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Research Article

On a Left Truncated Generalized Gaussian Distribution

K. Anithakumari[1*], K. Srinivas Rao[2], and P. R. S. Reddy[1]

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Abstract

The generalized Gaussian distribution is useful in analyzing the data sets arising at places like Image processing, Signal processing, Speech recognition, Statistical Quality Control, Industrial experimentation, and Biological experiments. In this paper, a Left Truncated Generalized Gaussian distribution is introduced. The various distributional properties such as the distribution function, the four moments, skewness, kurtosis, hazard function, survival function are derived. The distribution of the r^{th} order statistics, the median distribution also obtained. Some inferential aspects of the distribution are studied. An application of the suggested model is given with respect to the data on the length of fish in a catch.

Keywords: Left Truncate Distribution; Generalized Gaussian Distribution; Distributional Properties; Order Statistics

1. Introduction

The Gaussian distribution has found utility in a wide variety of fields. Most of the statistical inferences are developed based on Gaussian distribution due to its symmetric and bell shaped nature. There are some situations where the Gaussian assumption may not suit well to the data sets for which the random variable under study is lepto kurtic or platy kurtic even through it is symmetric and bell shaped. For this sort of situation a Generalized Gaussian distribution (GGD) was introduced as an alternative which includes platy-lepty-meso kurtic symmetric distributions. The generalized Gaussian distribution also includes the well known Gaussian distribution and Laplace distribution as particular cases. The generalized Gaussian distribution gained lot of importance in modeling the phenomenon arising at places like Atmospheric Science, Signal processing, Image processing and Speech recognition etc. (Varanasi et al. (1989), Choi, S. Cichocki et al. (2000), Wu et al. (1998), Armando et al. (2001)).

*Corresponding e-mail: anitha.kumari37@gmail.com
1 Department of Statistics, Sri Venkateswara University, Tirupati, Andhra Pradesh, India
2 Department of Statistics, Andhra University, Visakhapatnam, Andhra Pradesh, India

According to Farvardin et al. (1987), the generalized Gaussian distribution was first studied by Algazi et al. (1965) as a model useful for broad tailed processes. Farvardin et al. (1987) and Varanasi and Behnaam Aazhang (1989) have studied the large sample and small sample properties of estimators of the parameters involved in the generalized Gaussian distribution. Armando et al. (2001) developed a method to estimate the shape parameter in the generalized Gaussian distribution. Song (2006) suggested a method of estimating the shape parameter of generalized Gaussian distribution. Yunfei Chen et al. (2009) have studied the properties of moment based estimator for the shape parameter of generalized Gaussian distribution. Kurkin et al. (2010) studied on sample meridian of a location parameter using a generalized Gaussian distribution. Wakisaka et al. (2012) discussed a method for stable estimation of the kurtosis of a speech power spectrum. Song (2013) studied the sampling estimation of the sampling distributions of the maximum likelihood (ML) estimators of parameters in GGD under large sample theory. Guo et al. (2013) studied the estimation of the location parameters of GGD with a given shape parameters.

Using the generalized Gaussian distribution Bouman and Sauer (1993) considered the generalized Gaussian markov random field model (GGMRF), which was useful for image reconstruction. Hui and Neuhoff (1994) have used it in studying the asymptotic characteristics of optimum uniform scalar quantizers. Conte et al. (1995) established the use this distribution for detection of signals. Muhammed et al. (1996) studied the use of bi-Variate generalized Gaussian models in image and signal analysis. Batalamma et al. (1997) have used a mixture of generalized Gaussian distribution for image analysis. Sun et al. (1997) studied a lost function based on generalized Gaussian distribution for adaptive learning rules. Vinod and Bora (2004) have outlined an algorithm for collusion resistant multi-bit video watermarking. Lam (2004) has studied the statistical modeling of the wavelet modeling of using generalized Gaussian distribution. Srinivas et al. (2007) studied image segmentation methods using generalized Gaussian distribution. Fraysse et al. (2008) studied operational rate-distortion through entropy for uniform scalar optimization at low and high resolution images using generalized Gaussian distribution. Xu Mankun, et al. (2009) studied image processing through 2D generalized Gaussian distribution. Kumatani et al. (2010) have introduced a beamforming method for distant speech recognition with GGD. Duan et al. (2012) studied a systematical symmetrical distribution using GGD. Venkatraman (2013) have applied GGD in Signal processing. Zhang et al. (2013) studied the covariance estimation of a multivariate GGD. Dat et al. (2014) utilized the GGD in image analysis. Roenko et al. (2014) studied the shape parameter procedures for GGD. Yin et al. (2014) discussed the utility of the GGD in hyper spectrum reconstruction.

In all these papers, they considered that the variable under study has an infinite range i.e., the random variable can in theory assume any value over the range $-\infty$ *to* $+\infty$. But in many practical situations the random variables under study are constrained. For example, in inventory modeling the life time of commodity is non-negative. i.e., the lower limit of the variable is constrained to zero. Similarly exists in many other applications in quality control and financial modeling. There is a minimum warranty / grantee period for the risk to happen. Hence, the variables will have the lower bound. This practical situation can be better analyzed by considering the truncated distributions. In addition to this the majority of work reports in a literature regarding the estimation of the parameters of the truncated generalized Gaussian distribution are based on the samples of truncated nature. Little work has been done in terms of describing and analyzing the properties of truncated generalized Gaussian distribution. This motivated to develop the left truncated generalized Gaussian distribution which is useful for analyzing the data sets arising at image processing, speech recognition, quality control and financial modeling.

This paper presents the probability density function and distribution function of left truncated generalized Gaussian distribution. The cumulative distribution function (Area under probability curve) tables for standardized truncated generalized Gaussian distribution are computed and presented. The various distributional properties are derived. The distributions of the r^{th} order statistics, maximum order distribution, and median distribution are derived. The inferential aspects of the parameters are discussed using likelihood function. The goodness of fit of the proposed distribution is illustrated with the data on length of fish.

2. Left Truncated Generalized Gaussian Distribution

A Continuous random variable X is said to be a three parameter generalized Gaussian distribution if its probability density function (p.d.f) is of the form

$$f(x;\mu,\alpha,\beta)=\frac{\beta}{2\alpha\Gamma(\frac{1}{\beta})}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}} \quad ; \quad -\infty<x<\infty; \quad -\infty<\mu<\infty; \quad \alpha>0; \quad \beta>0$$

Consider that the range variable is finite say (A, ∞). Then the probability density function (p.d.f) of the left truncated three-parameter generalized Gaussian distribution is

$$f(x)=\frac{f(x;\mu,\alpha,\beta)}{1-F(A)}; \qquad A<x<\infty; \qquad A<\mu<\infty; \qquad \alpha>0; \qquad \beta>0$$

where $$F(A)=\int_{-\infty}^{A}\frac{\beta}{2\alpha\Gamma(\frac{1}{\beta})}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}dx \qquad (1)$$

The lower and upper truncation points are A and ∞ respectively.

Hence, the probability density function of three parameter left truncated generalized Gaussian distribution is

$$f(x)=\frac{\beta}{\alpha}\frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \qquad \text{for } A<\mu \qquad (2)$$

$$f(x)=\frac{\beta}{\alpha}\frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \quad \text{for } A\geq\mu \qquad (3)$$

3. Distributional Properties

The various distributional properties of the three-parameter left truncated generalized Gaussian distribution are discussed in this section.

