mu}{\alpha}\right)}+\frac{\beta}{\alpha}\frac{e^{-\left(\frac{B-\mu}{\alpha}\right)^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}$$

Equating $\frac{\partial LogL}{\partial\mu}$ to zero, we get

$$\frac{\beta}{\alpha}\sum_{i=1}^{n}\frac{\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}}{\left(\frac{x_i-\mu}{\alpha}\right)}+\frac{\beta}{\alpha}\frac{e^{-\left(\frac{B-\mu}{\alpha}\right)^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}=0 \qquad (3.5.7)$$

To derive maximum likelihood estimator of α, consider the derivative of Log L w. r. to α and equate it to zero. This implies

$$\frac{\partial Log\,L}{\partial\alpha}=-\frac{n}{\alpha}+\frac{\beta}{\alpha}\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}+\frac{\beta}{\alpha}\left(\frac{B-\mu}{\alpha}\right)\frac{e^{-\left(\frac{B-\mu}{\alpha}\right)^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}$$

Equating $\frac{\partial LogL}{\partial\alpha}$ to zero, we get

$$\frac{\beta}{\alpha}\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}+\frac{\beta}{\alpha}\left(\frac{B-\mu}{\alpha}\right)\frac{e^{-\left(\frac{B-\mu}{\alpha}\right)^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}-\frac{n}{\alpha}=0 \qquad (3.5.8)$$

Solving the equations (3.5.3), (3.5.7) and (3.5.8) simultaneously for μ, α and β. Using numerical methods like Newton Raphson's method, we can obtain the maximum likelihood estimators of the parameters μ, α and β.

Case (ii): For $B < \mu$

Let $x_1, x_2, \ldots x_n$ be a sample of size n drawn from a population having the probability density function of the form is given in equation (3.2.3), then the likely hood function of the sample is

$$L = \left(\frac{\beta}{\alpha}\right)^n \prod_{i=1}^{n} \frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\int_0^{\infty} e^{-x_i} x_i^{\frac{1}{\beta}-1} dx_i - \int_0^{\left|\frac{B-\mu}{\alpha}\right|^{\beta}} e^{-x_i} x_i^{\frac{1}{\beta}-1} dx_i} \tag{3.5.9}$$

Taking logarithms on both sides of (3.5.5.a), we get

$$Log L = n\log\beta - n\log\alpha - \sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta} - \log\sum_{i=1}^{n}\left(\int_0^{\infty} e^{-x_i} x_i^{\frac{1}{\beta}-1} dx_i - \int_0^{\left|\frac{B-\mu}{\alpha}\right|^{\beta}} e^{-x_i} x_i^{\frac{1}{\beta}-1} dx_i\right) \tag{3.5.10}$$

Since, Log L is not differentiable with respect to β for all values in the range $\beta > 0$, we obtain the estimate of β using the moment method of estimation in the equation (3.5.3).

For obtaining the maximum likelihood estimate of μ, we differentiate Log L with respect to μ and equate it to zero. But in equation (3.5.10) the function Log L is differentiable with respect to μ only when β is even. But in the case when β is odd we obtain the maximum likelihood estimator as in case of Laplace distribution (Keynes (1911)) i.e., when β is odd, we find μ which maximizes Log L. From eq. (3.5.10) log L is maximum if $\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}$ is minimum when β is odd. The function $\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}$ is minimum only when μ is the median. Therefore the MLE of μ is the median of the distribution when β is odd. In case of β being even, we differentiate Log L with respect to μ and equate it to zero. This implies

$$\frac{\partial Log L}{\partial\mu} = \frac{\beta}{\alpha}\sum_{i=1}^{n}\frac{\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}}{\left(\frac{x_i-\mu}{\alpha}\right)} + \frac{\beta}{\alpha}\frac{\left|\frac{B-\mu}{\alpha}\right|}{\left(\frac{B-\mu}{\alpha}\right)}\frac{e^{-\left(\frac{B-\mu}{\alpha}\right)^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}$$

Equating $\frac{\partial LogL}{\partial\mu}$ to zero, we get

$$\frac{\beta}{\alpha}\sum_{i=1}^{n}\frac{\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}}{\left(\frac{x_i-\mu}{\alpha}\right)}+\frac{\beta}{\alpha}\frac{\left|\frac{B-\mu}{\alpha}\right|}{\left(\frac{B-\mu}{\alpha}\right)}\frac{e^{-\left(\frac{B-\mu}{\alpha}\right)^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}=0 \qquad (3.5.11)$$

To derive maximum likelihood estimator of α, consider the derivative of Log L w. r. to α and equate it to zero. This implies

$$\frac{\partial Log\,L}{\partial\alpha}=-\frac{n}{\alpha}+\frac{\beta}{\alpha}\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}+\frac{\beta}{\alpha}\left|\frac{B-\mu}{\alpha}\right|\frac{e^{-\left|\frac{B-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}$$

Equating $\frac{\partial LogL}{\partial\alpha}$ to zero, we get

$$\frac{\beta}{\alpha}\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}+\frac{\beta}{\alpha}\left|\frac{B-\mu}{\alpha}\right|\frac{e^{-\left|\frac{B-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}-\frac{n}{\alpha}=0 \qquad (3.5.12)$$

Solving the equations (3.5.3), (3.5.11) and (3.5.12) simultaneously for μ, α and β. Using numerical methods like Newton Raphson's method, we can obtain the maximum likelihood estimators of the parameters μ, α and β.

CONCLUSION

In this chapter, we have introduced a right truncated generalized Gaussian distribution. The various distributional properties such as the probability density function, distributional function, moments, hazard function, and survival function are derived. The order statistics of the variate under study are also derived. The joint moments of the r^{th} order statistics are obtained. The parameters are obtained by using method of maximum likelihood estimation is presented. This distribution is useful for analyzing several data sets in management sciences, finance, quality control and agricultural experiments. The values of the cumulative distribution function of the proposed right truncated GGD are useful for further statistical inferences.