CHAPTER-V

TWO COMPONENT MIXTURE OF DOUBLY TRUNCATED THREE PARAMETER GENERALIZED GAUSSIAN DISTRIBUTION

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5.1. INTRODUCTION

In the earlier chapters 2, 3 and 4 we studied the properties of singly and doubly truncated generalized Gaussian distribution. These distributions are useful only when the population under consideration is homogenous. However, in many practical situations the populations under study are heterogeneous having different sources of generation. For example, in manpower modeling the complete length of employee in an organization is to be viewed as a mixture of two types of employees namely, committed and non-committed, each having a different distribution. To analyze these sort of situations one has to considered the random variable under study as a two component mixture of distributions. It can also be observed that the complete length of employee in an organization is having finite range 0 to A. Hence, in this chapter we develop the two component mixture of a doubly truncated three parameter generalized Gaussian distribution. The distributional properties like mean, variance, distribution function, skewness, kurtosis, etc are derived. The inferential aspects of this distribution is also studied.

5.2. TWO COMPONENT MIXTURE OF DOUBLY TRUNCATED THREE PARAMETER GENERALIZED GAUSSIAN DISTRIBUTION

A continuous random variable x is said to have a probability density function of the mixture of doubly truncated three parameter generalized Gaussian distribution, if

$$f(x) = p\ f_1(x) + (1-p)\ f_2(x) \qquad 0 < P < 1$$

where

$$f_1(x;\mu_1,\alpha_1,\beta_1)=\frac{\frac{\beta_1}{\alpha_1}e^{-\left|\frac{x-\mu_1}{\alpha_1}\right|^{\beta_1}}}{\gamma\left(\frac{1}{\beta_1},\left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\frac{1}{\beta_1},\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)} \quad \text{for } A<x<B,\ A<\mu_1<B, \alpha_1>0, \beta_1>0$$

and

$$f_2(x;\mu_2,\alpha_2,\beta_2)=\frac{\frac{\beta_2}{\alpha_2}e^{-\left|\frac{x-\mu_2}{\alpha_2}\right|^{\beta_2}}}{\gamma\left(\frac{1}{\beta_2},\left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2}\right)+\gamma\left(\frac{1}{\beta_2},\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)} \quad \text{for } A<x<B,\ A<\mu_2<B, \alpha_2>0, \beta_2>0$$

(5.2.1)

where p is the mixing parameter. Making the transformation in (5.2.1) we get the probability density function of standardized mixture of doubly truncated three parameter generalized Gaussian distribution as

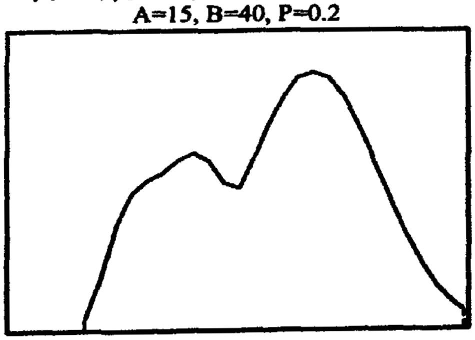
$$f(x)=\frac{p\,\frac{\beta_1}{\alpha_1}e^{-|y_1|^{\beta_1}}}{\gamma\left(\frac{1}{\beta_1},\left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\frac{1}{\beta_1},\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}+\frac{(1-p)\frac{\beta_2}{\alpha_2}e^{-|y_2|^{\beta_2}}}{\gamma\left(\frac{1}{\beta_2},\left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2}\right)+\gamma\left(\frac{1}{\beta_2},\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}$$

$$A<y<B,\ A<\mu_1<B;\ A<\mu_2<B;\ \beta_1>0,\ \alpha_1>0,\ \beta_2>0,\ \alpha_2>0,\ 0\le p\le 1 \qquad (5.2.2)$$

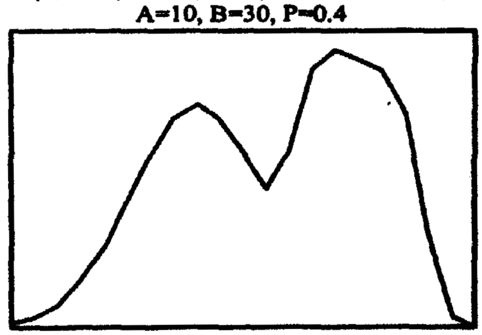
The various distributional properties of the mixture of three parameter generalized Gaussian distribution are discussed in this section.