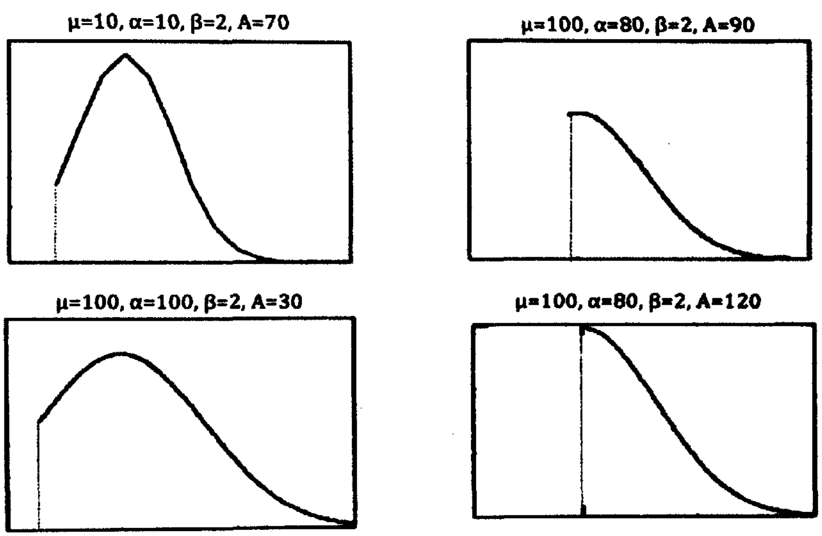

Fig. 1. The frequency curves for different values of the left truncated generalized Gaussian distribution.

The distribution function of X is given by

$$F(x)=\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \qquad for \quad A<\mu \qquad (4)$$

$$F(x)=\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \qquad for \quad A\geq\mu \qquad (5)$$

where, $\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)$ is an incomplete gamma function.

Different shapes of the frequency curves for given values of the parameter are shown in Figure 1

(i) The mean of the distribution is

$$E(X)=\mu+\alpha\left(\frac{\left(\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right) \qquad for \quad A<\mu \qquad (6)$$

$$E(X)=\mu+\alpha\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right) \qquad for \quad A\geq\mu \qquad (7)$$

(ii) The median M of the distribution can be obtained by solving the following equations (8) and (9).

$$\frac{\gamma\left(\frac{1}{\beta},\left(\frac{M-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\left(\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)\right)}=\frac{1}{2} \qquad for \quad A<\mu \qquad (8)$$

$$\frac{\gamma\left(\frac{1}{\beta},\left(\frac{M-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\left(\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)\right)}=\frac{1}{2} \qquad for\ A\geq\mu \qquad (9)$$

(iii) The mode of the distribution can be obtained by solving the following equation (10) for x.

$$f'(x)=\frac{-\frac{\beta}{\alpha}\left|\frac{x-\mu}{\alpha}\right|^{\beta-1}\left|\frac{x-\mu}{\alpha}\right|}{\left(\frac{x-\mu}{\alpha}\right)}=0 \qquad (10)$$

This model is a uni-model distribution.

(iv)The raw moments of the distribution are

$$\mu_r'=\sum_{j=0}^{r}\binom{r}{j}\alpha^{j}\mu^{r-j}\left(\frac{\Gamma\left(\frac{j+1}{\beta}\right)+\gamma\left(\frac{j+1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right) \qquad for\ A<\mu \qquad (11)$$

$$\mu_r'=\sum_{j=0}^{r}\binom{r}{j}\alpha^{j}\mu^{r-j}\left(\frac{\Gamma\left(\frac{j+1}{\beta}\right)-\gamma\left(\frac{j+1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right) \qquad for\ A\geq\mu \qquad (12)$$

(v) The central moments of this distribution are

$$\mu_r=\sum_{j=0}^{r}\binom{r}{j}\alpha^{j}(-D)^{r-j}\left(\frac{\Gamma\left(\frac{j+1}{\beta}\right)+\gamma\left(\frac{j+1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right) \qquad for\ A<\mu \qquad (13)$$

$$\mu_r = \sum_{j=0}^{r} \binom{r}{j} \alpha^j (-D)^{r-j} \left(\frac{\Gamma\left(\frac{j+1}{\beta}\right) - \gamma\left(\frac{j+1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right) \qquad for\ A \geq \mu \qquad (14)$$

where $D = \alpha \left(\frac{\Gamma\left(\frac{2}{\beta}\right) + \gamma\left(\frac{2}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right)$ *for* $A < \mu$

where $D = \alpha \left(\frac{\Gamma\left(\frac{2}{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right)$ *for* $A \geq \mu$

(vi)The hazard rate function of the distribution is $h(x) = \frac{f(x)}{1-F(x)}$

$$h(x) = \left(\frac{\frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{x-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right) \qquad for\ A < \mu \qquad (15)$$

$$h(x) = \left(\frac{\frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right) - \left(\gamma\left(\frac{1}{\beta}, \left(\frac{x-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)\right)} \right) \qquad for\ A > \mu \qquad (16)$$

(vii) The survival rate function S(x) is $S(x)=1-F(x)$

$$S(x)=1-\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \quad for\ A<\mu \tag{17}$$

$$S(x)=1-\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \quad for\ A\geq\mu \tag{18}$$

4. Order Statistics of Left Truncated Three-Parameter Generalized Gaussian Distribution

The simple explicit form of the distribution function as given in equation (4 & 5) leads us to derive the order statistics connected with this left truncated three parameter generalized Gaussian distribution.

$$f(x)=\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \quad A<\mu$$

$$f(x)=\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \quad A\geq\mu \tag{19}$$

Let $X_{1:n}\leq X_{2:n}\leq\ldots\leq X_{n:n}$ denote the order statistics obtained from a random sample of size n from the generalized truncated Gaussian distribution having the probability density function of the form given in (19).

The probability density function of sth order statistics is given by (David (1981)),

$$f_{s:n}(x)=D_{s:n}[F(x)]^{s-1}[1-F(x)]^{n-s}f(x)$$

where $D_{s:n} = \dfrac{n!}{(s-1)!(n-s)!}$ (20)

Substituting f(x) and F(x) values given in this equation (19) and (5) in the equation (20), we get the probability density function of the s[th] order statistics is given by

Case (i): For A ≥ μ

For A < x < 0

$$f_{s:n}(x) = D_{s:n} \frac{\frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \sum_{q=0}^{s-1} \binom{s-1}{q} (-1)^{q} \left[\frac{\gamma\left(\frac{1}{\beta}, \left(\frac{x-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{n}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right]^{n+q-s}$$

For 0 < x < ∞

$$f_{s:n}(x) = D_{s:n} \frac{\frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \sum_{q=0}^{n-s} \binom{n-s}{q} (-1)^{q} \left[\frac{\gamma\left(\frac{1}{\beta}, \left(\frac{x-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right]^{s+q-1} \quad (21)$$

Substituting f(x) and F(x) values given in this equation (19) and (4) in the equation (20), we get the probability density function of the s[th] order statistics is given by

Case (ii): For A < μ

For A < x < 0

$$f_{s:n}(x) = D_{s:n} \frac{\frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \sum_{q=0}^{s-1} \binom{s-1}{q} (-1)^{q} \left[\frac{\gamma\left(\frac{1}{\beta}, \left(\frac{x-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right]^{n+q-s}$$

For 0 < x < ∞

$$f_{s:n}(x) = D_{s:n} \frac{\frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \sum_{q=0}^{n-s} \binom{n-s}{q} (-1)^{q} \left[\frac{\gamma\left(\frac{1}{\beta}, \left(\frac{x-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right]^{s+q-1} \quad (22)$$

The probability density function of the first order statistics is obtained by substituting $s = 1$ in the equation (21).

