Chapter-3
Two-Warehouse Inventory Model for Deteriorating Items with Trade Credit and Stock-Dependent Demand

Abstract

In this study, two inventory models for deteriorating items with two warehouses and shortages are developed. It is observed that in supermarkets, the demand rate is usually influenced by the amount of the stock level i.e., the demand rate may go up or down with the on-hand inventory level. Influenced by this, we developed two models where the demand depends on the present stock level. In the first model, we assumed that the demand rate is stock-dependent till the stock of RW reaches to zero and after that it is constant. In the second model; the demand rate is stock-dependent till both warehouses have stock and after that it is constant. Trade credit is provided to the customers to attract them and boost up the demand. Due to different storage conditions, deterioration rate in two warehouses may be different. In addition, shortages are neither completely backlogged nor completely lost, assuming the backlogging rate to be inversely proportional to the waiting time for the next replenishment. At the end, particular cases are also provided with constant demand and allowable shortages. The associated cost minimization is illustrated by numerical example and its sensitivity analysis was also carried out by using MATHEMATICA – 5.2 for the feasibility and applicability of the models.

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3.1 Introduction

Studies on inventory control started with the EOQ (economic order quantity) formula which is derived based on the assumption that the retailer (buyer) has to pay fully for the items as soon as he receives them from a supplier. However, as a standard practice in the real markets, a supplier usually allows a certain fixed period (known as the credit period) for settling the amount of money that the retailer owes to him for the items supplied. This TC (trade credit) policy plays an important role in the business of many products and it serves the interests of both the supplier and retailer. The supplier usually expects the profit to increase since rising sales volumes compensate the capital losses incurred during the credit period. Also, the supplier finds an effective means of price discrimination which circumvents anti-trust measures. On the contrary, the retailer earns an interest by investing the sale proceeds earned during the credit period. At present, there are three kinds of trade credits, in the first case, the length of the credit period is fixed, in the second case, the length of the credit period is closely related to the ordering quantity. In the third case, the quantity should be paid immediately after receiving the items. In addition, a TC policy develops a relationship of mutual trust between the supplier and retailer. TC policies are a common and realistic industrial policy usually followed by wholesalers. For these reasons, policy is one kind of encouragement of the retailer to order large quantities because a delay of payments indirectly reduces inventory cost. Hence, the retailer may purchase more goods than that can be stored in his/her own warehouse (OW). Excess quantities are stored in a rented warehouse (RW). The proposed model is applicable for the business of small and medium-sized retailers since their storage capacities are small and limited. In general, the inventory holding charges in RW are higher than those in OW. To reduce the inventory costs, it will be economical to consume the goods of RW at the earliest. As a result, the stocks of OW will not be released until the stocks of RW are exhausted.

The model incorporating the facility of permissible delay in payment was first discussed by Haley and Higgins (1973). Goyal (1985) explored a single item EOQ model liquids, medicines, and materials, etc. in which the rate of deterioration is very large.
Therefore, the loss due to deterioration should not be ignored. Aggarwal and Jaggi (1995) extended Goyal’s (1985) model to allow for deteriorating items. Shortages are of great importance especially in a model that considers a delay in payment due to the fact that shortages can affect the quantity ordered to benefit from the delay in payment. Jamal et al. (1997) generalized the model of Aggarwal and Jaggi (1995) to allow for shortages and make it more applicable in the real world. In the above cited references, all the models were developed for a single warehouse and assumed that the available warehouse had unlimited capacity. However, this assumption is debatable in real life situations. A capacity can be acquired in the form of rented warehouse if the retailer’s existing storage capacity is insufficient to store the ordered quantity. Motivated by the trade credit policy given by supplier, first time Shah and Shah (1992) developed a deterministic inventory model under the permissible delay in payment with two storage facilities. In that model, the items were considered non-deteriorating without shortage and the time horizon was infinite. Chung and Huang (2004) extended Goyal’s (1985) model by considering limited that model, shortage was not allowed and the time horizon was infinite with constant demand rate. Ouyang et al. (2006) developed an inventory model for deteriorating items with permissible delay in payments. The purpose of that study was to find an optimal replenishment policy for minimizing the total relevant inventory cost. Chung and Huang (2007) presented a two-warehouse inventory model for deteriorating items under trade credit financing. In that model, shortages were not allowed and time horizon was infinite. The rate of deterioration in both warehouses was considered the same. Singh et al. (2008) presented a two-warehouse inventory model for deteriorating items with constant demand rate where shortages were allowed and partially backlogged. Singh and Jain (2009) proposed a deterministic inventory model with time varying deterioration rate and a linear trend in demand over a finite planning horizon. They assumed that the supplier offers a credit limit to the retailer during which no interest is charged. However, the retailer has the reserve capital with him to make the payments at the beginning of the transaction, but he decides to take the benefit of the credit limit. Each cycle has shortages, which have been partially backlogged to suit present day competition in the market. Numerical examples have been presented to explain the theory, while sensitivity of the optimal solution of the system has been studied with respect to various system parameters.
Recently, Geetha and Uthayakumar (2010) developed an Economic Order single storage facility. This model aids in minimizing the total inventory cost by finding an optimal replenishment policy. In that model, shortages were partially backlogged.

