CHAPTER-II

ON A LEFT TRUNCATED GENERALIZED GAUSSIAN DISTRIBUTION

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2.1. INTRODUCTION

In the previous chapter – 1, so many authors considered that the variable under study is having infinite range i.e., in theory, the random variable can assume any value over the range $-\infty$ *to* $+\infty$. But in many practical situations the random variables under study are constrained. For example, in inventory modeling the life time of commodity is non-negative. i.e., the lower limit of the variable is constrained to zero. Similarly, in many other applications in quality control and financial modeling, there is a minimum warranty / grantee period for the risk to happen. Hence, the variables will have the lower bound. This practical situation can be better analyzed by considering the truncated distributions. In addition to this the majority of work reports in a literature regarding the estimation of the parameters of the truncated generalized Gaussian distribution are based on the samples of truncated nature. Little work has been done in terms of describing and analyzing the properties of truncated generalized Gaussian distribution. This motivated to develop the left truncated generalized Gaussian distribution which is useful for analyzing the data sets arising at image processing, speech recognition, quality control and financial modeling.

In this chapter, we present the probability density function and distribution function of left truncated generalized Gaussian distribution. The cumulative distribution function (Area under probability curve) tables for standardized truncated generalized Gaussian distribution are computed and presented in Annex – I. The various distributional properties are derived. The distributions of the r^{th} order statistics, maximum order distribution, and median distribution are derived. The inferential aspects of the parameters are discussed using likelihood function. The goodness of fit of the proposed distribution is illustrated with the data on length of fish.

2.2. LEFT TRUNCATED GENERALIZED GAUSSIAN DISTRIBUTION

A Continuous random variable X is said to be a three parameter generalized Gaussian distribution if its probability density function (p.d.f) is of the form

$$f(x;\mu,\alpha,\beta)=\frac{\beta}{2\alpha\Gamma\left(\frac{1}{\beta}\right)}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}} \quad ; \quad -\infty<x<\infty; \quad -\infty<\mu<\infty; \quad \alpha>0; \quad \beta>0$$

(2.2.1)

Consider that the range variable is finite say (A, ∞). Then the probability density function (p.d.f) of a left truncated three parameter generalized Gaussian distribution is

$$f(x)=\frac{f(x;\mu,\alpha,\beta)}{1-F(A)}; \qquad A<x<\infty; \qquad A<\mu<\infty; \qquad \alpha>0; \qquad \beta>0$$

where $$F(A)=\int_{-\infty}^{A}\frac{\beta}{2\alpha\Gamma\left(\frac{1}{\beta}\right)}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}dx \tag{2.2.2}$$

The lower and upper truncation points are A and ∞ respectively.

Hence, the probability density function of three parameter left truncated generalized Gaussian distribution is

$$f(x)=\frac{\beta}{\alpha}\frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \qquad for\ A<\mu \tag{2.2.3}$$

$$f(x)=\frac{\beta}{\alpha}\frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \qquad for\ A\geq\mu \tag{2.2.4}$$

2.3. DISTRIBUTIONAL PROPERTIES

The various distributional properties of the left truncated generalized Gaussian distribution are discussed in this section. Different shapes of the frequency curves for given values of the parameter are shown in Figure 2.1

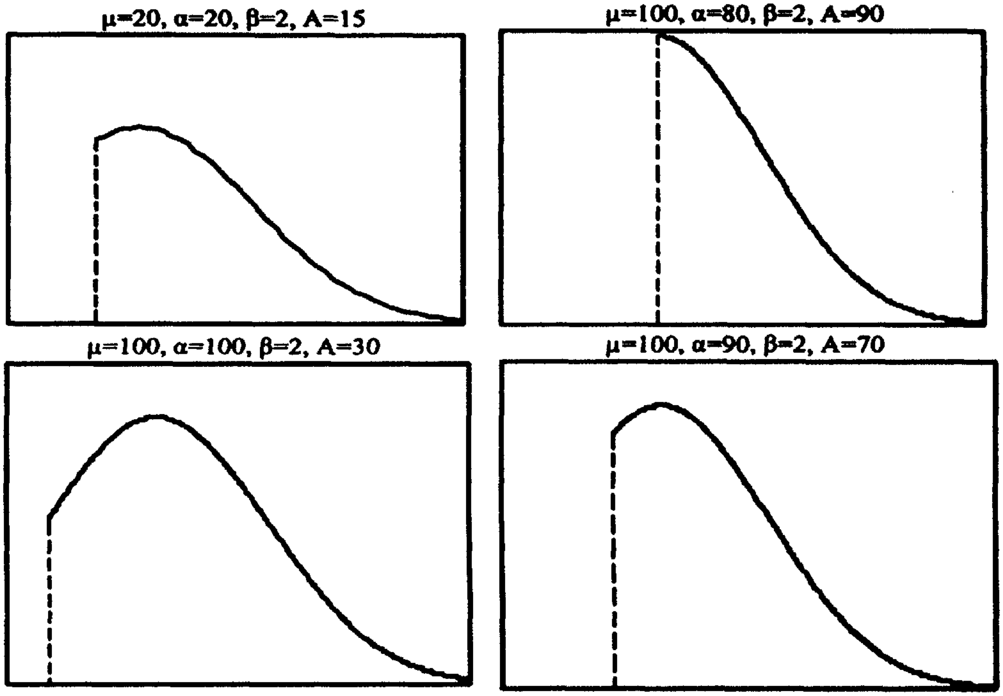

Figure 2.1:
The frequency curves for different values of the left truncated generalized Gaussian distribution

From figure 2.1 it is observed that this distribution is uni-model distribution.

The distribution function of X is given by

$$F(x)=\int_{A}^{x} f(t)dt$$

$$F(x)=\int_{A}^{x} \frac{\frac{\beta}{\alpha} e^{-\left|\frac{t-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\, dt$$

On simplification, we get

$$F(x)=\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \qquad for \quad A<\mu \qquad (2.3.1)$$

Similarly, we get

$$F(x)=\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \qquad for \quad A\geq\mu \qquad (2.3.2)$$

where, $\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)$ is an incomplete gamma function.