Hence, case (i): For A ≥ μ,
For A < x < 0

$$f_{1:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right]^{n+q-1}$$

For 0 < x < ∞

$$f_{1:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right]^{q}$$

The probability density function of the first order statistics is obtained by substituting $s=1$ in the equation (22)

Hence, case (ii): For A < μ,
For A < x < 0

$$f_{1:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right]^{n+q-1}$$

For 0 < x < ∞

$$f_{s:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right]^{q} \quad (23)$$

The probability density function of the n^{th} order statistics is obtained by substituting $s=n$ in equation (21)

Case (i): For A ≥ μ
For A < x < 0

$$f_{n:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right]^{q}$$

For $0 < x < \infty$

$$f_{n:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right]^{n+q-1} \tag{24}$$

The probability density function of the n^{th} order statistics is obtained by substituting $s = n$ in equation (22)

Case (ii): for $A < \mu$

For $A < x < 0$

$$f_{n:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^{q}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right]^{q}$$

For $0 < x < \infty$

$$f_{n:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right]^{n+q-1} \tag{25}$$

Distribution of the Median

Let n be odd. The distribution of the median is obtained by substituting $s = \frac{n+1}{2}$ in equation (21) and equation (22).

For $A < x < 0$

$$f_M(x)=\left(\frac{\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^2}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^\beta\right)}\right)\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q}(-1)^q\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^\beta\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^\beta\right)}\right]^{\frac{n-1}{2}+q} \qquad A\geq\mu$$

$$f_M(x)=\left(\frac{\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^2}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}\right)\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q}(-1)^q\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^\beta\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}\right]^{\frac{n-1}{2}+q} \qquad A<\mu$$

(26)

For $0 < x < \infty$

$$f_M(x)=\left(\frac{\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^2}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^\beta\right)}\right)\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q}(-1)^q\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^\beta\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^\beta\right)}\right]^{\frac{n-1}{2}+q} \qquad A\geq\mu$$

$$f_M(x)=\left(\frac{\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^2}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}\right)\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q}(-1)^q\left[\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^\beta\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}\right]^{\frac{n-1}{2}+q} \qquad A<\mu$$

(27)

5. Inferential Aspects of the Left Truncated Three-Parameter Generalized Gaussian Distribution

In this method, the theoretical moments of the population and the sample moments are equated correspondingly to deduce the estimators of the parameters.

Let $x_1, x_2, \ldots x_n$ be a sample of size n drawn from a population having the probability density function of the form given in equation (2 & 3) we have

$$f(x) = \frac{\beta}{\alpha} \frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \qquad A \geq \mu$$

$$f(x) = \frac{\beta}{\alpha} \frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \qquad A < \mu$$

This distribution is having three parameters μ, α and β. Hence we equate the first three moments of the population and the sample, which leads to the following equations.

$$\bar{x} = \mu + \alpha\left(\frac{\Gamma\left(\frac{2}{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right) \qquad A \geq \mu$$

$$\bar{x} = \mu + \alpha\left(\frac{\Gamma\left(\frac{2}{\beta}\right) + \gamma\left(\frac{2}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right) \qquad A < \mu \qquad (28)$$

And

$$s^2=\alpha^2\left(\left(\frac{\Gamma\left(\frac{3}{\beta}\right)-\gamma\left(\frac{3}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right)-\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right)^2\right) \quad \text{for } A\geq\mu$$

$$s^2=\alpha^2\left(\left(\frac{\Gamma\left(\frac{3}{\beta}\right)+\gamma\left(\frac{3}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)-\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)^2\right) \quad \text{for } A<\mu \qquad (29)$$

$$\beta_2=\frac{3S_1^2\left(2S_2-S_1^2\right)+S_4-4S_1S_3}{\left(S_2-S_1^2\right)^2} \quad \text{for } A<\mu$$

$$\beta_2=\frac{3P_1^2\left(2P_2-P_1^2\right)+P_4-4P_1P_3}{\left(P_2-P_1^2\right)^2} \quad \text{for } A\geq\mu \qquad (30)$$

where $\bar{x}=\frac{1}{n}\sum_{i=1}^{n}x_i$ and $s^2=\frac{1}{n}\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2$ and $\beta_2=\frac{n\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^4}{\left(\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2\right)^2}$

For given values of A, solving the above equations (28), (29) and (30) simultaneously by using Newtons-Raphson method, we can obtain the estimators for the parameters μ, α and β.
Sample mean $\overline{X}$ is an unbiased estimator for the parameter μ.
Variance of $\overline{X}$ is

$$\operatorname{var}\left(\overline{X}\right)=\operatorname{var}\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)$$
$$=\frac{1}{n}\alpha^2\left(\left(\frac{\Gamma\left(\frac{3}{\beta}\right)-\gamma\left(\frac{3}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right)-\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right)^2\right) \quad \text{for } A\geq\mu$$

$$=\frac{1}{n}\alpha^2\left(\left(\frac{\Gamma\left(\frac{3}{\beta}\right)+\gamma\left(\frac{3}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)-\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right)^2\right) \quad \text{for } A<\mu \tag{31}$$

6. Maximum Likelihood Method of Estimation

Let $x_1, x_2, \ldots x_n$ be a sample of size n drawn from a population having the probability density function of the form is given in equation (3), then the likely hood function of the sample is

$$L=\left(\frac{\beta}{\alpha}\right)^n\prod_{i=1}^{n}\frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\int_0^{\infty}e^{-x}x^{\frac{1}{\beta}-1}dx-\int_0^{\left(\frac{A-\mu}{\alpha}\right)^{\beta}}e^{-x}x^{\frac{1}{\beta}-1}dx} \tag{32}$$

Taking logarithms on both sides of (32), we get

$$LogL=n\log\beta-n\log\alpha-\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}-\log\sum_{i=1}^{n}\left(\int_0^{\infty}e^{-x_i}x_i^{\frac{1}{\beta}-1}dx-\int_0^{\left(\frac{A-\mu}{\alpha}\right)^{\beta}}e^{-x_i}x_i^{\frac{1}{\beta}-1}dx\right) \tag{33}$$

Since, Log L is not differentiable with respect to β for all values in the range $\beta>0$, we obtain the estimate of β using the moment method of estimation using the equation (30).

For obtaining the maximum likelihood estimate of μ, we differentiate log L with respect to μ and equate it to zero. But in equation (33) the function log L is differentiable with respect to μ only when β is even. But in the case when β is odd we obtain the maximum likelihood estimator as in case of Laplace distribution (Keynes (1911)) i.e., when β is odd, we find μ which maximizes log L.