Different shapes of the frequency curve for given values of the parameter are shown in figure 5.1

$\mu_1=20$, $\mu_2=30$, $\beta_1=4$, $\beta_2=2$, $\alpha_1=4$, $\alpha_2=6$, A=15, B=40, P=0.2

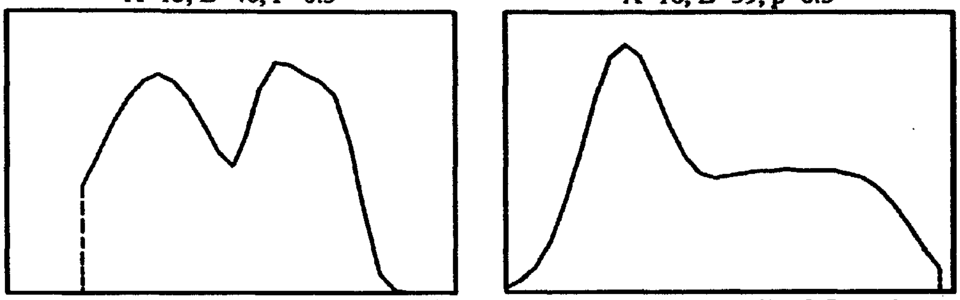


$\mu_1=25$, $\mu_2=18$, $\beta_1=4$, $\beta_2=2$, $\alpha_1=3$, $\alpha_2=4$, A=10, B=30, P=0.4



$\mu_1=30,\ \mu_2=20,\ \beta_1=4,\ \beta_2=2,\ \alpha_1=4,\ \alpha_2=6,$ $A=15,\ B=40,\ P=0.3$

$\mu_1=30,\ \mu_2=18,\ \beta_1=4.\ \beta_2=2,\ \alpha_1=8,\ \alpha_2=4,$ $A=10,\ B=39,\ p=0.5$



The frequency curves of mixture of the three parameter generalized Gaussian distribution

5.3. DISTRIBUTIONAL PROPERTIES

The distribution function of x is

$$F_x(x)=\int_A^x f(t)\,dt$$

$$= p\int_A^x f_1(t)dt+(1-p)\int_A^x f_2(t)dt$$

On simplification, we get

For $x>\max(\mu_1,\mu_2)$

$$F(x)=p\left(\frac{\gamma\left(\frac{1}{\beta_1},\left(\frac{x-\mu_1}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}{\gamma\left(\frac{1}{\beta_1},\left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}\right)+(1-p)\left(\frac{\gamma\left(\frac{1}{\beta_2},\left(\frac{x-\mu_2}{\alpha_2}\right)^{\beta_2}\right)+\gamma\left(\frac{1}{\beta_2},\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}\right)$$

where $\gamma\left(\frac{1}{\beta_1},\left(\frac{x-\mu_1}{\alpha_1}\right)^{\beta_1}\right),\gamma\left(\frac{1}{\beta_2},\left(\frac{x-\mu_2}{\alpha_2}\right)^{\beta_2}\right)$ are incomplete gamma functions.

For $x<\min(\mu_1,\mu_2)$

$$F(x)=p\left(1-\frac{\gamma\left(\frac{1}{\beta_1},\left(\frac{x-\mu_1}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}{\gamma\left(\frac{1}{\beta_1},\left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}\right)+(1-p)\left(1-\frac{\gamma\left(\frac{1}{\beta_2},\left(\frac{x-\mu_2}{\alpha_2}\right)^{\beta_2}\right)+\gamma\left(\frac{1}{\beta_2},\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}\right)$$