As pointed out by Levin et al. (1972) 'at times, the presence of inventory has a motivational effect on the people around it. It is common belief that large piles of goods displayed in a supermarket will lead the customers to buy more'. In the last several years, many researchers have given considerable attention to the situation where the demand rate is dependent on the level of the on-hand inventory. Gupta and Vrat (1986) were the first to develop models for stock-dependent consumption rate. Mandal and Phaujdar (1989) then developed an economic production quantity model for deteriorating items with constant production rate and linearly stock-dependent demand. Some of the other related works in this area are: Dutta et al. (1998), Dye (2002), Chung (2003), Zhou and Yang (2005), Alfares (2007), Goyal and Chang (2009) and Yang et al. (2010).

In most of the inventory models researchers assumed the rate of deterioration for both warehouses were same. However, this assumption is not applicable for all the products due to different holding facilities in two-warehouse and due to many other conditions. In most of the inventory models developed so far for two warehouses it has been observed that the effect of shortages is not taken into consideration. Due to high holding cost of inventory, it is not advisable to store higher quantity in store and due to high demand it is always possible that the shortage occurs in inventory systems. So it is worthwhile to consider shortage to maintain their goodwill in the market. That’s why the payment in which the rate of deterioration in different warehouses is different and shortages are permitted which is completely or partially backlogged.

In this chapter, the researcher has proposed deterministic inventory models for deteriorating items with two storage facilities including the conditions of allowable shortage and permissible delay in payments. In this study, the rate of deterioration in both warehouses was considered to be different and the replenishment occurs instantaneously at an infinite rate. The two mathematical models considered are:

In model- 1, the demand rate was considered to be stock-dependent till the stock of
In model 2, the demand rate is stock-dependent till both warehouses have stock and after that it is constant.

In both models, shortages are neither completely backlogged nor completely lost, assuming the backlogging rate to be inversely proportional to the waiting time for the next replenishment. At the end, particular cases are also provided with constant demand and allowable shortages. Finally, numerical example is presented to illustrate the model and the sensitivity analysis of the optimal solution with respect to parameters of the system was also carried out by using MATHEMATICA-5.2, which is followed by concluding remarks.

3.2 Assumptions and Notations

In developing the mathematical models of the inventory system the following assumptions were used:

1. Replenishment rate is infinite, and lead-time is zero.
2. The time horizon of the inventory system is infinite.
3. The owned warehouse (OW) has a fixed capacity of W units, the rented warehouse (RW) has unlimited capacity.
4. The goods of OW are consumed only after consuming the goods kept in RW.
5. The unit inventory costs (including holding cost and deterioration cost) per unit time in RW are higher than those in OW, that is, \( F + \beta C > H + \alpha C \).
6. To guarantee the optimal solution exists, we assumed that the maximum deteriorating quantity for times in OW, \( \alpha W \), is less than the demand rate \( D(t) \), that, \( \alpha W < D(t) \).
7. No payment to the supplier is outstanding at the time of placing an order i.e. \( M < t_c \).

In addition, the following notations were used throughout this chapter:

\[
I_r(t) = \text{the level of positive inventory in RW of time } t.
\]

\[
I_o(t) = \text{the level of positive inventory in OW of time } t.
\]
\[ D(t) \quad = \quad \text{the demand rate.} \]
\[ A \quad = \quad \text{the replenishment cost, } \$/\text{order.} \]
\[ P \quad = \quad \text{the purchasing cost, } \$/\text{unit.} \]
\[ W \quad = \quad \text{the capacity of the owned warehouse.} \]
\[ H \quad = \quad \text{the holding cost, } \$/\text{unit per unit time in OW.} \]
\[ F \quad = \quad \text{the holding cost, } \$/\text{unit per unit time in RW, where } F > H. \]
\[ s \quad = \quad \text{the shortage cost, } \$/\text{unit.} \]
\[ c_d \quad = \quad \text{the opportunity cost, } \$/\text{unit.} \]
\[ c_d \quad = \quad \text{the deterioration cost, } \$/\text{unit.} \]
\[ \alpha \quad = \quad \text{the deterioration rate in OW, where } 0 \leq \alpha < 1. \]
\[ \beta \quad = \quad \text{the deterioration rate in RW, where } 0 \leq \beta < 1. \]
\[ t_1 \quad = \quad \text{the time at which the inventory level reaches zero in RW.} \]
\[ t_2 \quad = \quad \text{the time at which the inventory level reaches zero in OW.} \]
\[ I_e \quad = \quad \text{interest which can be earned, } \$/\text{year.} \]
\[ i_e \quad = \quad \text{interest charges which invested in inventory, } \$/\text{year,} \quad i_e \geq i_e. \]
\[ M \quad = \quad \text{permissible delay in settling the accounts, and } 0 < M < 1. \]