From eq. (33) log L is maximum if $\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}$ is minimum when β is odd. The function $\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}$ is minimum only when μ is the median. Therefore the MLE of μ is the median of the distribution when β is odd. In case of β being even, we differentiate log L with respect to μ and equate it to zero. This implies

$$\frac{\beta}{\alpha}\frac{\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}}{\left(\frac{x_i-\mu}{\alpha}\right)}-\frac{\beta}{\alpha}\frac{e^{-\left(\frac{A-\mu}{\alpha}\right)^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}=0 \quad (34)$$

To derive maximum likelihood estimator of α, consider the derivative of L w. r. to α and equate it to zero. This implies

$$\frac{n}{\alpha}-\frac{\beta}{\alpha}\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}+\frac{\beta}{\alpha}\left(\frac{A-\mu}{\alpha}\right)\frac{e^{-\left(\frac{A-\mu}{\alpha}\right)^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}=0 \quad (35)$$

Solving the equations (30), (34) and (35) simultaneously for μ, α and β. Using numerical methods like Newton Raphson's method, we can obtain the maximum likelihood estimators of the parameters μ, α and β.

7. Application

The proposed left truncated generalized Gaussian distribution (GGD) is having several applications in analyzing the data sets arising at quality control, Industrial experiments, Agricultural experiments and man power modeling. As an illustration consider the fitting of a left truncated GGD to the data on length of fish. The data on length of fish of catch is given in Table 1.

Table 1 Observed and Expected Frequencies of Fish Length

Length of the Fish (cm)	Observed Frequencies	Expected Frequencies
1	6	3
2	11	8
3	15	13
4	17	16
5	17	17
6	15	16
7	11	14
8	6	9
9	2	4

From this data we observed that the length of fish is a non negative random variable. Hence, we consider the left truncation point is zero. By assuming that the Variate under study follows a left truncated GGD of the form distribution with the probability density function given in section 2 solving the equations of maximum likelihood estimators using Mathcad, we obtained the estimates of the parameters as $\hat{\mu}=4.55, \hat{\alpha}=3.42, \hat{\beta}=3.0$. The estimated frequencies are also given in Table 1. Using χ^2 test for goodness of fit of the distribution, the calculated χ^2 values is 0.6563, at 5% level of significance. Comparing the calculated χ^2 value with the critical value of $\chi^2=9.488$ at 4 degrees of freedom and 5% level of significance we observed that the data gives a good fit to the left truncated generalized Gaussian distribution.

8. Conclusion

In this paper, we have introduced a left truncated Generalized Gaussian Distribution. GGD has lot of applications in analyzing several data sets as an alternative to the Gaussian distribution where the variable under study is lepty or platy or meso kurtic and symmetric. A left truncated GGD includes GGD as limiting case as when the truncated point tends to infinite. The various distributional properties such as distribution function, moments, hazard function, and survival function are derived. It is observed that the hazard function is sometimes increases and decreases depending upon the truncation parameter. It also includes a J-type distribution when the truncation point is greater than its location parameter. The order statistics of the Variate under study are also derived. The joint moments of the r^{th} order statistics are obtained. The method of obtaining maximum likelihood estimators of the parameters using Mathcad code is presented. The asymptotic properties of the estimators are studied. The utility of the distribution for fitting the data on length of fish is given. This distribution is useful for analyzing several data sets in management sciences, finance, quality control and agricultural experiments. The values of the cumulative distribution function of the proposed left truncated GGD are useful for further statistical inferences. It is also possible to obtain other inferential aspects such as testing of hypothesis.

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Annex 1 Standard table for a left truncated generalized Gaussian distribution

z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3	0.000011	0.00001	0.00001	0.000009	0.000009	0.000008	0.000008	0.000007	0.000007	0.000006
-2.9	0.000021	0.000019	0.000018	0.000017	0.000016	0.000015	0.000014	0.000013	0.000013	0.000012
-2.8	0.000037	0.000035	0.000033	0.000031	0.00003	0.000028	0.000026	0.000025	0.000023	0.000022
-2.7	0.000067	0.000063	0.00006	0.000056	0.000053	0.00005	0.000047	0.000045	0.000042	0.00004
-2.6	0.000118	0.000112	0.000106	0.0001	0.000094	0.000089	0.000084	0.00008	0.000075	0.000071
-2.5	0.000203	0.000193	0.000183	0.000173	0.000164	0.000155	0.000147	0.000139	0.000132	0.000125
-2.4	0.000344	0.000327	0.00031	0.000295	0.00028	0.000265	0.000252	0.000239	0.000226	0.000215
-2.3	0.000572	0.000544	0.000517	0.000492	0.000468	0.000445	0.000423	0.000402	0.000382	0.000362
-2.2	0.000931	0.000888	0.000846	0.000806	0.000768	0.000731	0.000696	0.000663	0.000631	0.000601
-2.1	0.00149	0.001423	0.001358	0.001296	0.001237	0.001181	0.001126	0.001074	0.001025	0.000977
-2	0.002339	0.002238	0.00214	0.002047	0.001957	0.001871	0.001788	0.001709	0.001633	0.00156
-1.9	0.003605	0.003455	0.003311	0.003172	0.003039	0.00291	0.002787	0.002668	0.002554	0.002444
-1.8	0.005455	0.005238	0.005028	0.004827	0.004632	0.004444	0.004264	0.00409	0.003922	0.00376
-1.7	0.008105	0.007796	0.007499	0.007211	0.006933	0.006664	0.006405	0.006155	0.005913	0.00568
-1.6	0.011826	0.011397	0.010981	0.010579	0.010189	0.009812	0.009448	0.009095	0.008754	0.008424
-1.5	0.016947	0.016362	0.015793	0.015242	0.014707	0.014189	0.013686	0.013199	0.012726	0.012269
-1.4	0.023857	0.023074	0.022312	0.021571	0.020852	0.020152	0.019473	0.018814	0.018173	0.017551
-1.3	0.032996	0.031968	0.030967	0.029992	0.029043	0.028119	0.027219	0.026344	0.025492	0.024663
-1.2	0.044843	0.043522	0.042233	0.040975	0.039747	0.03855	0.037382	0.036243	0.035133	0.034051
-1.1	0.059897	0.058233	0.056606	0.055015	0.053459	0.051938	0.050452	0.049	0.047581	0.046196
-1	0.07865	0.076595	0.074581	0.072608	0.070675	0.068782	0.066928	0.065113	0.063337	0.061598
-0.9	0.101546	0.099059	0.096616	0.094218	0.091864	0.089555	0.087288	0.085065	0.082884	0.080746
-0.8	0.12895	0.125998	0.123095	0.120238	0.117429	0.114666	0.11195	0.10928	0.106656	0.104078
-0.7	0.161099	0.157667	0.154283	0.150948	0.147661	0.144422	0.141232	0.138089	0.134995	0.131948
-0.6	0.198072	0.194159	0.190294	0.186477	0.182707	0.178985	0.175312	0.171686	0.168109	0.16458
-0.5	0.23975	0.235378	0.231051	0.226768	0.22253	0.218338	0.214192	0.210092	0.206039	0.202032
-0.4	0.285804	0.281015	0.276266	0.271557	0.266887	0.262259	0.257672	0.253127	0.248625	0.244166
-0.3	0.335687	0.330546	0.325437	0.320361	0.315318	0.310309	0.305335	0.300397	0.295495	0.290631
-0.2	0.388649	0.383239	0.377852	0.372489	0.36715	0.361837	0.35655	0.351291	0.34606	0.340858
-0.1	0.443769	0.438189	0.432621	0.427066	0.421526	0.416002	0.410494	0.405004	0.399532	0.39408
0	0.5	0.494358	0.488718	0.483079	0.477444	0.471814	0.466189	0.460571	0.454961	0.44936
0.1	0.556231	0.55064	0.545039	0.539429	0.533811	0.528186	0.522556	0.516921	0.511282	0.505642
0.2	0.611351	0.60592	0.600468	0.594996	0.589506	0.583998	0.578474	0.572934	0.567379	0.561811
0.3	0.664313	0.659142	0.65394	0.648709	0.64345	0.638163	0.63285	0.627511	0.622148	0.616761
0.4	0.714196	0.709369	0.704505	0.699603	0.694665	0.689691	0.684682	0.679639	0.674563	0.669454
0.5	0.76025	0.755834	0.751375	0.746873	0.742328	0.737741	0.733113	0.728443	0.723734	0.718985