The mean of the distribution is

$$E(X)=\int_{A}^{\infty} x \;\; f(x)\, dx$$

$$E(X)=\int_{A}^{\infty} x\frac{\beta}{\alpha}\frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\, dx$$

On simplification, we get

$$E(X)=\mu+\alpha\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right) \qquad for \quad A<\mu \qquad (2.3.3)$$

Similarly, we get

$$E(X)=\mu+\alpha\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right) \qquad for \quad A\geq\mu \qquad (2.3.4)$$

Let M be the median of the distribution, then we have

$$\int_A^M f(x)\,dx = \frac{1}{2}$$

$$\int_A^M \frac{\beta}{\alpha} \frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\,dx = \frac{1}{2}$$

On simplification, we get

$$\frac{\gamma\left(\frac{1}{\beta},\left(\frac{M-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} = \frac{1}{2} \qquad for \quad A < \mu \qquad (2.3.5)$$

Similarly, we get

$$\frac{\gamma\left(\frac{1}{\beta},\left(\frac{M-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} = \frac{1}{2} \qquad for \quad A \geq \mu \qquad (2.3.6)$$

The median M of the distribution can be obtained by solving the equations (2.3.5) and (2.3.6).

For obtaining the mode of the distribution consider the probability density function of the distribution.

$$f(x) = K_1 e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}$$

where $K_1 = \dfrac{\frac{\beta}{\alpha}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}$

Taking logarithms on both sides, we get

$$\log f(x) = \log(K_1) - \left|\frac{x-\mu}{\alpha}\right|^{\beta}$$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}\log f(x)=\frac{-\frac{\beta}{\alpha}\left|\frac{x-\mu}{\alpha}\right|^{\beta-1}\left|\frac{x-\mu}{\alpha}\right|}{\left(\frac{x-\mu}{\alpha}\right)}$$

$$\frac{d}{dx}\log f(x)=0$$

$$\Rightarrow\frac{-\frac{\beta}{\alpha}\left|\frac{x-\mu}{\alpha}\right|^{\beta-1}\left|\frac{x-\mu}{\alpha}\right|}{\left(\frac{x-\mu}{\alpha}\right)}=0 \qquad (2.3.7)$$

Solving equation (2.3.7), we get x = μ.

Thus, $x=\mu$ is the unique solution which indicates this distribution is uni-model.

$$\frac{d^2}{dx^2}\log f(x)=-\frac{\beta}{\alpha^2}\left(\frac{\beta\left|\frac{x-\mu}{\alpha}\right|^{\beta}-\left(\frac{x-\mu}{\alpha}\right)}{\left(\frac{x-\mu}{\alpha}\right)^2}\right)<0$$

This distribution reaches its maximum at the point $x=\mu$

The raw moments of the distribution are

$$\mu_r'=\int_A^{\infty}x^r\ f(x)\,dx$$

$$\mu_r'=\int_A^{\infty}x^r\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\,dx$$

On simplification, we get

$$\mu_r'=\sum_{j=0}^{r}\binom{r}{j}\alpha^j\mu^{r-j}\left(\frac{\Gamma\left(\frac{j+1}{\beta}\right)+\gamma\left(\frac{j+1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right) \qquad for\ A<\mu \qquad (2.3.8)$$

The first four non central moments are obtained by substituting r = 1, 2, 3, 4 in equation (2.3.8)

$$\mu_1' = \mu + \alpha \left(\frac{\Gamma\left(\frac{2}{\beta}\right) + \gamma\left(\frac{2}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right)$$

$$\mu_2' = \mu^2 + 2\alpha\mu \left(\frac{\Gamma\left(\frac{2}{\beta}\right) + \gamma\left(\frac{2}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right) + \alpha^2 \left(\frac{\Gamma\left(\frac{3}{\beta}\right) + \gamma\left(\frac{3}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right)$$

$$\mu_3' = \mu^3 + 3\,\alpha\,\mu^2 \left(\frac{\Gamma\left(\frac{2}{\beta}\right) + \gamma\left(\frac{2}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right) + 3\,\alpha^2\mu \left(\frac{\Gamma\left(\frac{3}{\beta}\right) + \gamma\left(\frac{3}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right) + \alpha^3 \left(\frac{\Gamma\left(\frac{4}{\beta}\right) + \gamma\left(\frac{4}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right),$$

$$\mu_4' = \mu^4 + 4\alpha\mu^3 \left(\frac{\Gamma\left(\frac{2}{\beta}\right) + \gamma\left(\frac{2}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right) + 6\alpha^2\mu^2 \left(\frac{\Gamma\left(\frac{3}{\beta}\right) + \gamma\left(\frac{3}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right)$$
$$+ 4\alpha^3\mu \left(\frac{\Gamma\left(\frac{4}{\beta}\right) + \gamma\left(\frac{4}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right) + \alpha^4 \left(\frac{\Gamma\left(\frac{5}{\beta}\right) + \gamma\left(\frac{5}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right)$$

Similarly for $A \geq \mu$, the rth non central moment is

$$\mu_r' = \sum_{j=0}^{r} \binom{r}{j} \alpha^j \mu^{r-j} \left(\frac{\Gamma\left(\frac{j+1}{\beta}\right) - \gamma\left(\frac{j+1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right) \qquad for\ A \geq \mu \qquad (2.3.9)$$

The first four non central moments are obtained by substituting r = 1, 2, 3, 4 in equation (2.3.9)

$$\mu_1' = \mu + \alpha \left(\frac{\Gamma\left(\frac{2}{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right)$$

$$\mu_2' = \mu^2 + 2\alpha\mu \left(\frac{\Gamma\left(\frac{2}{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right) + \alpha^2 \left(\frac{\Gamma\left(\frac{3}{\beta}\right) - \gamma\left(\frac{3}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right)$$

$$\mu_3' = \mu^3 + 3\alpha\mu^2 \left(\frac{\Gamma\left(\frac{2}{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right) + 3\alpha^2\mu \left(\frac{\Gamma\left(\frac{3}{\beta}\right) - \gamma\left(\frac{3}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right) + \alpha^3 \left(\frac{\Gamma\left(\frac{4}{\beta}\right) - \gamma\left(\frac{4}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right)$$

$$\mu_4' = \mu^4 + 4\alpha\mu^3 \left(\frac{\Gamma\left(\frac{2}{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right) + 6\alpha^2\mu^2 \left(\frac{\Gamma\left(\frac{3}{\beta}\right) - \gamma\left(\frac{3}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right)$$

$$+ 4\alpha^3\mu \left(\frac{\Gamma\left(\frac{4}{\beta}\right) - \gamma\left(\frac{4}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right) + \alpha^4 \left(\frac{\Gamma\left(\frac{5}{\beta}\right) - \gamma\left(\frac{5}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right)$$