\[ TC_1(t, T) = \text{the total average inventory cost per unit time for } M \leq t \text{ in case-1.} \]
\[ TC_2(t, T) = \text{the total average inventory cost per unit time for } M > t \text{ in case-2.} \]

### 3.3 Inventory Model with Stock-Dependent Demand till the Stock of RW Reaches to Zero and after that it is Constant

Many Inventory models were developed for the static environment where the product’s demand rate was assumed to be constant over the planning horizon. However, it has been observed that in practical situations, constant demand can be justified only for the maturity phase of the product. In the present section, a deterministic two-warehouse inventory model under permissible delay in payment was developed. Here, the demand rate was assumed to be stock-dependent only when the stock is kept in rented warehouse and after that the demand rate is constant. Shortages are allowed and partially backlogged.
Besides the earlier mentioned assumptions and notations, some other assumptions and notations for the proposed model are follows:

\[ D(t) \]

- The demand rate is deterministic and is a known function of instantaneous stock level; the function is given by:

\[
D(t) = \begin{cases} 
 a + b l(t), & 0 \leq t \leq t_1 \\
 a, & t_1 \leq t \leq T
\end{cases},
\]

where \( a \) and \( b > 0 \).

- Shortages are allowed and partially backlogged. Unsatisfied demand is backlogged, and the fraction of shortages backordered is 

\[
\frac{1}{1 + \delta(T - t)}
\]

where \( \delta \) is a positive constant.

### 3.3.1 Formulation and Solution of the Model

Here, the deterministic inventory model for deteriorating items with two-warehouses where shortages occur at the end of the cycle is being discussed. For a L2 system (see fig. 3.1(a)), at time \( t=0 \), a lot size of \( S \) units enters into the L2 system in which \( W \) units are kept in OW and \( S-W \) units in RW. The goods of OW are consumed only when RW is empty. During the time interval \([0, t_1]\), the inventory \( S-W \) in RW decreases due to demand and deterioration and it vanishes at \( t=t_1 \). In OW, the inventory \( W \) decreases during \([0, t_1]\) due to deterioration only, but during \([t_1, t_2]\) the inventory is depleted due to both demand and deterioration. At time \( t=t_2 \), the inventory in OW reaches to zero and thereafter the shortages occur during the time interval \([t_2, T]\). The shortage quantity is supplied to customers at the beginning of the next cycle. The objective of the inventory system is to determine the timings of \( t_1, t_2 \) and \( T \) in order to keep the total relevant cost per unit of time as low as possible. As to the L1 system (see fig. 3.1(b)), the firm receives \( W \) units in OW at \( t=0 \). The inventory \( W \) depleted due to both demand and deterioration, and reaches zero at \( t=t_2 \), and thereafter the shortages occurs during \([t_2, T]\). Note that the L1 system here is, in fact, equivalent to the L2 system at \( t_2=0 \).
For a $L_2$ system, the inventory level at RW during the time interval $[0, t_1]$ is depleted by the combined effect of demand and deterioration, the inventory level at time $t \in [0, t_1]$, $I_r(t)$, is governing by the following differential equation:

$$I'_r(t) = -(a + bl_r(t)) - \beta I_r(t), \quad 0 \leq t \leq t_1$$

$$I_r(t_1) = 0$$

With the boundary condition the. Solving the differential equation (1), one have

$$I_r(t) = \frac{a}{\beta} \left[ e^{(b+\beta) t} - 1 \right], \quad 0 \leq t \leq t_1$$
During the time interval $[0, t_1)$, as the demand is meet from RW, the stock at OW decreases due to deterioration only. Thus, the inventory level at time $t \in [0, t_1)$, $I_0(t)$ is governed by the following differential equation:

$$I_0'(t) = -\alpha I_0(t), \quad 0 \leq t \leq t_1 \quad \ldots (3.3)$$

$$I_0(0) = W$$

With the initial condition. Again, during the time interval $[t_1, t_2]$, the inventory level at OW is depleted by the combined effect of demand and deterioration, the inventory level at time $t \in [t_1, t_2)$, $I_0(t)$, is governed by the following differential equation:

$$I_0'(t) = -\alpha - \alpha I_0(t), \quad t_1 \leq t \leq t_2 \quad \ldots (3.4)$$

$$I_0(t_2) = 0$$

With the boundary condition. Solving the differential equation (3.3) and (3.4), one can have

$$I_0(t) = We^{-\alpha t}, \quad 0 \leq t \leq t_1 \quad \ldots (3.5)$$

$$I_0(t) = \frac{a}{\alpha} \left[ e^{\alpha t} - 1 \right], \quad t_1 \leq t \leq t_2 \quad \ldots (3.6)$$

Due to continuity of at $t = t_1$, if follows equations (3.5) and (3.6), one can have

$$I_0(t_1) = We^{-\alpha t_1} = \frac{a}{\alpha} \left[ e^{\alpha t_1} - 1 \right] \quad \ldots (3.7)$$

This implies that

$$t_2 = t_1 + \frac{1}{\alpha} \ln \left( 1 + \frac{\alpha W e^{\alpha t_1}}{a} \right)$$

which shows that $t_2$ is a function of $t_1$. 
Furthermore, during the period \([t_2, T]\), the behavior of the inventory system can be described by
\[
I_0(t) = -\frac{a}{1 + \delta (T - t)}, \quad t_2 \leq t \leq T
\]  ...(3.8)

With initial condition \(I_0(t_2) = 0\), one can have
\[
I_0(t) = -\frac{a}{\delta} \ln \left[1 + \delta (T - t)\right] - \delta (t - t_2), \quad t_2 \leq t \leq T
\]  ...(3.9)

From the equations (3.2), (3.5), (3.6) and (3.9), the total cost per cycle consists of the elements:

1. Ordering cost per cycle = \(C\)
2. Holding cost per cycle in RW
\[
HO_{rw} = \int_0^{t_2} I_0(t) dt = \left(\frac{F}{b + \beta}\right) e^{(b + \beta)(t - (b + \beta)t_2)} - 1
\]
3. Holding cost per cycle in OW
\[
HO_{ow} = H \left(\int_0^{t_2} I_0(t) dt + \int_{t_2}^{t_2} I_0(t) dt\right) = H \left[\frac{W}{\alpha} 1 - e^{-\alpha} + \frac{a}{\alpha} (e^{\alpha(t_2 - t_1)} - \alpha (t_2 - t_1) - 1)\right]
\]
4. Shortage cost per cycle
\[
SC = s \left(\int_{t_2}^{t_2} I_0(t) dt = \frac{sa}{\delta^2} \delta (T - t_2) + I_0 (1 + \delta (T - t_2))\right)
\]

The number of deteriorated items in RW in \([0, t_1]\) is
\[
D_r = I_r(0) - \int_0^{t_2} D(t) dt = \frac{ab}{b + \beta} e^{(b + \beta)t_2} - 1 - b + \beta t_2
\]
and the number of deteriorated items in OW in \([0, t_2]\) is
\[
D_0 = I_0(0) - \int_{t_2}^{t_2} D(t) dt = W - \alpha (t_2 - t_1)
\]
5. Deterioration cost per cycle
\[ DC = c_d (D_s + D_b) = c_d \left\{ \frac{ab}{(b+\beta)^2} e^{(b+\beta)t_1} -1 - (b+\beta) t_2 + W - a(t_2 - t_1) \right\} \]

6. Opportunity cost due to lost sale per cycle

\[ OP = \pi a \int_{t_2}^{T} \left\{ 1 - \frac{1}{1+\delta(T-t_2)} \right\} dt = \frac{\pi a}{\delta} \delta(T-t_2) - \ln\left[ 1+\delta(T-t_2) \right] \]

The model is developed under two conditions depending on whether the permissible delay period is less than or equal to the inventory period and the permissible delay period is greater than the inventory period.

3.3.2 Case I: when \( M \leq t_2 \) (i.e. when the permissible delay period is less than the inventory period)

In this situation, since the length of period with positive stock is larger than the permissible delay period, the buyer can use the sale revenue to earn interest at an annual rate \( I_r \) in \((0, \infty)\). The interest earn \( IE_t \) is

\[ IE_t = \int_{0}^{t} \left( t_1 - t \right) a + b I(t) \right) dt + \int_{t_1}^{t_2} \left( t_2 - t \right) \alpha dt \]

\[ = \frac{PLd}{2(b+\beta)} \left\{ 2b + (b+\beta)^2 \left(t_1 - t_2\right)^2 + b(b+\beta)^2 t_1^2 + 2b e^{b+\beta t_1} \left( -1 - (b+\beta) t_1 \right) \right\} \]

\[ \ldots \]