For $\mu_1 < x < \mu_2$

$$F(x)=p\left(\frac{\gamma\left(\frac{1}{\beta_1},\left(\frac{x-\mu_1}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}{\gamma\left(\frac{1}{\beta_1},\left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}\right)+(1-p)\left(1-\frac{\gamma\left(\frac{1}{\beta_2},\left(\frac{x-\mu_2}{\alpha_2}\right)^{\beta_2}\right)+\gamma\left(\frac{1}{\beta_2},\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}\right)$$

(5.3.1)

The Mean of the distribution is

$$E(x)=\int_A^B x\; f(x)dx$$

$$=p\int_A^B f_1(x)dx+(1-p)\int_A^B f_2(x)dx$$

On simplification, we get

$$E(x)=p\mu_1+(1-p)\mu_2+\frac{p\alpha_1\left\{\gamma\left(\frac{2}{\beta_1},\left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\frac{2}{\beta_1},\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)\right\}}{\gamma\left(\frac{1}{\beta_1},\left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\frac{1}{\beta_1},\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}$$

$$+\frac{(1-p)\alpha_2\left\{\gamma\left(\frac{2}{\beta_1},\left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2}\right)+\gamma\left(\frac{2}{\beta_2},\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)\right\}}{\gamma\left(\frac{1}{\beta_2},\left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2}\right)+\gamma\left(\frac{1}{\beta_2},\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)} \qquad (5.3.2)$$

The mode of two component mixture of three parameter generalized Gaussian distribution is obtained by differentiating $f(x)$ with respect to x and equating to zero, we have

$$f(x)=p\; f_1(x)+(1-p)\; f_2(x)$$

Differentiating $f(x)$ with respect to x and equating to zero, we get

$$p\ f_1'(x)+(1-p)\ f_2'(x)=0 \tag{5.3.3}$$

Where $f_1'(x)=\dfrac{-\dfrac{\beta_1}{\alpha_1}\left|\dfrac{x-\mu_1}{\alpha_1}\right|^{\beta_1}}{\left(\dfrac{x-\mu_1}{\alpha_1}\right)}$

and

$$f_2'(x)=\frac{-\dfrac{\beta_2}{\alpha_2}\left|\dfrac{x-\mu_2}{\alpha_2}\right|^{\beta_2}}{\left(\dfrac{x-\mu_2}{\alpha_2}\right)}$$

Solving the equation (5.3.3), we get the mode of the distribution.

The raw moments of the distribution are

$$\mu_r'=\int_A^B x^r\ f(x)\,dx$$

$$=p\int_A^B x^r\ f_1(x)dx+(1-p)\int_A^B x^r\ f_2(x)dx$$

On simplification, we get

$$\mu'_r=p\left(\frac{\sum_{j=0}^{r}\binom{r}{j}\alpha_1^{\,j}\mu_1^{\,r-j}\left[\gamma\left(\dfrac{j+1}{\beta_1},\left(\dfrac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\dfrac{j+1}{\beta_1},\left|\dfrac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)\right]}{\gamma\left(\dfrac{1}{\beta_1},\left(\dfrac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\dfrac{1}{\beta_1},\left|\dfrac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}\right)$$

$$+(1-p)\left(\frac{\sum_{j=0}^{r}\binom{r}{j}\alpha_2^{\,j}\mu_2^{\,r-j}\left[\gamma\left(\dfrac{j+1}{\beta_2},\left(\dfrac{B-\mu_2}{\alpha_2}\right)^{\beta_2}\right)+\gamma\left(\dfrac{j+1}{\beta_1},\left|\dfrac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)\right]}{\gamma\left(\dfrac{1}{\beta_1},\left(\dfrac{B-\mu_2}{\alpha_2}\right)^{\beta_2}\right)+\gamma\left(\dfrac{1}{\beta_2},\left|\dfrac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}\right) \tag{5.3.4}$$