The central moments of this distribution are

$$\mu_r = \int_A^\infty (x - \mu - D)^r f(x)dx$$

$$= \int_A^\infty (x - \mu - D)^r \frac{\frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} dx$$

where $D = \alpha \left(\frac{\Gamma\left(\frac{2}{\beta}\right) + \gamma\left(\frac{2}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right)$

On simplification, we get

$$\mu_r = \sum_{j=0}^{r}\binom{r}{j}\alpha^{j}(-D)^{r-j}\left(\frac{\Gamma\left(\frac{j+1}{\beta}\right)+\gamma\left(\frac{j+1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right) \qquad \text{for } A<\mu \qquad (2.3.10)$$

The first four central moments are obtained by substituting r = 1, 2, 3, 4 in equation (2.3.10)

$$\mu_1 = 0$$

$$\mu_2 = \alpha^2\left(\left(\frac{\Gamma\left(\frac{3}{\beta}\right)+\gamma\left(\frac{3}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)-\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)^{2}\right)$$

$$\begin{aligned}\mu_3 = {} & 2\alpha^3\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)^{3}-3\alpha^3\left(\frac{\Gamma\left(\frac{3}{\beta}\right)+\gamma\left(\frac{3}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)\\ & \left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)+\alpha^3\left(\frac{\Gamma\left(\frac{4}{\beta}\right)+\gamma\left(\frac{4}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)\end{aligned}$$

$$\begin{aligned}\mu_4 = {} & 6\alpha^4\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)^{2}\left(\frac{\Gamma\left(\frac{3}{\beta}\right)+\gamma\left(\frac{3}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)\\ & -3\alpha^4\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)^{4}-4\alpha^4\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)\\ & \left(\frac{\Gamma\left(\frac{4}{\beta}\right)+\gamma\left(\frac{4}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)+\alpha^4\left(\frac{\Gamma\left(\frac{5}{\beta}\right)+\gamma\left(\frac{5}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)\end{aligned}$$

Similarly for $A \geq \mu$, the r[th] non central moment is

$$\mu_r = \sum_{j=0}^{r} \binom{r}{j} \alpha^j (-D)^{r-j} \left(\frac{\Gamma\left(\frac{j+1}{\beta}\right) - \gamma\left(\frac{j+1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right) \qquad \text{for } A \geq \mu \qquad (2.3.11)$$

where $D = \alpha \left(\frac{\Gamma\left(\frac{2}{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right)$

The first four central moments are obtained by substituting r = 1, 2, 3, 4 in equation (2.3.11)

$\mu_1 = 0$

$$\mu_2 = \alpha^2 \left(\left(\frac{\Gamma\left(\frac{3}{\beta}\right) - \gamma\left(\frac{3}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right) - \left(\frac{\Gamma\left(\frac{2}{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right)^2 \right)$$

$$\mu_3 = 2\alpha^3 \left(\frac{\Gamma\left(\frac{2}{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right)^3 - 3\alpha^3 \left(\frac{\Gamma\left(\frac{3}{\beta}\right) - \gamma\left(\frac{3}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right)$$
$$\left(\frac{\Gamma\left(\frac{2}{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right) + \alpha^3 \left(\frac{\Gamma\left(\frac{4}{\beta}\right) - \gamma\left(\frac{4}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right)$$

$$\mu_4 = 6\alpha^4\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right)^2\left(\frac{\Gamma\left(\frac{3}{\beta}\right)-\gamma\left(\frac{3}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right)$$

$$-3\alpha^4\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right)^4-4\alpha^4\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right)$$

$$\left(\frac{\Gamma\left(\frac{4}{\beta}\right)-\gamma\left(\frac{4}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right)+\alpha^4\left(\frac{\Gamma\left(\frac{5}{\beta}\right)-\gamma\left(\frac{5}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right)$$

The skewness of the distribution is

$$\beta_1 = \frac{\left(2S_1^3-3S_1S_2+S_3\right)^2}{\left(S_2-S_1^2\right)^3} \qquad for \quad A<\mu$$

$$\beta_1 = \frac{\left(2P_1^3-3P_1P_2+P_3\right)^2}{\left(P_2-P_1^2\right)^3} \qquad for \ A\geq\mu$$

Kurtosis of the distribution is

$$\beta_2 = \frac{3S_1^2\left(2S_2-S_1^2\right)+S_4-4S_1S_3}{\left(S_2-S_1^2\right)^2} \qquad for \ A<\mu$$

$$\beta_2 = \frac{3P_1^2\left(2P_2-P_1^2\right)+P_4-4P_1P_3}{\left(P_2-P_1^2\right)^2} \qquad for \ A\geq\mu$$

where $S_1 = \left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)$; $\qquad S_2 = \left(\frac{\Gamma\left(\frac{3}{\beta}\right)+\gamma\left(\frac{3}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)$;

$$S_3 = \left(\frac{\Gamma\left(\frac{4}{\beta}\right) + \gamma\left(\frac{4}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right); \qquad S_4 = \left(\frac{\Gamma\left(\frac{5}{\beta}\right) + \gamma\left(\frac{5}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right).$$

and

$$P_1 = \left(\frac{\Gamma\left(\frac{2}{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right); \qquad P_2 = \left(\frac{\Gamma\left(\frac{3}{\beta}\right) - \gamma\left(\frac{3}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right);$$

$$P_3 = \left(\frac{\Gamma\left(\frac{4}{\beta}\right) - \gamma\left(\frac{4}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right); \qquad P_4 = \left(\frac{\Gamma\left(\frac{5}{\beta}\right) - \gamma\left(\frac{5}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right).$$

The hazard rate function of the distribution is

$$h(x) = \frac{f(x)}{1-F(x)}$$

$$h(x) = \frac{\beta}{\alpha} \frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)} \qquad \text{for } A < \mu \qquad (2.3.12)$$

$$h(x) = \frac{\beta}{\alpha} \frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)} \qquad \text{for } A > \mu \qquad (2.3.13)$$