Z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.6	0.801928	0.797968	0.793961	0.789908	0.785808	0.781662	0.77747	0.773232	0.768949	0.764622
0.7	0.838901	0.83542	0.831891	0.828314	0.824688	0.821015	0.817293	0.813523	0.809706	0.805941
0.8	0.87105	0.868052	0.865005	0.861911	0.858768	0.855578	0.852339	0.849052	0.845717	0.842333
0.9	0.898454	0.895922	0.893344	0.89072	0.88805	0.885334	0.882571	0.879762	0.876905	0.874002
1	0.92135	0.919254	0.917116	0.914935	0.912712	0.910445	0.908136	0.905782	0.903384	0.900941
1.1	0.940103	0.938402	0.936663	0.934887	0.933072	0.931218	0.929325	0.927392	0.925419	0.923405
1.2	0.955157	0.953804	0.952419	0.951	0.949548	0.948062	0.946541	0.944985	0.943394	0.941767
1.3	0.967004	0.965949	0.964867	0.963757	0.962618	0.96145	0.960253	0.959025	0.957767	0.956478
1.4	0.976143	0.975337	0.974508	0.973656	0.972781	0.971881	0.970957	0.970008	0.969033	0.968032
1.5	0.983053	0.982449	0.981827	0.981186	0.980527	0.979848	0.979148	0.978429	0.977688	0.976926
1.6	0.988174	0.987731	0.987274	0.986801	0.986314	0.985811	0.985293	0.984758	0.984207	0.983638
1.7	0.991895	0.991576	0.991246	0.990905	0.990552	0.990188	0.989811	0.989421	0.989019	0.988603
1.8	0.994545	0.99432	0.994087	0.993845	0.993595	0.993336	0.993067	0.992789	0.992501	0.992204
1.9	0.996395	0.99624	0.996078	0.99591	0.995736	0.995556	0.995368	0.995173	0.994972	0.994762
2	0.997661	0.997556	0.997446	0.997332	0.997213	0.99709	0.996961	0.996828	0.996689	0.996545
2.1	0.99851	0.99844	0.998367	0.998291	0.998212	0.998129	0.998043	0.997953	0.99786	0.997762
2.2	0.999069	0.999023	0.998975	0.998926	0.998874	0.998819	0.998763	0.998704	0.998642	0.998577
2.3	0.999428	0.999399	0.999369	0.999337	0.999304	0.999269	0.999232	0.999194	0.999154	0.999112
2.4	0.999656	0.999638	0.999618	0.999598	0.999577	0.999555	0.999532	0.999508	0.999483	0.999456
2.5	0.999797	0.999785	0.999774	0.999761	0.999748	0.999735	0.99972	0.999705	0.99969	0.999673
2.6	0.999882	0.999875	0.999868	0.999861	0.999853	0.999845	0.999836	0.999827	0.999817	0.999807
2.7	0.999933	0.999929	0.999925	0.99992	0.999916	0.999911	0.999906	0.9999	0.999894	0.999888
2.8	0.999963	0.99996	0.999958	0.999955	0.999953	0.99995	0.999947	0.999944	0.99994	0.999937
2.9	0.999979	0.999978	0.999977	0.999975	0.999974	0.999972	0.99997	0.999969	0.999967	0.999965
3	0.999989	0.999988	0.999987	0.999987	0.999986	0.999985	0.999984	0.999983	0.999982	0.999981

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On a Right Truncated Generalized Gaussian distribution

K. Anithakumari*[1], K. Srinivas Rao[2] and PRS Reddy[3]
[1, 3]Department of Statistics, SV University, Tirupati, AP, INDIA
[2]Department of Statistics, Andhra University, Visakhapatnam, AP, INDIA

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Abstract

Generalized Gaussian distribution is useful in analyzing several data sets arising at places at image processing, speech recognition, signal processing, statistical quality control, agricultural experimentation, industrial experimentation and biological experiments. In this paper, a right truncated generalized Gaussian distribution is introduced. The various distribution properties such as the distribution function, moments, skewness, kurtosis, hazard function and survival function are derived. The distribution of the r^{th} order statistics and the median distribution are also derived. Some inferential aspects of the distribution are also studied.

Keywords: right truncated distribution, generalized Gaussian distribution, distributional properties, order statistics.

Introduction

Generalized Gaussian distribution has been proposed for modeling atmospheric noise, subband encoding of audio-video signals, impulse noise, blind signals separation etc. Varanasi M.K et al[1], Choi, S. Cichocki et al[2], Wu et al[3], Armando et al[4]. Edgeworth[5] has considered the possibility of polynomial transformations to normality. Kameda, T[6] has pioneered the idea of probability plot to indicate the form of transformation. Johnson, N. L[7] has introduced a system of frequency curves generated by method of translation analogy to pearsonian system of distributions using log-normal and or unit normal distribution. By choosing an initial distribution Gram-Charlier series distributions are generated by Edgeworth using the normal distribution. Plucinska[8] used generalized gamma distributions one for negative and one for positive values of arguments to construct a new class of distribution functions. She also developed in 1965 the distributions by reflecting the generalized gamma distribution about the origin. Borgi.O[9] has also considered similar reflection of the standard gamma distribution. The main difficulty in these distributions is that the density is zero in general at the point of symmetry. Srinivasa Rao et al[10] have generated a class of symmetric distributions using Laplace distribution. Anithakumari et al[11] developed and analyzed a left truncated generalized Gaussian distribution. There it is assumed that the variate on the study follows a generalized Gaussian distribution and constrained with a finite value on the left end. This distribution work well in some cases where there is a minimum threshold for the variate under study. However, in some other datasets arising at quality control, Agricultural experiments, reliability study, the variable under study is having constrained on the right end. i.e., there is an upper bound for the variable. For example, in man power modeling there is an upper bound for the complete length of service known as age of superannuation. For these sort of situations it is needed to consider right truncated generalized Gaussian distribution.

In this paper, we develop and analyze a right truncated generalized Gaussian distribution. The various distributional properties such as the probability density function, the distribution function, the four moments, the skewness, the kurtosis, the hazard function and survival function are derived. The order statistics of the distribution are also studied. Some inferential properties related to the parameters of the distribution are discussed. A numerical illustration is also presented.