The r^{th} central moments of this distribution is

$$\mu_r = \int_A^B (x-\mu-D)^r f(x)dx$$

$$= p\int_A^B (x-\mu-D_1)^r f_1(x)dx + (1-p)\int_A^B (x-\mu-D_2)^r f_2(x)dx$$

where $D_1 = \alpha_1 \left(\dfrac{\gamma\left(\frac{2}{\beta_1}, \left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right) + \gamma\left(\frac{2}{\beta_1}, \left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}{\gamma\left(\frac{1}{\beta_1}, \left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right) + \gamma\left(\frac{1}{\beta_1}, \left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)} \right)$

where $D_2 = \alpha_2 \left(\dfrac{\gamma\left(\frac{2}{\beta_2}, \left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2}\right) + \gamma\left(\frac{2}{\beta}, \left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)} \right)$

On simplification, we get

$$\mu_r = p\left(\frac{\sum_{j=0}^{r}\binom{r}{j}\alpha_1^{\,j}(-D_1)^{r-j}\left[\gamma\left(\frac{j+1}{\beta_1}, \left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta}\right) + \gamma\left(\frac{j+1}{\beta}, \left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)\right]}{\gamma\left(\frac{1}{\beta_1}, \left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right) + \gamma\left(\frac{1}{\beta_1}, \left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)} \right)$$
$$+ (1-p)\left(\frac{\sum_{j=0}^{r}\binom{r}{j}\alpha_2^{\,j}(-D_2)^{r-j}\left[\gamma\left(\frac{j+1}{\beta_2}, \left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2}\right) + \gamma\left(\frac{j+1}{\beta_2}, \left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)\right]}{\gamma\left(\frac{1}{\beta_2}, \left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2}\right) + \gamma\left(\frac{1}{\beta_2}, \left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta}\right)} \right) \quad (5.3.5)$$

Skewness of the distribution is

$$\beta_1 = p\left[\frac{\left(2S_1^3 - 3S_1S_2 + S_3\right)^2}{\left(S_2 - S_1^2\right)^3}\right] + (1-p)\left[\frac{\left(2Q_1^3 - 3Q_1Q_2 + Q_3\right)^2}{\left(Q_2 - Q_1^2\right)^3}\right]$$

Kurtosis of the distribution is

$$\beta_2 = p\left[\frac{3S_1^2\left(2S_2 - S_1^2\right) + S_4 - 4S_1S_3}{\left(S_2 - S_1^2\right)^2}\right] + (1-p)\left[\frac{3Q_1^2\left(2Q_2 - Q_1^2\right) + Q_4 - 4Q_1Q_3}{\left(Q_2 - Q_1^2\right)^2}\right]$$

where

$$S_1 = \left[\frac{\gamma\left(\frac{2}{\beta_1}, \left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right) + \gamma\left(\frac{2}{\beta_1}, \left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}{\gamma\left(\frac{1}{\beta_1}, \left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right) + \gamma\left(\frac{1}{\beta_1}, \left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}\right], S_2 = \left[\frac{\gamma\left(\frac{3}{\beta_1}, \left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right) + \gamma\left(\frac{3}{\beta_1}, \left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}{\gamma\left(\frac{1}{\beta_1}, \left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right) + \gamma\left(\frac{1}{\beta_1}, \left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}\right]$$

$$S_3 = \left[\frac{\gamma\left(\frac{4}{\beta_1}, \left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right) + \gamma\left(\frac{4}{\beta_1}, \left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}{\gamma\left(\frac{1}{\beta_1}, \left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right) + \gamma\left(\frac{1}{\beta_1}, \left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}\right], S_4 = \left[\frac{\gamma\left(\frac{5}{\beta_1}, \left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right) + \gamma\left(\frac{5}{\beta_1}, \left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}{\gamma\left(\frac{1}{\beta_1}, \left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right) + \gamma\left(\frac{1}{\beta_1}, \left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}\right]$$

and

$$Q_1 = \left[\frac{\gamma\left(\frac{2}{\beta_2}, \left(\frac{B-\mu_2}{\alpha_1}\right)^{\beta_2}\right) + \gamma\left(\frac{2}{\beta_2}, \left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}{\gamma\left(\frac{1}{\beta_2}, \left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2}\right) + \gamma\left(\frac{1}{\beta_2}, \left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}\right], \quad Q_2 = \left[\frac{\gamma\left(\frac{3}{\beta_2}, \left(\frac{B-\mu_2}{\alpha_1}\right)^{\beta_2}\right) + \gamma\left(\frac{3}{\beta_2}, \left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}{\gamma\left(\frac{1}{\beta_2}, \left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2}\right) + \gamma\left(\frac{1}{\beta_2}, \left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}\right]$$