The survival rate function $S(x)$ is

$$S(x) = 1 - F(x)$$

$$S(x) = 1 - \frac{\gamma\left(\frac{1}{\beta}, \left(\frac{x-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \qquad \text{for } A < \mu \qquad (2.3.14)$$

$$S(x)=1-\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \qquad \textit{for } A\geq\mu \qquad (2.3.15)$$

2.4. ORDER STATISTICS OF LEFT TRUNCATED THREE PARAMETER GENERALIZED GAUSSIAN DISTRIBUTION

The simple explicit form of the distribution function as given in equation (2.3.1 and 2.3.2) leads us to derive the order statistics connected with this left truncated three parameters generalized Gaussian distribution.

$$f(x)=\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \qquad A<\mu$$

$$f(x)=\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \qquad A\geq\mu$$

(2.4.1)

Let $X_{1,n}\leq X_{2,n}\leq\ldots\leq X_{n,n}$ denote the order statistics obtained from a random sample of size n from the generalized truncated Gaussian distribution having the probability density function of the form given in (2.4.1). The probability density function of s^{th} order statistics is given by [David (1981)],

$$f_{s,n}(x)=D_{s,n}[F(x)]^{s-1}[1-F(x)]^{n-s}f(x)$$

where $D_{s,n}=\frac{n!}{(s-1)!(n-s)!}$ (2.4.2)

Substituting f(x) and F(x) values given in equation (2.4.1) and (2.3.2) in the equation (2.4.2), we get the probability density function of the s^{th} order statistics is as

Case (i): For $A\geq\mu$

For $A<x<0$

$$f_{s:n}(x)=D_{s:n}\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\sum_{q=0}^{s-1}\binom{s-1}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right]^{n+q-s}$$

For $0<x<\infty$

$$f_{s:n}(x)=D_{s:n}\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\sum_{q=0}^{n-s}\binom{n-s}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right]^{s+q-1}$$

(2.4.3)

Substituting f(x) and F(x) values given in this equation (2.4.1) and (2.3.1) in the equation (2.4.2), we get the probability density function of the s[th] order statistics is as

Case (ii): For $A<\mu$

For $A<x<0$

$$f_{s:n}(x)=D_{s:n}\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\sum_{q=0}^{s-1}\binom{s-1}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right]^{n+q-s}$$

For $0<x<\infty$

$$f_{s:n}(x)=D_{s:n}\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\sum_{q=0}^{n-s}\binom{n-s}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right]^{s+q-1} \quad (2.4.4)$$

The probability density function of the first order statistics is obtained by substituting $s=1$ in the equation (2.4.3)

Hence, Case (i): For $A\geq\mu$

For $A<x<0$

$$f_{1:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right]^{n-1}$$

For $0<x<\infty$

$$f_{1:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right]^{q}$$

The probability density function of the first order statistics is obtained by substituting $s=1$ in the equation (2.4.4)

Hence, Case (ii): For $A<\mu$

For $A<x<0$

$$f_{1:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right]^{n-1}$$

For $0<x<\infty$

$$f_{1:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right]^{q}$$

(2.4.5)

The probability density function of the n^{th} order statistics is obtained by substituting $s=n$ in equation (2.4.3)

Case (i): For $A\geq\mu$

For $A<x<0$

$$f_{nn}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right]^{q}$$

For $0<x<\infty$

$$f_{n:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right]^{n-1} \qquad (2.4.6)$$

The probability density function of the n[th] order statistics is obtained by substituting $s=n$ in equation (2.4.4)

Case (ii): For $A<\mu$

For $A<x<0$

$$f_{nn}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right]^{q}$$

For $0<x<\infty$

$$f_{nn}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right]^{n-1} \qquad (2.4.7)$$

The g[th] moment of sth order statistics is

$$\alpha^{(g)}{}_{sn}=\int_{A}^{\infty}x^{g}f_{sn}(x)$$

$$= \frac{D_{sn}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\left(\int_{0}^{\infty}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}\sum_{q=0}^{n-s}\binom{n-s}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right]^{s+q-1}\right)dx$$

$$- \frac{D_{sn}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\left(\int_{0}^{A}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}\sum_{q=0}^{s-1}\binom{s-1}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right]^{n+q-s}\right)dx$$

$$A \geq \mu$$

$$= \frac{D_{sn}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\left(\int_{0}^{\infty}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}\sum_{q=0}^{n-s}\binom{n-s}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right]^{s+q-1}\right)dx$$

$$- \frac{D_{sn}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\left(\int_{0}^{A}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}\sum_{q=0}^{s-1}\binom{s-1}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right]^{n+q-s}\right)dx$$

$$A < \mu$$

(2.4.8)

Distribution of the Median

Let n be odd. The distribution of the median is obtained by substituting $s = \frac{n+1}{2}$ in equation (2.4.3) and equation (2.4.4).

For A < x < 0

$$f_M(x)=\left(\frac{\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^2}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^\beta\right)}\right)\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q}(-1)^q\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^\beta\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^\beta\right)}\right]^{\frac{n-1}{2}+q} \quad A\geq\mu$$

$$f_M(x)=\left(\frac{\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^2}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}\right)\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q}(-1)^q\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^\beta\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}\right]^{\frac{n-1}{2}+q} \quad A<\mu$$

For $0 < x < \infty$

$$f_M(x)=\left(\frac{\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^2}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^\beta\right)}\right)\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q}(-1)^q\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^\beta\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^\beta\right)}\right]^{\frac{n-1}{2}+q} \quad A\geq\mu$$

$$f_M(x)=\left(\frac{\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^2}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}\right)\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q}(-1)^q\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^\beta\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}\right]^{\frac{n-1}{2}+q} \quad A<\mu$$

(2.4.9)

Joint Moments of Order Statistics

The joint probability density function of the order statistics $X_{s:n}$ and $X_{s':n}$, $s<s'$ as given by [DAVID (1981)] is

$$f_{s,s':n}(x,y)=D_{s,s':n}[F(x)]^{s-1}[F(y)-F(x)]^{s'-s-1}[1-F(y)]^{n-s'}f(x)f(y)$$

where $D_{s,s':n}=\dfrac{n\,!}{(s-1)\,!(s'-s-1)\,!(n-s')\,!}$ (2.4.10)

and $F(x)$ is the cumulative density function of the left truncated three parameter generalized Gaussian distribution.