Right Truncated Generalized Gaussian distribution

A Continuous random variable X is said to be a three parameter generalized Gaussian distribution if its probability density function (p.d.f) is of the form

$$f(x;\mu,\alpha,\beta)=\frac{\beta}{2\alpha\Gamma\left(\frac{1}{\beta}\right)}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}};\ -\infty<x<\infty;$$

$$-\infty<\mu<\infty;\alpha>0;\beta>0$$

Consider that the range variable is finite say $(-\infty,B)$. Then the probability density function (p.d.f) of a right truncated three parameter generalized Gaussian distribution is

$$f(x)=\frac{f(x;\mu,\alpha,\beta)}{F(B)};\ -\infty<x<B;\ -\infty<\mu<B;\ \alpha>0;\ \beta>0$$

Where $F(B)=\int_{-\infty}^{B}\frac{\beta}{2\alpha\Gamma\left(\frac{1}{\beta}\right)}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}dx$ (1)

The lower and upper truncation points are $-\infty$ *and* B respectively. Hence, the probability density function of three parameter right truncated generalized Gaussian distribution is

$$f(x)=\frac{\beta}{\alpha}\frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)} \quad for \quad B<\mu \qquad (2)$$

$$f(x)=\frac{\beta}{\alpha}\frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)} \quad for \quad B\geq\mu \qquad (3)$$

where, $\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)$ is an incomplete gamma function.

Distributional Properties

The various distributional properties of the right truncated generalized Gaussian distribution are discussed in this section

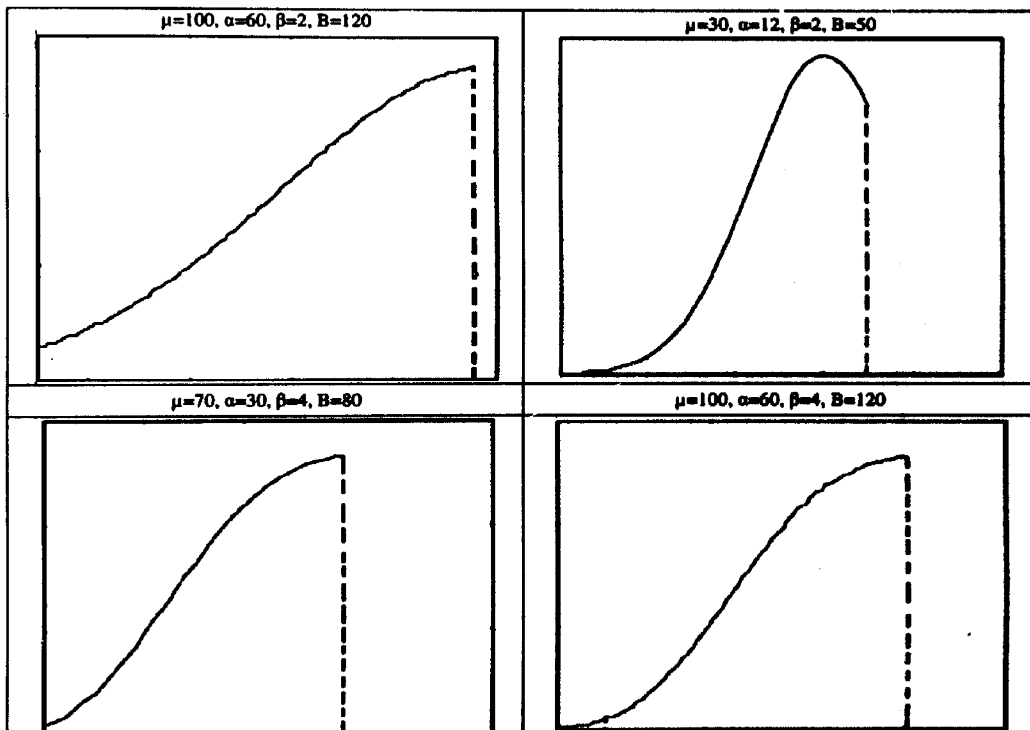

Figure-1
The frequency curves for different values of the right truncated generalized Gaussian distribution.

From figure1 it is observed that this distribution is uni-model distribution.

(i) The distribution function of X is given by

$$F(x)=\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\quad for\quad B<\mu \tag{4}$$

$$F(x)=\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\quad for\quad B\geq\mu \tag{5}$$

where, $\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)$ is an incomplete gamma function.

(ii) The mean of the distribution is

$$E(X)=\mu+\alpha\left(\frac{\left(\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)\quad for\quad B<\mu \tag{6}$$

$$E(X)=\mu+\alpha\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)\quad for\quad B\geq\mu \tag{7}$$

(iii) The median M of the distribution can be obtained by solving the equation (8) and (9)

$$\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{M-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\left(\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)\right)}=\frac{1}{2}\quad for\quad B<\mu \tag{8}$$

$$\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{M-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\left(\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)\right)}=\frac{1}{2}\quad for\quad B\geq\mu \tag{9}$$

(iv) The mode of the distribution can be obtained by solving the following equation (10) for x

$$f'(x)=\frac{-\frac{\beta}{\alpha}\left|\frac{x-\mu}{\alpha}\right|^{\beta-1}\left|\frac{x-\mu}{\alpha}\right|}{\left(\frac{x-\mu}{\alpha}\right)}=0 \tag{10}$$

This model is uni-model distribution

(v) The raw moments of the distribution are

$$\mu'_r=\sum_{j=0}^{r}\binom{r}{j}\alpha^{j}\mu^{r-j}\left(\frac{\Gamma\left(\frac{j+1}{\beta}\right)-\gamma\left(\frac{j+1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)\quad for B<\mu \tag{11}$$

Similarly for $B\geq\mu$, the r[th] non central moment is

$$\mu'_r=\sum_{j=0}^{r}\binom{r}{j}\alpha^{j}\mu^{r-j}\left(\frac{\Gamma\left(\frac{j+1}{\beta}\right)+\gamma\left(\frac{j+1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)\quad for B\geq\mu \tag{12}$$

(vi) The central moments of this distribution are

$$\mu_r=\sum_{j=0}^{r}\binom{r}{j}\alpha^{j}(-D)^{r-j}\left(\frac{\Gamma\left(\frac{j+1}{\beta}\right)-\gamma\left(\frac{j+1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)\quad for B<\mu \tag{13}$$

where $D=\alpha\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)$

Similarly for $B\geq\mu$, the r[th] non central moment is

$$\mu_r = \sum_{j=0}^{r}\binom{r}{j}\alpha^j(-D)^{r-j}\left(\frac{\Gamma\left(\frac{j+1}{\beta}\right)+\gamma\left(\frac{j+1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right) \text{ for } B\geq\mu \quad (14)$$

where $D = \alpha\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)$

(vii) The skewness of the distribution is

$$\beta_1 = \frac{\left(2S_1^3-3S_1S_2+S_3\right)^2}{\left(S_2-S_1^2\right)^3} \quad for \quad B<\mu$$

$$\beta_1 = \frac{\left(2P_1^3-3P_1P_2+P_3\right)^2}{\left(P_2-P_1^2\right)^3} \quad for \quad B\geq\mu$$

(viii) Kurtosis of the distribution is

$$\beta_2 = \frac{3S_1^2\left(2S_2-S_1^2\right)+S_4-4S_1S_3}{\left(S_2-S_1^2\right)^2} \quad for \ B<\mu$$

$$\beta_2 = \frac{3P_1^2\left(2P_2-P_1^2\right)+P_4-4P_1P_3}{\left(P_2-P_1^2\right)^2} \quad for \ B\geq\mu$$

where $S_1 = \left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)$,

$S_3 = \left(\frac{\Gamma\left(\frac{4}{\beta}\right)-\gamma\left(\frac{4}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)$, $S_4 = \left(\frac{\Gamma\left(\frac{5}{\beta}\right)-\gamma\left(\frac{5}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right)$

and $P_1 = \left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)$, $P_4 = \left(\frac{\Gamma\left(\frac{5}{\beta}\right)+\gamma\left(\frac{5}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)$