(3.10)

However beyond the fixed credit period, an interest with an annual rate \( I_r \) is to be pay on unsold stock and interest payable \( IP \) is given by

\[ IP = \int_{M}^{t} I_b(t) dt = \frac{PLd}{a} e^{\alpha(t-M)} -1 - \alpha(t_2 - M) \]

\[ \ldots \] \( (3.11) \)

Therefore total average cost per unit time is

\[ TC, C, T, \quad I_r, DC, \quad HC, \quad SC, \quad OP, \quad DC, \quad IP, \quad IE_t \]
\[
= \frac{1}{T} \left\{ A + \frac{Fa}{(b+\beta)^2} e^{(b+\beta)\delta} - (b+\beta)\delta - 1 \right\} + H \left[ \frac{W}{\alpha} \left( 1 - e^{-\alpha t_1} \right) \right] + \frac{a}{\alpha^2} \left( \frac{e^{(b+\beta)\delta}}{e^{(b+\beta)\delta} - 1} \right) - (b+\beta)\delta - 1 \right\} + \frac{c_d}{(b+\beta)^2} e^{(b+\beta)\delta} - (b+\beta)\delta - 1 \right\} + W
- a(t_2 - t_1) \right\} \left( e^{(b+\beta)\delta} - 1 \right) \left( e^{(b+\beta)\delta} - 1 \right)
\]
\[
+ \frac{PI}{\alpha^2} \left\{ 2b + (b+\beta)^2 (t_2 - t_1)^2 + \beta(b+\beta)^2 t_2^2 + 2be^{(b+\beta)\delta} - 1 \right\} \left( e^{(b+\beta)\delta} - 1 \right) \left( e^{(b+\beta)\delta} - 1 \right)
\]
\[
+ \frac{PI}{\alpha^2} \left\{ e^{\alpha(t-M)} \right\} \left( e^{(b+\beta)\delta} - 1 \right) \left( e^{(b+\beta)\delta} - 1 \right)
\]
\[
(3.12)
\]

Now, for minimizing the total average cost per unit time, the optimal values of

\[ t_2 \quad t_1 \]

and \( T, t \) can be obtained by solving the following equations simultaneously:

\[
\frac{\partial TC_1(t_1, T)}{\partial t_1} = 0 \quad \text{and} \quad \frac{\partial TC_1(t_1, T)}{\partial T} = 0
\]

...(3.13)

Provided they satisfy the sufficient conditions:

\[
\left. \frac{\partial^2 TC_1(t_1, T)}{\partial t_1^2} \right|_{t_1, T} > 0, \left. \frac{\partial^2 TC_1(t_1, T)}{\partial T^2} \right|_{t_1, T} > 0
\]

\[
\left. \left( \frac{\partial^2 TC_1(t_1, T)}{\partial t_1^2} \right) \left( \frac{\partial^2 TC_1(t_1, T)}{\partial T^2} \right) + \left. \frac{\partial^2 TC_1(t_1, T)}{\partial t_1 \partial T} \right|_{t_1, T} \right)^2 > 0
\]

and

Equation (3.13) is equivalent to

\[
\left\{ \frac{Fa}{(b+\beta)} e^{(b+\beta)\delta} - 1 \right\} + H \left[ \frac{W}{\alpha} \left( 1 - e^{-\alpha t_1} \right) \right] + \frac{a}{\alpha^2} \left( \frac{e^{(b+\beta)\delta}}{e^{(b+\beta)\delta} - 1} \right) - (b+\beta)\delta - 1 \right\} + \frac{c_d}{(b+\beta)^2} e^{(b+\beta)\delta} - (b+\beta)\delta - 1 \right\} + W
- a(t_2 - t_1) \right\} \left( e^{(b+\beta)\delta} - 1 \right) \left( e^{(b+\beta)\delta} - 1 \right)
\]
\[
\left\{ e^{\alpha(t-M)} \right\} \left( e^{(b+\beta)\delta} - 1 \right) \left( e^{(b+\beta)\delta} - 1 \right)
\]
\[
(3.14)
\]
and
\[ -\frac{1}{T^2} \left\{ A + \frac{Fa}{(b+\beta)^2} \left( e^{(b+\beta)t} - (b+\beta)t_1 - 1 \right) + H \left[ \frac{W}{\alpha} \left( 1 - e^{-t_1} \right) + \frac{a}{\alpha^2} \left( e^{t_1(M-1)} - \alpha \ t_1 - t \right) - 1 \right] \right. \]