Following the heuristic arguments of Balakrishna and Kocharlakota (1985) and considering

For $A \geq \mu$

$$U(x)=\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}$$

For $A < \mu$

$$U(x)=\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}$$

We can express the joint probability density function of $X_{s:n}$ and $X_{s':n}$ as

$$f_{s,s':n}(x,y)=D_{s,s':n}[U(x)]^{s-1}[U(y)-U(x)]^{s'-s-1}[1-U(y)]^{n-s'}f(x)f(y)$$

(2.4.11)

The region $\{(x,y): A<x<y<\infty\}$ can be split in to three mutually exclusive regions:

$R_1=\{(x,y): A<x<y<0\}$

$R_2=\{(x,y): 0<x<y<\infty\}$

$R_3=\{(x,y): A<x<0, 0<y<\infty\}$

With this splitting of the region the product moments can be obtained as

$$E(X_s,X_{s'})=D_{s,s':n}\left\{\iint_{R_1} xy[U(-x)]^{s-1}[U(-y)-U(-x)]^{s'-s-1}[1-U(-y)]^{n-s'}f(-x)f(-y)dxdy\right.$$

$$+\iint_{R_2} xy[1-U(x)]^{s-1}[U(y)-U(x)]^{s'-s-1}[U(y)]^{n-s'}f(x)f(y)dxdy$$

$$\left.+\iint_{R3}(-x)y[U(-x)]^{s-1}[1-U(y)-U(-x)]^{s'-s-1}[U(y)]^{n-s'}f(-x)f(-y)dxdy\right\}$$

$$E(X_s,X_{s'})=D_{s,s':n}\left\{\iint_{R_1} xy\sum_{i=0}^{s'-s-1}\sum_{j=0}^{n-s'}\binom{s'-s-1}{i}\binom{n-s'}{j}(-1)^{s'-s-1-i+j}[U(-x)]^{s'-2-i}[U(-y)]^{i+j}f(x)f(y)dxdy\right.$$

$$+\iint_{R_2} xy\sum_{i=0}^{s-1}\sum_{j=0}^{s'-s-1}\binom{s-1}{i}\binom{s'-s-1}{j}(-1)^{i+j}[U(x)]^{s'-s-1+i-j}[U(y)]^{n-s'+j}f(x)f(y)dxdy$$

$$\left.+\iint_{R3}(-x)y\sum_{i=0}^{s'-s-1}\sum_{j=0}^{s'-s-1-i}\binom{s'-s-1}{i}\binom{s'-s-1-i}{j}(-1)^{i+j}[U(-x)]^{s-1}[U(-x)]^{s+i-1}[U(y)]^{n-s'+j}f(x)f(y)dxdy\right\}$$

(2.4.12)

Let $\psi(a,b)=\int_0^\infty\int_0^\infty xy[U(x)]^a[U(y)]^b f(x)f(y)dxdy$ (2.4.13)

Substituting (2.4.13) in equation (2.4.12), we get

$$E(X_s,X_{s'})=D_{s,s':n}\left\{\sum_{i=0}^{s'-s-1}\sum_{j=0}^{n-s'}\binom{s'-s-1}{i}\binom{n-s'}{j}(-1)^{s'-s-1-i+j}\psi(s'-2-i,i+j)\right.$$

$$+\sum_{i=0}^{s-1}\sum_{j=0}^{s'-s-1}\binom{s-1}{i}\binom{s'-s-1}{j}(-1)^{i+j}\psi(s'-s-1+i-j,n-s'+j)$$

$$\left.+\sum_{i=0}^{s'-s-1}\sum_{j=0}^{s'-s-1-i}\binom{s'-s-1}{i}\binom{s'-s-1-i}{j}(-1)^{i+j}\psi(s-1+i,n-s'+j)\right\}$$

(2.4.14)

These distributions and moments of the order statistics are very useful in obtaining the optimal estimators for the scale and location parameters.

2.5. INFERENTIAL ASPECTS OF THE LEFT TRUNCATED THREE PARAMETER GENERALIZED GAUSSIAN DISTRIBUTION

Method of Moments

In this method, the theoretical moments of the population and the sample moments are equated correspondingly to deduce the estimators of the parameters.

Let $x_1, x_2, \ldots x_n$ be a sample of size n drawn from a population having the probability density function of the form given in equation (2.4.1), we have

$$f(x) = \frac{\beta}{\alpha} \frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \qquad A \geq \mu$$

$$f(x) = \frac{\beta}{\alpha} \frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \qquad A < \mu$$

This distribution is having three parameters μ, α and β. Hence we equate the first three moments of the population and the sample, which leads to the following equations.

$$\bar{x} = \mu + \alpha\left(\frac{\Gamma\left(\frac{2}{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right) \qquad A \geq \mu$$

$$\bar{x} = \mu + \alpha\left(\frac{\Gamma\left(\frac{2}{\beta}\right) + \gamma\left(\frac{2}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right) \qquad A < \mu \qquad (2.5.1)$$

and

$$s^2=\alpha^2\left(\left(\frac{\Gamma\left(\frac{3}{\beta}\right)-\gamma\left(\frac{3}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right)-\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right)^2\right) \qquad for\ A\geq\mu$$

$$s^2=\alpha^2\left(\left(\frac{\Gamma\left(\frac{3}{\beta}\right)+\gamma\left(\frac{3}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)-\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)^2\right) \qquad for\ A<\mu$$

(2.5.2)

$$\beta_2=\frac{3S_1^2\left(2S_2-S_1^2\right)+S_4-4S_1S_3}{\left(S_2-S_1^2\right)^2} \qquad for\ A<\mu$$

$$\beta_2=\frac{3P_1^2\left(2P_2-P_1^2\right)+P_4-4P_1P_3}{\left(P_2-P_1^2\right)^2} \qquad for\ A\geq\mu$$

(2.5.3)

where $\bar{x}=\frac{1}{n}\sum_{i=1}^{n}x_i$ and $s^2=\frac{1}{n}\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2$ and $\beta_2=\frac{n\sum_{i=1}^{n}(x_i-\bar{x})^4}{\left(\sum_{i=1}^{n}(x_i-\bar{x})^2\right)^2}$

For given values of A, solving the above equations (2.5.1), (2.5.2) and (2.5.3) simultaneously by using Newtons-Raphson method, we can obtain the estimators for the parameters μ, α and β.

Sample mean $\overline{X}$ is an unbiased estimator for the parameter μ.