(ix) The hazard rate function of the distribution is

$$h(x) = \frac{f(x)}{1-F(x)}$$

$$h(x) = \frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)} \quad for \ B<\mu \quad (15)$$

$$h(x) = \frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)} \quad for \ B\geq\mu \quad (16)$$

(x) The survival rate function $S(x)$ is $S(x) = 1-F(x)$

$$S(x) = 1-\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)} \quad for \ B<\mu \quad (17)$$

$$S(x) = 1-\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)} \quad for \ B\geq\mu \quad (18)$$

Order Statistics of Right Truncated Three Parameter Generalized Gaussian distribution

The simple explicit form of the distribution function as given in equation (4) and (5) leads us to derive the order statistics connected with this right truncated three parameter generalized Gaussian distribution.

$$f(x) = \frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)} \quad B<\mu$$

$$f(x) = \frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)} \quad B\geq\mu \quad (19)$$

Let $X_{1n}\leq X_{2n}\leq\ldots\leq X_{nn}$ denote the order statistics obtained from a random sample of size n from the generalized truncated Gaussian distribution having the probability density function of the form given in (19). The probability density function of s^{th} order statistics[12] is given by,

$$f_{s:n}(x) = D_{s:n}\left[F(x)\right]^{s-1}\left[1-F(x)\right]^{n-s}f(x)$$

where $D_{s:n} = \frac{n!}{(s-1)!(n-s)!}$ (20)

Substituting f(x) and F(x) values given in this equation (19) and (5) in the equation (20), we get the probability density function of the s[th] order statistics is given by

Case (i): For $B \geq \mu$

For $-\infty < x < 0$

$$f_{s:n}(x)=D_{s:n}\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\sum_{q=0}^{s-1}\binom{s-1}{q}(-1)^{q}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right]^{n+q-s}$$

For $0 < x < B$

$$f_{s:n}(x)=D_{s:n}\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\sum_{q=0}^{n-s}\binom{n-s}{q}(-1)^{q}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right]^{s+q-1} \qquad (21)$$

Substituting f(x) and F(x) values given in this equation (19) and (4) in the equation (20), we get the probability density function of the s[th] order statistics is given by

Case (ii): For $B < \mu$

For $-\infty < x < 0$

$$f_{s:n}(x)=D_{s:n}\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\sum_{q=0}^{s-1}\binom{s-1}{q}(-1)^{q}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{n+q-s}$$

For $0 < x < B$

$$f_{s:n}(x)=D_{s:n}\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\sum_{q=0}^{n-s}\binom{n-s}{q}(-1)^{q}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{s+q-1} \qquad (22)$$

The probability density function of the first order statistics is obtained by substituting $s = 1$ in the equation (21)

Hence, case (i): For $B \geq \mu$,

For $-\infty < x < 0$

$$f_{1:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right]^{n-1}$$

For $0 < x < B$

$$f_{1:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^{q}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right]^{q}$$

The probability density function of the first order statistics is obtained by substituting $s = 1$ in the equation (22)

Hence, case (ii): For $B < \mu$,

For $-\infty < x < 0$

$$f_{1:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{n-1}$$

For $0<x<B$

$$f_{1:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^{q}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{q} \quad (23)$$

The probability density function of the nth order statistics is obtained by substituting $s=n$ in equation (21)

Case (i): For $B \geq \mu$

For $-\infty<x<0$

$$f_{n:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^{q}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right]^{q}$$

For $0<x<B$

$$f_{n:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right]^{n-1} \quad (24)$$

The probability density function of the nth order statistics is obtained by substituting $s=n$ in equation (22)

Case (ii): for $B<\mu$

For $-\infty<x<0$

$$f_{n:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^{q}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{q}$$

For $0<x<B$

$$f_{n:n}(x)=\frac{n\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right]^{n-1} \quad (25)$$

Distribution of the Median: Let n be odd. The distribution of the median is obtained by substituting $s=\frac{n+1}{2}$ in equation (21) and equation (22).

For $-\infty<x<0$

$$f_{M}(x)=\left(\frac{\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^{2}}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right)\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q}(-1)^{q}\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}\right]^{\frac{n-1}{2}+q} \quad B\geq\mu$$

$$f_M(x)=\left(\frac{\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^2}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right)\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q}$$ For

$$(-1)^q\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right]^{\frac{n-1}{2}+q} \quad B<\mu$$

$0<x<B$

$$f_M(x)=\left(\frac{\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^2}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)}\right)\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q} \tag{26}$$

$$(-1)^q\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)}\right]^{\frac{n-1}{2}+q} \quad B\geq\mu$$

$$f_M(x)=\left(\frac{\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^2}\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right)\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q} \tag{27}$$

$$(-1)^q\left[\frac{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{x-\mu}{\alpha}\right|^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right]^{\frac{n-1}{2}+q} \quad B$$

Inferential Aspects of the Right Truncated Three Parameter Generalized Gaussian distribution

Method of Moments: In this method, the theoretical moments of the population and the sample moments are equated correspondingly to deduce the estimators of the parameters. Let $x_1, x_2, \ldots x_n$ be a sample of size n drawn from a population having the probability density function of the form given in equation (2 and 3), we have

$$f(x)=\frac{\beta}{\alpha}\frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)} \quad B\geq\mu$$

$$f(x)=\frac{\beta}{\alpha}\frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)} \quad B<\mu$$

This distribution is having three parameters μ, α and β. Hence we equate the first three moments of the population and the sample, which leads to the following equations.

$$\bar{x}=\mu+\alpha\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)}\right) \quad B\geq\mu$$

$$\bar{x}=\mu+\alpha\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right) \quad B<\mu \tag{28}$$

and

$$s^2=\alpha^2\left(\left(\frac{\Gamma\left(\frac{3}{\beta}\right)+\gamma\left(\frac{3}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)}\right)-\left(\frac{\Gamma\left(\frac{2}{\beta}\right)+\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)}\right)^2\right) \quad \text{for } B\geq\mu$$

$$s^2=\alpha^2\left(\left(\frac{\Gamma\left(\frac{3}{\beta}\right)-\gamma\left(\frac{3}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right)-\left(\frac{\Gamma\left(\frac{2}{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right)^2\right) \quad \text{for } B<\mu \tag{29}$$

$$\beta_2 = \frac{3S_1^2(2S_2 - S_1^2) + S_4 - 4S_1S_3}{(S_2 - S_1^2)^2} \qquad for\ B < \mu$$
$$\beta_2 = \frac{3P_1^2(2P_2 - P_1^2) + P_4 - 4P_1P_3}{(P_2 - P_1^2)^2} \qquad for\ B \geq \mu \qquad (30)$$

Where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$ and

$$\beta_2 = \frac{n\sum_{i=1}^{n}(x_i - \bar{x})^4}{\left(\sum_{i=1}^{n}(x_i - \bar{x})^2\right)^2}$$