\[ + \frac{s + \pi \delta}{\delta^2} \left[ \delta \ T - t \right] - \ln \left[ 1 + \delta \ T - t \right] \left] \right. \]

\[ + c_0 \left\{ \frac{ab}{(b+\beta)^2} \left( e^{(b+\beta)t} - 1 - (b+\beta)t_1 \right) + W - a \ t_2 - t_1 \right\} \]

\[ - \frac{PL_a}{2(b+\beta)^3} \left( 2b + (b+\beta)^3 \ t_1 - t_2 \right)^2 + \beta(b+\beta)^2 t_1^2 + 2be^{b+\beta} \left( -1 + (b+\beta)t_1 \right) \]

\[ + \frac{PL_a}{\alpha^2} \left( e^{\alpha t_1 - M} - 1 - \alpha \ t_2 - M \right) \left[ \frac{s + \pi \delta}{\delta} \left[ T - t_2 \right] \right] - \ln \left[ 1 + \delta \ T - t_2 \right] = 0 \]

(3.15)

To acquire the optimal values of \( t_1 \) and \( T \) that minimizes \( TC_1(t_1, T) \), the following algorithm to find the optimal values of \( t_1 \) and \( T \), say \((t_1^*, T^*)\) was developed:

**Algorithm- 1**

**Step 1.** Perform (i) –(iv)

(i) Start with \( t_{1, (1)} = t_2 \).

(ii) Substituting \( t_{1, (1)} \) into equation (3.14) evaluate \( T_{1, (2)} \).

(iii) Using \( T_{1, (2)} \), determine \( T_{1, (3)} \) from equation (3.15) (using MATHEMATICA – 5.2).

(iv) Repeat (ii) and (iii) until no change occurs in the values of \( t_1 \) and \( T \).

\[ t_1, t_2 \]

**Step 2.** Compare \( t_1 \) and \( t_2 \).

(i) If \( t_1 \leq t_2 \), \( t_1 \) is feasible, then go to Step 3.

(ii) If \( t_2 < t_1 \), \( t_1 \) is not feasible. Set \( t_1 = t_2 \) and evaluate the corresponding values of \( T \) from equation (3.15), then go to step 3.
Step 3. Compute the corresponding TC ($T^*$).

3.3.3 Case-II: when $M > t_2$ (i.e. when the permissible delay period is greater than the inventory period)

Since $M > t_2$, the buyer pays no interest but earns interest at an annual rate during the period $(0, M)$, interest earned in this case, denoted by $I_e$, is given by:

$$I_e = PI_e \left[ \int_0^{t_1-t} (a + bI(t)) \, dt + \int_{t_1}^M (t - t_1) \, dt \cdot \left( a + bI(t) \right) \right]$$

$$= \frac{PI_e a}{2b+\beta} \left[ 2b+\beta \right] \left[ (t_1-t_2)^2 + 2b+\beta \right] \left[ t_1^2 + 2b+\beta \right] \left[ -1 + (b+\beta) \right]$$

$$+ (M-t_2) 2b+\beta \left[ (t_1-t_2)^2 + 2b+\beta \right] \left[ -1 + (b+\beta) \right]$$

$$= (2.16)$$

![Inventory level](image)

![Ordering quantity](image)

![Time](image)

Lost sale
Fig. 3.2 \(L_2\) inventory system (Two-warehouse inventory system) when \(M > t_2\)

Then the total average cost per unit time is

\[
TC_2(t_1, T) = \left(\frac{1}{T}\right) \left[ OC + HO_{rw} + HO_{ow} + SC + OP + DC - IE_2 \right]
\]

\[
= \frac{1}{T} \left\{ A + \frac{Fa}{(b+\beta)^2} e^{(b+\beta)t_1} - (b+\beta)t_1 - 1 \right\}
\]

\[
+ H \left[ \frac{W}{\alpha} 1 - e^{-\alpha t_1} + \frac{a}{\alpha^2} e^{a(t_1-t_2)} - \alpha \ (t_2-t_1) - 1 \right]
\]

\[
+ \frac{s + \pi \delta}{8} \frac{a}{\delta} \ (T - t_2) - \ln \left[ 1 + \frac{\delta}{T} (T - t_2) \right] \}
\]

\[
+ c_d \left\{ \frac{ab}{(b+\beta)^2} e^{(b+\beta)t_1} - 1 - (b+\beta)t_1 \right\} + W - a(t_2-t_1) \}
\]

\[
- \frac{PL_a}{2(b+\beta)^3} \left[ (2(b+\beta)^3 t_1 - t_2)^2 + \beta(b+\beta)^2 t_2^2 + 2b + 2be^{(b+\beta)t_1} (-1 + (b+\beta)t_1) \right]
\]

\[
+ (M - t_2) \left[ (2(b+\beta)^3 t_1 - t_2)^2 + \beta(b+\beta)^2 t_2^2 + 2b + 2be^{(b+\beta)t_1} (-1 + (b+\beta)t_1) \right] \}
\]