$$Q_3 = \left[\frac{\gamma\left(\frac{4}{\beta_2}, \left(\frac{B-\mu_2}{\alpha_1}\right)^{\beta_2}\right) + \gamma\left(\frac{4}{\beta_2}, \left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}{\gamma\left(\frac{1}{\beta_2}, \left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2}\right) + \gamma\left(\frac{1}{\beta_2}, \left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}\right] Q_4 = \left[\frac{\gamma\left(\frac{5}{\beta_2}, \left(\frac{B-\mu_2}{\alpha_1}\right)^{\beta_2}\right) + \gamma\left(\frac{5}{\beta_2}, \left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}{\gamma\left(\frac{1}{\beta_2}, \left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2}\right) + \gamma\left(\frac{1}{\beta_2}, \left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}\right]$$

5.4. INFERENTIAL ASPECTS OF THE TWO COMPONENT MIXTURE OF DOUBLY TRUNCATED THREE PARAMETER GENERALIZED GAUSSIAN DISTRIBUTION

Method of Moments

In this method, the theoretical moments of the population and the sample moments are equated correspondingly to deduce the estimators of the parameters.

Let $x_1, x_2, \ldots x_n$ be a sample of size n drawn from a population having the p.d.f of the form given in equation (5.2.1) we have,

$$f(x) = p\ f_1(x) + (1-p)\ f_2(x)$$

$$= p\left[\frac{\frac{\beta_1}{\alpha_1} e^{-\left|\frac{x-\mu_1}{\alpha_1}\right|^{\beta_1}}}{\gamma\left(\frac{1}{\beta_1},\left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\frac{1}{\beta_1},\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}\right]+(1-p)\left[\frac{\frac{\beta_2}{\alpha_2} e^{-\left|\frac{x-\mu_2}{\alpha_2}\right|^{\beta_2}}}{\gamma\left(\frac{1}{\beta_2},\left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2}\right)+\gamma\left(\frac{1}{\beta_2},\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}\right]$$

(5.4.1)

This distribution is having parameters $\mu_1, \mu_2, \alpha_1, \alpha_2, \beta_1$ *and* β_2. Hence we equate the first three moments of the population and the sample, which leads to the following equations.

$$\bar{x} = p\mu_1 + (1-p)\mu_2 + \frac{p\alpha_1\left[\gamma\left(\frac{2}{\beta_1},\left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\frac{2}{\beta_1},\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)\right]}{\gamma\left(\frac{1}{\beta_1},\left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\frac{1}{\beta_1},\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}$$

$$+\frac{(1-p)\alpha_2\left[\gamma\left(\frac{2}{\beta_1},\left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2}\right)+\gamma\left(\frac{2}{\beta_2},\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)\right]}{\gamma\left(\frac{1}{\beta_2},\left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2}\right)+\gamma\left(\frac{1}{\beta_2},\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)} \qquad (5.4.2)$$

and

$$s^2 = p\,\alpha_2^2\left[\frac{\gamma\left(\frac{3}{\beta_1},\left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\frac{3}{\beta_1},\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}{\gamma\left(\frac{1}{\beta_1},\left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\frac{1}{\beta_1},\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}-\left[\frac{\gamma\left(\frac{2}{\beta_1},\left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\frac{2}{\beta_1},\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}{\gamma\left(\frac{1}{\beta_1},\left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\frac{1}{\beta_1},\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}\right]^2\right]$$

$$+(1-p)\,\alpha_2^2\left[\frac{\gamma\left(\frac{3}{\beta_2},\left(\frac{B-\mu_2}{\alpha_1}\right)^{\beta_2}\right)+\gamma\left(\frac{3}{\beta_2},\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}{\gamma\left(\frac{1}{\beta_2},\left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2}\right)+\gamma\left(\frac{1}{\beta_2},\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}-\left[\frac{\gamma\left(\frac{2}{\beta_2},\left(\frac{B-\mu_2}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\frac{2}{\beta_2},\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_1}\right)}{\gamma\left(\frac{1}{\beta_2},\left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_1}\right)+\gamma\left(\frac{1}{\beta_2},\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}\right]^2\right]$$