Variance of $\overline{X}$ is

$$\operatorname{var}\left(\overline{X}\right)=\operatorname{var}\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)$$

$$=\frac{1}{n}\alpha^2\left[\left(\frac{\Gamma\left(\frac{3}{\beta}\right)-\gamma\left(\frac{3}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right)-\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right)^2\right] \qquad for\ A\geq\mu$$

$$=\frac{1}{n}\alpha^2\left[\left(\frac{\Gamma\left(\frac{3}{\beta}\right)+\gamma\left(\frac{3}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)-\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)^2\right] \qquad for\ A<\mu$$

(2.5.4)

Maximum Likelihood Method of Estimation

Case (i): For $A\geq\mu$

Let $x_1,x_2,...x_n$ be a sample of size n drawn from a population having the probability density function of the form is given in equation (2.2.4), then the likelihood function of the sample is

$$L=\left(\frac{\beta}{\alpha}\right)^n\prod_{i=1}^{n}\frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\int_0^\infty e^{-x}x^{\frac{1}{\beta}-1}dx-\int_0^{\left(\frac{A-\mu}{\alpha}\right)^{\beta}}e^{-x}x^{\frac{1}{\beta}-1}dx} \qquad (2.5.5)$$

Taking logarithms on both sides of (2.5.5), we get

$$LogL=n\log\beta-n\log\alpha-\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}-\log\sum_{i=1}^{n}\left(\int_0^\infty e^{-x_i}x_i^{\frac{1}{\beta}-1}dx-\int_0^{\left(\frac{A-\mu}{\alpha}\right)^{\beta}}e^{-x_i}x_i^{\frac{1}{\beta}-1}dx\right) \qquad (2.5.6)$$

Since, Log L is not differentiable with respect to β for all values in the range $\beta>0$, we obtain the estimate of β using the moment method of estimation in the equation (2.5.3).

For obtaining the maximum likelihood estimate of μ, we differentiate Log L with respect to μ and equate it to zero. But in equation (2.5.6) the function Log L is differentiable with respect to μ only when β is even. But in the case when β is odd we obtain the maximum likelihood estimator as in case of Laplace distribution (Keynes

(1911)) i.e., when β is odd, we find μ which maximizes log L. From equation (2.5.6) log L is maximum if $\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}$ is minimum when β is odd. The function $\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}$ is minimum only when μ is the median. Therefore the MLE of μ is the median of the distribution when β is odd. In case of β being even, we differentiate Log L with respect to μ and equate it to zero. This implies

$$\frac{\partial Log\,L}{\partial \mu}=\frac{\beta}{\alpha}\sum_{i=1}^{n}\frac{\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}}{\left(\frac{x_i-\mu}{\alpha}\right)}-\frac{\beta}{\alpha}\frac{e^{-\left(\frac{A-\mu}{\alpha}\right)^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}$$

Equating $\frac{\partial LogL}{\partial \mu}$ to zero, we get

$$\frac{\beta}{\alpha}\sum_{i=1}^{n}\frac{\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}}{\left(\frac{x_i-\mu}{\alpha}\right)}-\frac{\beta}{\alpha}\frac{e^{-\left(\frac{A-\mu}{\alpha}\right)^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}=0 \qquad (2.5.7)$$

To derive maximum likelihood estimator of α, consider the derivative by L with respect to α and equate it to zero. This implies

$$\frac{\partial Log\,L}{\partial \alpha}=-\frac{n}{\alpha}+\frac{\beta}{\alpha}\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}-\frac{\beta}{\alpha}\left(\frac{A-\mu}{\alpha}\right)\frac{e^{-\left(\frac{A-\mu}{\alpha}\right)^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}$$

Equating $\frac{\partial LogL}{\partial \alpha}$ to zero, we get

$$\frac{\beta}{\alpha}\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}-\frac{\beta}{\alpha}\left(\frac{A-\mu}{\alpha}\right)\frac{e^{-\left(\frac{A-\mu}{\alpha}\right)^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}-\frac{n}{\alpha}=0 \qquad (2.5.8)$$

Solving the equations (2.5.3), (2.5.7) and (2.5.8) simultaneously for μ, α and β. Using numerical methods like Newton Raphson's method, we can obtain the maximum likelihood estimators of the parameters μ, α and β.

Case (ii): For $A < \mu$

Let $x_1, x_2, \ldots x_n$ be a sample of size n drawn from a population having the probability density function of the form is given in equation (2.2.3), then the likelihood function of the sample is

$$L = \left(\frac{\beta}{\alpha}\right)^n \prod_{i=1}^{n} \frac{e^{-\left|\frac{x_i-\mu}{\alpha}\right|^\beta}}{\int_0^\infty e^{-x_i} x_i^{\frac{1}{\beta}-1} dx_i + \int_0^{\left|\frac{A-\mu}{\alpha}\right|^\beta} e^{-x_i} x_i^{\frac{1}{\beta}-1} dx_i} \tag{2.5.9}$$

Taking logarithms on both sides of (2.5.9), we get

$$LogL = n\log\beta - n\log\alpha - \sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^\beta - \log\sum_{i=1}^{n}\left(\int_0^\infty e^{-x_i} x_i^{\frac{1}{\beta}-1} dx_i + \int_0^{\left|\frac{A-\mu}{\alpha}\right|^\beta} e^{-x_i} x_i^{\frac{1}{\beta}-1} dx_i\right) \tag{2.5.10}$$

Since, Log L is not differentiable with respect to β for all values in the range $\beta > 0$, we obtain the estimate of β using the moment method of estimation in the equation (2.5.3).