For given values of A, solving the above equations (28), (29) and (30) simultaneously by using Newtons-Raphson method, we can obtain the estimators for the parameters μ, α and β. Sample mean $\overline{X}$ is an unbiased estimator for the parameter μ. Variance of $\overline{X}$ is

$$var(\overline{X}) = var\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)$$

$$= \frac{1}{n}\alpha^2\left[\left(\frac{\Gamma\left(\frac{3}{\beta}\right) + \gamma\left(\frac{3}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right)}\right) - \left(\frac{\Gamma\left(\frac{2}{\beta}\right) + \gamma\left(\frac{2}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right)}\right)^2\right] \quad for\ B \geq \mu$$

$$= \frac{1}{n}\alpha^2\left[\left(\frac{\Gamma\left(\frac{3}{\beta}\right) - \gamma\left(\frac{3}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right) - \left(\frac{\Gamma\left(\frac{2}{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^\beta\right)}{\Gamma\left(\frac{1}{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^\beta\right)}\right)^2\right] \quad for\ B < \mu \qquad (31)$$

Maximum Likelihood Method of Estimation

Case (i): For $B \geq \mu$

Let $x_1, x_2, \ldots x_n$ be a sample of size n drawn from a population having the probability density function of the form is given in equation (3), then the likelihood function of the sample is

$$L = \left(\frac{\beta}{\alpha}\right)^n \prod_{i=1}^{n} \frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\int_0^\infty e^{-x_i} x_i^{\frac{1}{\beta}-1} dx_i + \int_0^{\left(\frac{B-\mu}{\alpha}\right)^\beta} e^{-x_i} x_i^{\frac{1}{\beta}-1} dx_i}$$

(32)

Taking logarithms on both sides of (32), we get

$$Log L = n\log\beta - n\log\alpha - \sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^\beta - \log\sum_{i=1}^{n}\left(\int_0^\infty e^{-x_i} x_i^{\frac{1}{\beta}-1} dx_i + \int_0^{\left(\frac{B-\mu}{\alpha}\right)^\beta} e^{-x_i} x_i^{\frac{1}{\beta}-1} dx_i\right) \qquad (33)$$

Since, Log L is not differentiable with respect to β for all values in the range $\beta > 0$, we obtain the estimate of β using the moment method of estimation using the equation (30).

For obtaining the maximum likelihood estimate of μ, we differentiate Log L with respect to μ and equate it to zero. But in equation (33) the function Log L is differentiable with respect to μ only when β is even. But in the case when β is odd we obtain the maximum likelihood estimator as in case of Laplace distribution (Keynes (1911)) i.e., when β is odd, we find μ which maximizes log L. From equation (33) Log L is maximum if $\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^\beta$ is minimum when β is odd. The function $\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^\beta$ is minimum only when μ is the median. Therefore the MLE of μ is the median of the distribution when β is odd. In case of β being even, we differentiate Log L with respect to μ and equate it to zero.

$$\frac{\beta}{\alpha}\sum_{i=1}^{n}\frac{\left|\frac{x_i-\mu}{\alpha}\right|^\beta}{\left(\frac{x_i-\mu}{\alpha}\right)} + \frac{\beta}{\alpha}\frac{e^{-\left(\frac{B-\mu}{\alpha}\right)^\beta}}{\Gamma\left(\frac{1}{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right)} = 0 \qquad (34)$$

To derive maximum likelihood estimator of α, consider the derivative of Log L w. r. to α and equate it to zero. This implies

$$\frac{\beta}{\alpha}\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}+\frac{\beta}{\alpha}\left(\frac{B-\mu}{\alpha}\right)\frac{e^{-\left(\frac{B-\mu}{\alpha}\right)^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)+\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)}-\frac{n}{\alpha}=0 \quad (35)$$

Solving the equations (30), (34) and (35) simultaneously for μ, α and β. Using numerical methods like Newton Raphson's method, we can obtain the maximum likelihood estimators of the parameters μ, α and β.

Case (ii): For $B<\mu$

Let $x_1, x_2, \ldots x_n$ be a sample of size n drawn from a population having the probability density function of the form is given in equation (3), then the likely hood function of the sample is

$$L=\left(\frac{\beta}{\alpha}\right)^{n}\prod_{i=1}^{n}\frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\int_0^{\infty}e^{-x_i}x_i^{\frac{1}{\beta}-1}dx_i-\int_0^{\left|\frac{B-\mu}{\alpha}\right|^{\beta}}e^{-x_i}x_i^{\frac{1}{\beta}-1}dx_i} \quad (36)$$

Taking logarithms on both sides of (36), we get

$$LogL=n\log\beta-n\log\alpha-\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}-\log\sum_{i=1}^{n}\left(\int_0^{\infty}e^{-x_i}x_i^{\frac{1}{\beta}-1}dx_i-\int_0^{\left|\frac{B-\mu}{\alpha}\right|^{\beta}}e^{-x_i}x_i^{\frac{1}{\beta}-1}dx_i\right) \quad (37)$$

Since, Log L is not differentiable with respect to β for all values in the range $\beta>0$, we obtain the estimate of β using the moment method of estimation using the equation (30).

For obtaining the maximum likelihood estimate of μ, we differentiate Log L with respect to μ and equate it to zero. But in equation (37) the function Log L is differentiable with respect to μ only when β is even. But in the case when β is odd we obtain the maximum likelihood estimator as in case of Laplace distribution (Keynes (1911)) i.e., when β is odd, we find μ which maximizes Log L. From eq. (37) log L is maximum if $\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}$ is minimum when β is odd. The function $\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}$ is minimum only when μ is the median. Therefore the MLE of μ is the median of the distribution when β is odd. In case of β being even, we differentiate Log L with respect to μ and equate it to zero.

$$\frac{\beta}{\alpha}\sum_{i=1}^{n}\frac{\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}}{\left(\frac{x_i-\mu}{\alpha}\right)}+\frac{\beta}{\alpha}\frac{\left|\frac{B-\mu}{\alpha}\right|}{\left(\frac{B-\mu}{\alpha}\right)}\frac{e^{-\left(\frac{B-\mu}{\alpha}\right)^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}=0 \quad (38)$$

To derive maximum likelihood estimator of α, consider the derivative of Log L w. r. to α and equate it to zero.

$$\frac{\beta}{\alpha}\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\alpha}\right|^{\beta}+\frac{\beta}{\alpha}\left|\frac{B-\mu}{\alpha}\right|\frac{e^{-\left|\frac{B-\mu}{\alpha}\right|^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}-\frac{n}{\alpha}=0 \quad (39)$$

Solving the equations (30), (38) and (39) simultaneously for μ, α and β. Using numerical methods like Newton Raphson's method, we can obtain the maximum likelihood estimators of the parameters μ, α and β.

Conclusion

In this paper, we have introduced right truncated generalized Gaussian distribution. Generalized Gaussian distribution is useful in analyzing several data sets arising at places at image processing, speech recognition, signal processing, statistical quality control, agricultural experimentation, industrial experimentation and biological experiments. The various distributional properties such as distribution function, moments, skewness, kurtosis, hazard function and survival function are derived. It is observed that the hazard function is sometimes increases and decreases depending upon the truncation parameter. The order statistics of the variate under study are also derived. This distribution is useful for analyzing several data sets in management science, finance, quality control and agricultural experiments. Some inferential aspects of the distribution, method of moments, and maximum likelihood estimation are also derived.

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375250