(5.4.3)

$$\beta_2 = p\left[\frac{3S_1^2\left(2S_2-S_1^2\right)+S_4-4S_1S_3}{\left(S_2-S_1^2\right)^2}\right]+(1-p)\left[\frac{3Q_1^2\left(2Q_2-Q_1^2\right)+Q_4-4Q_1Q_3}{\left(Q_2-Q_1^2\right)^2}\right]$$

(5.4.4)

Where $\bar{x}=\frac{1}{n}\sum_{i=1}^{n}x_i$ **and** $s^2=\frac{1}{n}\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2$ **and** $\beta_2=\frac{n\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^4}{\left(\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2\right)^2}$

For given values of A and B, solving the above equations (5.4.2), (5.4.3) and (5.4.4) simultaneously by using Newtons-Raphson method, we can obtain the estimators for the parameters.

Sample mean $\overline{X}$ is unbiased estimator for the parameter μ.

Variance of $\overline{X}$ is

$$\mathrm{var}\left(\overline{X}\right)=\mathrm{var}\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)$$

$$
=\frac{1}{n}\left\{\begin{array}{l}
p\,\alpha_2^2\left[\dfrac{\gamma\left(\frac{3}{\beta_1},\left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\frac{3}{\beta_1},\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}{\gamma\left(\frac{1}{\beta_1},\left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\frac{1}{\beta_1},\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}-\left[\dfrac{\gamma\left(\frac{2}{\beta_1},\left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\frac{2}{\beta_1},\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}{\gamma\left(\frac{1}{\beta_1},\left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\frac{1}{\beta_1},\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}\right]^2\right] \\
+(1-p)\,\alpha_2^2\left[\dfrac{\gamma\left(\frac{3}{\beta_2},\left(\frac{B-\mu_2}{\alpha_1}\right)^{\beta_2}\right)+\gamma\left(\frac{3}{\beta_2},\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}{\gamma\left(\frac{1}{\beta_2},\left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2}\right)+\gamma\left(\frac{1}{\beta_2},\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}-\left[\dfrac{\gamma\left(\frac{2}{\beta_2},\left(\frac{B-\mu_2}{\alpha_1}\right)^{\beta_2}\right)+\gamma\left(\frac{2}{\beta_2},\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}{\gamma\left(\frac{1}{\beta_2},\left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2}\right)+\gamma\left(\frac{1}{\beta_2},\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}\right]^2\right]
\end{array}\right.
$$

(5.4.5)

Maximum Likelihood Method of Estimation

Let $x_1, x_2, \ldots x_n$ be a sample of size n drawn from a population having the probability density function of the form is given in equation (5.2.1), then the likelihood function of the sample is

$$
L = p^n\left(\frac{\beta_1}{\alpha_1}\right)^n \prod_{i=1}^{n} \frac{e^{-\left|\frac{x_i-\mu_1}{\alpha_1}\right|^{\beta_1}}}{\int_0^{\left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}} e^{-x_i} x_i^{\frac{1}{\beta_1}-1} dx_i + \int_0^{\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}} e^{-x_i} x_i^{\frac{1}{\beta_1}-1} dx_i}
$$

$$
+(1-p)^n\left(\frac{\beta_2}{\alpha_2}\right)^n \prod_{i=1}^{n} \frac{e^{-\left|\frac{x_i-\mu_2}{\alpha_2}\right|^{\beta_2}}}{\int_0^{\left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2}} e^{-x_i} x_i^{\frac{1}{\beta_2}-1} dx_i + \int_0^{\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}} e^{-x_i} x_i^{\frac{1}{\beta_2}-1} dx}
$$