For obtaining the maximum likelihood estimate of μ, we differentiate Log L with respect to μ and equate it to zero. But in equation (2.5.10) the function Log L is differentiable with respect to μ only when β is even. But in the case when β is odd we obtain the maximum likelihood estimator as in case of Laplace distribution (Keynes (1911)) i.e., when β is odd, we find μ which maximizes Log L. From eq. (2.5.6.a) log L is maximum if $\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^\beta$ is minimum when β is odd. The function $\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^\beta$ is minimum only when μ is the median. Therefore the MLE of μ is the median of the distribution when β is odd. In case of β being even, we differentiate Log L with respect to μ and equate it to zero. This implies

$$\frac{\partial Log\,L}{\partial\mu} = \frac{\beta}{\alpha}\sum_{i=1}^{n}\frac{\left|\frac{x_i-\mu}{\alpha}\right|^\beta}{\left(\frac{x_i-\mu}{\alpha}\right)} - \frac{\beta}{\alpha}\frac{\left|\frac{A-\mu}{\alpha}\right|}{\left(\frac{A-\mu}{\alpha}\right)}\frac{e^{-\left|\frac{A-\mu}{\alpha}\right|^\beta}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}$$

Equating $\frac{\partial LogL}{\partial\mu}$ to zero, we get

$$\frac{\beta}{\alpha}\sum_{i=1}^{n}\frac{\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}}{\left(\frac{x_i-\mu}{\alpha}\right)}-\frac{\beta}{\alpha}\frac{\left|\frac{A-\mu}{\alpha}\right|}{\left(\frac{A-\mu}{\alpha}\right)}\frac{e^{-\left|\frac{A-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}=0 \qquad (2.5.11)$$

To derive maximum likelihood estimator of α, consider the derivative of Log L with respect to α and equate it to zero. This implies

$$\frac{\partial Log\, L}{\partial\alpha}=-\frac{n}{\alpha}+\frac{\beta}{\alpha}\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}-\frac{\beta}{\alpha}\left|\frac{A-\mu}{\alpha}\right|\frac{e^{-\left|\frac{A-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}$$

Equating $\frac{\partial LogL}{\partial\alpha}$ to zero, we get

$$\frac{\beta}{\alpha}\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}-\frac{\beta}{\alpha}\left|\frac{A-\mu}{\alpha}\right|\frac{e^{-\left|\frac{A-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}-\frac{n}{\alpha}=0 \qquad (2.5.12)$$

Solving the equations (2.5.3), (2.5.11) and (2.5.12) simultaneously for μ, α and β. Using numerical methods like Newton Raphson's method, we can obtain the maximum likelihood estimators of the parameters μ, α and β.

APPLICATION

The proposed left truncated Generalized Gaussian Distribution (GGD) is having several applications in analyzing the data sets arising at quality control, Industrial experiments, Agricultural experiments and man power modeling. As an illustration consider the fitting of a left truncated GGD to the data on length of fish. The data on length of fish of catch is given in Table 1. From this data we observed that the length of fish is a non negative random variable. Hence, we consider the left truncation point is zero. By assuming that the variate under study follows a left truncated GGD of the form with the probability density function given in section 2 and solving the equations of maximum likelihood estimators using Mathcad, we obtained the estimates of the parameters as $\hat{\mu}=4.55$, $\hat{\alpha}=3.42$, $\hat{\beta}=3.0$. The estimated frequencies are also given in

Table 1. Using χ^2 test for goodness of fit of the distribution, the calculated χ^2 values is 0.6563, at 5% level of significance. Comparing the calculated χ^2 value with the critical value of $\chi^2 = 9.488$ with 4 degrees of freedom at 5% level of significance we observed that the data gives a good fit to the left truncated generalized Gaussian distribution.

TABLE 1

Observed and Expected Frequencies of Fish Length

Length of the Fish (cm)	Observed Frequencies	Expected Frequencies
1	6	3
2	11	8
3	15	13
4	17	16
5	17	17
6	15	16
7	11	14
8	6	9
9	2	4

CONCLUSION

In this chapter, we have introduced a left truncated generalized Gaussian distribution. GGD has lot of applications in analyzing several data sets as an alternative to the Gaussian distribution where the variable under study is lepty or platy or meso kurtic and symmetric. A left truncated GGD includes GGD as limiting case as when the truncated point tends to infinite. The various distributional properties such as distribution function, moments, hazard function, and survival function are derived. It is observed that the hazard function is sometimes increases and decreases depending upon the truncation parameter. It also includes a J-type distribution when the truncation point is greater than its location parameter. The order statistics of the variate under study are also derived. The joint moments of the r^{th} order statistics are obtained. The method of obtaining maximum likelihood estimators of the parameters using Mathcad is presented. The asymptotic properties of the estimators are studied. The utility of the distribution for fitting the data on length of fish is given. This distribution is useful for analyzing several data sets in management sciences, finance, quality control and agricultural experiments. The values of the cumulative distribution function of the proposed left truncated GGD are useful for further statistical inferences. It is also possible to obtain other inferential aspects such as testing of hypothesis.

Annex – I

Standard table for a left truncated generalized Gaussian distribution

(μ=0, α=1, β=2, A= - 4)