... (3.17)

Now, for minimizing the total average cost per unit time, the optimal values of

\(t_1^*\) and \(T\) (say) and ( ) can be obtained by solving the following equations simultaneously:

\[
\frac{\partial TC_2(t_1, T)}{\partial t_1} = 0 \text{ and } \frac{\partial TC_2(t_1, T)}{\partial T} = 0
\]

... (3.18)

Provided they satisfy the sufficient conditions:

\[
\frac{\partial^2 TC_2(t_1, T)}{\partial t_1^2} \bigg|_{(t_1, T)} > 0, \quad \frac{\partial^2 TC_2(t_1, T)}{\partial t_1^2} \bigg|_{(t_1, T)} > 0
\]
\[
\left( \frac{\partial^2 T C_2}{\partial t^2} \bigg|_{t_1, T} \right) + \left( \frac{\partial^2 T C_2}{\partial T^2} \bigg|_{t_1} \right) + \frac{\partial^2 (t_1, T)}{\partial t \partial T} \bigg|_{t_1, T} > 0.
\]

and Equation (3.18) is equivalent to

\[
\left\{ \frac{F a}{(b + \beta)} e^{b\beta t_1} - 1 \right\} + H \left[ W e^{-\alpha t_1} + \frac{a}{\alpha} 1 - e^{\alpha (t_1 - 1)} \right] + c_d \left[ \frac{a b}{b + \beta} e^{b\beta t_1} - 1 \right] + a
\]

\[- \frac{P I_a}{2(2 + b + \beta)} \left[ 2(b + \beta) (t_1 - t_2) + \beta t_1 + b e^{b\beta t_1} + (M - t_2) \right] \left[ (b + \beta) t_1 + b t_1 e^{b\beta t_1} \right] \right\} = 0.
\]

(3.19)

and

\[
\frac{s + \pi \delta a}{\delta^2} \delta T - t_2 - H \left[ 1 + \delta T - t_2 \right] \right\} + c_f \left[ \frac{ab}{(b + \beta)^2} \left( e^{b\beta t_1} - 1 - (b + \beta) t_1 \right) + W - a(t_2 - t_1) \right]
\]

\[- \frac{1}{T} \left\{ A + \frac{Fa}{(b + \beta)^2} e^{b\beta t_1} - (b + \beta) t_1 - 1 \right\} + H \left[ \frac{W}{\alpha} (1 - e^{-\alpha}) + \frac{a}{\alpha} (e^{\alpha (t_1 - 1)} - \alpha t_2 - t_1 - 1) \right]
\]

\[- \frac{P_{f_a}}{2(b + \beta)^2} \left[ 2(b + \beta)^2 (t_1 - t_2)^2 + \beta (b + \beta)^2 t_1^2 + 2b + 2b e^{b\beta t_1} (-1 + (b + \beta) t_1) \right] \left[ (b + \beta) t_1 + b t_1 e^{b\beta t_1} \right] \right\} + \left( \frac{s + \pi \delta a}{\delta^2} \delta T - t_2 \right) = 0.
\]

(3.20)

To acquire the optimal values of \( t_1 \) and \( T \) that minimizes \( TC_1 \), the researcher developed the following algorithm to find the optimal values of \( t_1 \) and \( T \):
**Algorithm- 2**

**Step 1.** Perform (i) – (iv)

(i) Start with \( t_{1,0} = t. \)

(ii) Substituting \( t_{1,0} \) into equation (3.19) evaluate \( T_{(1)}. \)

(iii) Using \( T_{(1)} \) determine \( T_{1, 0} \) from equation (3.20) (using \textit{MATHEMATICA} – 5.2).

(iv) Repeat (ii) and (iii) until no change occurs in the values of \( t_1 \) and \( T. \)

\[
\begin{array}{c}
\frac{t_1}{t_2} \\
\frac{t_1}{t_2}
\end{array}
\]

**Step 2.** Compare \( t_1 \) and \( t_2 \).

(i) If \( t_2 \geq t_1 \) is feasible, then go to Step 3.