(5.4.6)

Taking logarithms on both sides of (5.4.6), we get

$$Log\,L = n\log p + n\log\beta_1 - n\log\alpha_1 - \sum_{i=1}^{n}\left|\frac{x_i-\mu_1}{\alpha_1}\right|^{\beta_1} - \log\sum_{i=1}^{n}\left(\int_0^{\left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}} e^{-x_i} x_i^{\frac{1}{\beta_1}-1}dx_i + \int_0^{\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}} e^{-x_i} x_i^{\frac{1}{\beta_1}-1}dx_i\right)$$

$$+ n\log(1-p) + n\log\beta_2 - n\log\alpha_2 - \sum_{i=1}^{n}\left|\frac{x_i-\mu_2}{\alpha_2}\right|^{\beta_2} - \log\sum_{i=1}^{n}\left(\int_0^{\left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2}} e^{-x_i} x_i^{\frac{1}{\beta_2}-1}dx_i + \int_0^{\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}} e^{-x_i} x_i^{\frac{1}{\beta_1}-1}dx_i\right)$$

(5.4.7)

For obtaining the maximum likelihood estimates of the parameters we differentiate log L with respect to μ_1, μ_2 and α_1, α_2. But (5.4.7) is not differentiable with respect to β_1, β_2 for all values in the range $\beta_1 > 0, \beta_2 > 0$, we obtain the estimate of $\beta1$, $\beta2$ using the moment method of estimation using the equation (5.4.4).

To derive maximum likelihood estimator of μ_1 and μ_2, consider the derivative of L w. r.to μ_1, μ_2 and equate it to zero. This implies

$$\frac{\partial LogL}{\partial\mu_1} = \frac{-\beta_1}{\alpha_1}\frac{\sum_{i=1}^{n}\left|\frac{x_i-\mu_1}{\alpha_1}\right|^{\beta_1}}{\left(\frac{x_i-\mu_1}{\alpha_1}\right)} - \frac{\beta_1}{\alpha_1}\frac{\frac{\left|\frac{A-\mu_1}{\alpha_1}\right|}{\left(\frac{A-\mu_1}{\alpha_1}\right)}e^{-\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}} - e^{-\left|\frac{B-\mu_1}{\alpha_1}\right|^{\beta_1}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}$$

Equating $\frac{\partial LogL}{\partial\mu_1}$ to zero, we get

$$\frac{-\beta_1}{\alpha_1}\frac{\sum_{i=1}^{n}\left|\frac{x_i-\mu_1}{\alpha_1}\right|^{\beta_1}}{\left(\frac{x_i-\mu_1}{\alpha_1}\right)} - \frac{\beta_1}{\alpha_1}\frac{\frac{\left|\frac{A-\mu_1}{\alpha_1}\right|}{\left(\frac{A-\mu_1}{\alpha_1}\right)}e^{-\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}} - e^{-\left|\frac{B-\mu_1}{\alpha_1}\right|^{\beta_1}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)} = 0$$

(5.4.8)

and

$$\frac{\partial LogL}{\partial \mu_2}=\frac{-\beta_2}{\alpha_2}\frac{\sum_{i=1}^{n}\left|\frac{x_i-\mu_2}{\alpha_2}\right|^{\beta_2}}{\left(\frac{x_i-\mu_2}{\alpha_2}\right)}-\frac{\beta_2}{\alpha_2}\frac{\frac{\left|\frac{A-\mu_2}{\alpha_2}\right|}{\left(\frac{A-\mu_2}{\alpha_2}\right)}e^{-\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}}-e^{-\left|\frac{B-\mu_2}{\alpha_2}\right|^{\beta_2}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}$$

Equating $\frac{\partial LogL}{\partial \mu_2}$ to zero, we get

$$\frac{-\beta_2}{\alpha_2}\frac{\sum_{i=1}^{n}\left|\frac{x_i-\mu_2}{\alpha_2}\right|^{\beta_2}}{\left(\frac{x_i-\mu_2}{\alpha_2}\right)}-\frac{\beta_2}{\alpha_2}\frac{\frac{\left|\frac{A-\mu_2}{\alpha_2}\right|}{\left(\frac{A-\mu_2}{\alpha_2}\right)}e^{-\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}}-e^{-\left|\frac{B-\mu_2}{\alpha_2}\right|^{\beta_2}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}=0 \qquad (5.4.9)$$