Z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3	0.000011	0.00001	0.00001	0.000009	0.000009	0.000008	0.000008	0.000007	0.000007	0.000006
-2.9	0.000021	0.000019	0.000018	0.000017	0.000016	0.000015	0.000014	0.000013	0.000013	0.000012
-2.8	0.000037	0.000035	0.000033	0.000031	0.00003	0.000028	0.000026	0.000025	0.000023	0.000022
-2.7	0.000067	0.000063	0.00006	0.000056	0.000053	0.00005	0.000047	0.000045	0.000042	0.00004
-2.6	0.000118	0.000112	0.000106	0.0001	0.000094	0.000089	0.000084	0.00008	0.000075	0.000071
-2.5	0.000203	0.000193	0.000183	0.000173	0.000164	0.000155	0.000147	0.000139	0.000132	0.000125
-2.4	0.000344	0.000327	0.00031	0.000295	0.00028	0.000265	0.000252	0.000239	0.000226	0.000215
-2.3	0.000572	0.000544	0.000517	0.000492	0.000468	0.000445	0.000423	0.000402	0.000382	0.000362
-2.2	0.000931	0.000888	0.000846	0.000806	0.000768	0.000731	0.000696	0.000663	0.000631	0.000601
-2.1	0.00149	0.001423	0.001358	0.001296	0.001237	0.001181	0.001126	0.001074	0.001025	0.000977
-2	0.002339	0.002238	0.00214	0.002047	0.001957	0.001871	0.001788	0.001709	0.001633	0.00156
-1.9	0.003605	0.003455	0.003311	0.003172	0.003039	0.00291	0.002787	0.002668	0.002554	0.002444
-1.8	0.005455	0.005238	0.005028	0.004827	0.004632	0.004444	0.004264	0.00409	0.003922	0.00376
-1.7	0.008105	0.007796	0.007499	0.007211	0.006933	0.006664	0.006405	0.006155	0.005913	0.00568
-1.6	0.011826	0.011397	0.010981	0.010579	0.010189	0.009812	0.009448	0.009095	0.008754	0.008424
-1.5	0.016947	0.016362	0.015793	0.015242	0.014707	0.014189	0.013686	0.013199	0.012726	0.012269
-1.4	0.023857	0.023074	0.022312	0.021571	0.020852	0.020152	0.019473	0.018814	0.018173	0.017551
-1.3	0.032996	0.031968	0.030967	0.029992	0.029043	0.028119	0.027219	0.026344	0.025492	0.024663
-1.2	0.044843	0.043522	0.042233	0.040975	0.039747	0.03855	0.037382	0.036243	0.035133	0.034051
-1.1	0.059897	0.058233	0.056606	0.055015	0.053459	0.051938	0.050452	0.049	0.047581	0.046196
-1	0.07865	0.076595	0.074581	0.072608	0.070675	0.068782	0.066928	0.065113	0.063337	0.061598
-0.9	0.101546	0.099059	0.096616	0.094218	0.091864	0.089555	0.087288	0.085065	0.082884	0.080746
-0.8	0.12895	0.125998	0.123095	0.120238	0.117429	0.114666	0.11195	0.10928	0.106656	0.104078
-0.7	0.161099	0.157667	0.154283	0.150948	0.147661	0.144422	0.141232	0.138089	0.134995	0.131948
-0.6	0.198072	0.194159	0.190294	0.186477	0.182707	0.178985	0.175312	0.171686	0.168109	0.16458
-0.5	0.23975	0.235378	0.231051	0.226768	0.22253	0.218338	0.214192	0.210092	0.206039	0.202032
-0.4	0.285804	0.281015	0.276266	0.271557	0.266887	0.262259	0.257672	0.253127	0.248625	0.244166
-0.3	0.335687	0.330546	0.325437	0.320361	0.315318	0.310309	0.305335	0.300397	0.295495	0.290631
-0.2	0.388649	0.383239	0.377852	0.372489	0.36715	0.361837	0.35655	0.351291	0.34606	0.340858
-0.1	0.443769	0.438189	0.432621	0.427066	0.421526	0.416002	0.410494	0.405004	0.399532	0.39408
0	0.5	0.494358	0.488718	0.483079	0.477444	0.471814	0.466189	0.460571	0.454961	0.44936
0.1	0.556231	0.55064	0.545039	0.539429	0.533811	0.528186	0.522556	0.516921	0.511282	0.505642

Z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.2	0.611351	0.60592	0.600468	0.594996	0.589506	0.583998	0.578474	0.572934	0.567379	0.561811
0.3	0.664313	0.659142	0.65394	0.648709	0.64345	0.638163	0.63285	0.627511	0.622148	0.616761
0.4	0.714196	0.709369	0.704505	0.699603	0.694665	0.689691	0.684682	0.679639	0.674563	0.669454
0.5	0.76025	0.755834	0.751375	0.746873	0.742328	0.737741	0.733113	0.728443	0.723734	0.718985
0.6	0.801928	0.797968	0.793961	0.789908	0.785808	0.781662	0.77747	0.773232	0.768949	0.764622
0.7	0.838901	0.83542	0.831891	0.828314	0.824688	0.821015	0.817293	0.813523	0.809706	0.805841
0.8	0.87105	0.868052	0.865005	0.861911	0.858768	0.855578	0.852339	0.849052	0.845717	0.842333
0.9	0.898454	0.895922	0.893344	0.89072	0.88805	0.885334	0.882571	0.879762	0.876905	0.874002
1	0.92135	0.919254	0.917116	0.914935	0.912712	0.910445	0.908136	0.905782	0.903384	0.900941
1.1	0.940103	0.938402	0.936663	0.934887	0.933072	0.931218	0.929325	0.927392	0.925419	0.923405
1.2	0.955157	0.953804	0.952419	0.951	0.949548	0.948062	0.946541	0.944985	0.943394	0.941767
1.3	0.967004	0.965949	0.964867	0.963757	0.962618	0.96145	0.960253	0.959025	0.957767	0.956478
1.4	0.976143	0.975337	0.974508	0.973656	0.972781	0.971881	0.970957	0.970008	0.969033	0.968032
1.5	0.983053	0.982449	0.981827	0.981186	0.980527	0.979848	0.979148	0.978429	0.977688	0.976926
1.6	0.988174	0.987731	0.987274	0.986801	0.986314	0.985811	0.985293	0.984758	0.984207	0.983638
1.7	0.991895	0.991576	0.991246	0.990905	0.990552	0.990188	0.989811	0 989421	0.989019	0.988603
1.8	0.994545	0.99432	0.994087	0.993845	0.993595	0.993336	0.993067	0.992789	0.992501	0.992204
1.9	0.996395	0.99624	0.996078	0.99591	0.995736	0.995556	0.995368	0.995173	0.994972	0.994762
2	0.997661	0.997556	0.997446	0.997332	0.997213	0.99709	0.996961	0.996828	0.996689	0.996545
2.1	0.99851	0.99844	0.998367	0.998291	0.998212	0.998129	0.998043	0.997953	0.99786	0.997762
2.2	0.999069	0.999023	0.998975	0.998926	0.998874	0.998819	0.998763	0.998704	0.998642	0.998577
2.3	0.999428	0.999399	0.999369	0.999337	0.999304	0.999269	0.999232	0.999194	0.999154	0.999112
2.4	0.999656	0.999638	0.999618	0.999598	0.999577	0.999555	0.999532	0.999508	0.999483	0.999456
2.5	0.999797	0.999785	0.999774	0.999761	0.999748	0.999735	0.99972	0.999705	0.99969	0.999673
2.6	0.999882	0.999875	0.999868	0.999861	0.999853	0.999845	0.999836	0.999827	0.999817	0.999807
2.7	0.999933	0.999929	0.999925	0.99992	0.999916	0.999911	0.999906	0.9999	0.999894	0.999888
2.8	0.999963	0.99996	0.999958	0.999955	0.999953	0.99995	0.999947	0.999944	0.99994	0.999937
2.9	0.999979	0.999978	0.999977	0.999975	0.999974	0.999972	0.99997	0.999969	0.999967	0.999965
3	0.999989	0.999988	0.999987	0.999987	0.999986	0.999985	0.999984	0.999983	0.999982	0.999981