To derive maximum likelihood estimator of α1 and α2, consider the derivative of L with respect to α1, α2 and equate it to zero

$$\frac{\partial LogL}{\partial \alpha_1}=-\frac{n}{\alpha_1}+\frac{\beta_1}{\alpha_1}\sum_{i=1}^{n}\left|\frac{x_i-\mu_1}{\alpha_1}\right|^{\beta_1}-\frac{\beta_1}{\alpha_1}\frac{\left|\frac{A-\mu_1}{\alpha_1}\right|e^{-\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}}-\left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}e^{-\left|\frac{B-\mu_1}{\alpha_1}\right|^{\beta_1}}}{\gamma\left(\frac{1}{\beta_1},\left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\frac{1}{\beta_1},\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}$$

Equating $\frac{\partial LogL}{\partial \alpha_1}$ to zero, we get

$$\frac{\beta_1}{\alpha_1}\sum_{i=1}^{n}\left|\frac{x_i-\mu_1}{\alpha_1}\right|^{\beta_1}-\frac{\beta_1}{\alpha_1}\frac{\left|\frac{A-\mu_1}{\alpha_1}\right|e^{-\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}}-\left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}e^{-\left|\frac{B-\mu_1}{\alpha_1}\right|^{\beta_1}}}{\gamma\left(\frac{1}{\beta_1},\left(\frac{B-\mu_1}{\alpha_1}\right)^{\beta_1}\right)+\gamma\left(\frac{1}{\beta_1},\left|\frac{A-\mu_1}{\alpha_1}\right|^{\beta_1}\right)}-\frac{n}{\alpha_1}=0 \qquad (5.4.10)$$

and

$$\frac{\partial LogL}{\partial \alpha_2} = -\frac{n}{\alpha_2} + \frac{\beta_2}{\alpha_2}\sum_{i=1}^{n}\left|\frac{x_i-\mu_2}{\alpha_2}\right|^{\beta_2} - \frac{\beta_2}{\alpha_2}\frac{\left|\frac{A-\mu_2}{\alpha_2}\right| e^{-\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}} - \left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2} e^{-\left|\frac{B-\mu_2}{\alpha_2}\right|^{\beta_2}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2}\right)+\gamma\left(\frac{1}{\beta_2},\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)}$$

Equating $\frac{\partial LogL}{\partial \alpha_2}$ to zero, we get

$$\frac{\beta_2}{\alpha_2}\sum_{i=1}^{n}\left|\frac{x_i-\mu_2}{\alpha_2}\right|^{\beta_2} - \frac{\beta_2}{\alpha_2}\frac{\left|\frac{A-\mu_2}{\alpha_2}\right| e^{-\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}} - \left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2} e^{-\left|\frac{B-\mu_2}{\alpha_2}\right|^{\beta_2}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu_2}{\alpha_2}\right)^{\beta_2}\right)+\gamma\left(\frac{1}{\beta_2},\left|\frac{A-\mu_2}{\alpha_2}\right|^{\beta_2}\right)} - \frac{n}{\alpha_2} = 0 \tag{5.4.11}$$

Solving the equations (5.4.8), (5.4.9), (5.4.10) and (5.4.11) simultaneously using numerical methods like Newton Raphson's method, we can obtain the maximum likelihood estimators of the parameters μ_1, μ_2, α_1, and α_2.

CONCLUSION

In this chapter, we have introduced the two component mixture of a doubly truncated three parameter generalized Gaussian distribution. The distributional properties like the probability density function, distribution function, mean, variance, the four moments, skewness and kurtosis derived. The parameters are obtained by using method of maximum likelihood estimation is presented. This distribution is useful for analyzing several data sets in management sciences, finance, quality control and agricultural experiments etc.