(ii) \( t_2 < t_1 \) is not feasible. Set \( t_1 = t_2 \) and evaluate the corresponding values of \( T \) from equation (3.20), then go to step 3.

\[
\begin{array}{c}
t_1,
T^*
\end{array}
\]

**Step 3.** Compute the corresponding TC ( )

### 3.4 Inventory Model with Stock-Dependent Demand till Both Warehouses has Stock and after that it is Constant

A look at the literature available on inventory reveals that several models have been formulated in a static environment, where the demand for the item under consideration is assumed to be constant, for the sake of simplicity. However, it is observed that in practical situations, constant demand can be justified only for the maturity phase of the product. In many real-life situations, for certain types of consumer goods (e.g., fruits, vegetables, doughnut and others), the consumption rate is sometimes influenced by the stock-level. It is usually observed that a large pile of goods on shelf in a supermarket will lead the customer to buy more and then generate higher demand. The consumption rate may go up or down with the on-hand stock level. These phenomena attract many marketing researchers to investigate inventory models related to stock-level.

In the present section, a deterministic two-warehouse inventory model with stock-dependent demand rate is developed in which inventory is depleted not only by demand but also by deterioration. Shortages are allowed and partially backlogged.
Besides the earlier mentioned assumptions and notations, some other assumptions and notations for the proposed model are:

- The demand rate \( D(t) \) is deterministic and is a known function of instantaneous stock level; the function \( D(t) \) is given by:
  \[
  D = \begin{cases} 
  a + bt, & 0 \leq t \leq t_2 \\
  a, & t_2 \leq t \leq T 
  \end{cases}
  \]

  where \( a \) and \( b > 0 \).

- Shortages are allowed and partially backlogged. Unsatisfied demand is backlogged, and the fraction of shortages backordered is

  \[
  \frac{1}{1 + \delta(T - t)}
  \]

  where \( \delta > 0 \) is a positive constant.

### 3.4.1 Formulation and Solution of the Model

As shown in Fig. 3.1, the following time intervals separately, \([0, t_2]\), \([t_2, t_1]\) and \([t_1, T]\) were considered. During the interval \([0, t_2]\), the inventory levels are positive at RW deterioration. At OW, the inventory is only depleted by the effect of deterioration. Hence the inventory level at RW and OW are governed by the following differential equations:

\[
I_r'(t) = -a + bI_r(t) - \beta I_r(T), \quad 0 \leq t \leq t_1 
\]

\[
I_0(t) = 0, \quad 0 \leq t \leq t_1
\]

With the boundary condition and

\[
I_r'(t) = -\alpha I_0(t), \quad 0 \leq t \leq t_1
\]

\[
I_0(0) = W
\]

With the initial condition, respectively. Solving equations (3.21) and (3.22), one can get the inventory level as follows:

\[
I_r(t) = \frac{a}{b + \beta} \left[ e^{\beta t} \left( b + \beta ight) I_0(t) - 1 \right], \quad 0 \leq t \leq t_1
\]

\[
I_0(t) = We^{-\alpha t}, \quad 0 \leq t \leq t_1
\]

and

\[
I_0(t) = We^{-\alpha t}, \quad 0 \leq t \leq t_1
\]
During the interval \([t_1, t_2]\), the inventory in OW is depleted due to the combined effects of demand and deterioration. Hence, the inventory level at OW is governed by the following differential equation:

\[
I_0'(t) = -a + b l_0(t) - \alpha I_0(t),
\]

\[t_1 \leq t \leq t_2\]  \hspace{1cm} \text{ ...(3.25)}

\[I_0(t_2) = 0\]

with the boundary condition \(I_0(t_1)\). Solving the differential equation (3.25), one can get the inventory level as:

\[
I_0(t) = \frac{a}{b + \alpha} \left[ e^{(b+\alpha)(t-t_1)} - e^{(b+\alpha)(t_1-t)} \right]
\]

\[t_1 \leq t \leq t_2\]  \hspace{1cm} \text{ ...(3.26)}

Due to continuity of \(I_0(t)\) at \(t = t_1\), from equations (3.24) and (3.26), one can have

\[
We^{-\alpha t_1} = \frac{a}{b + \alpha} \left[ e^{(b+\alpha)(t_1-t_2)} - 1 \right]
\]

\[\text{ ...(3.27)}\]

This implies that

\[
t_2 = t_1 + \frac{1}{b + \alpha} \ln \left( 1 + \frac{(b + \alpha)We^{-\alpha t_1}}{a} \right)
\]

\[\text{ ...(3.28)}\]

which shows that \(t_2\) is a function of \(t_1\).

Furthermore, at time \([t_2, T]\), the inventory level reaches zero in OW and shortage occurs. During \([t_2, T]\), the inventory level only depends on demand, and some demand is lost while a fraction \((\delta(T-t))\) of the demand is backlogged, where \(\delta\). The inventory level is governed by the following differential equation:

\[
I_0'(t) = -\frac{a}{1 + \delta(T-t)},
\]

\[t_2 \leq t \leq T\]  \hspace{1cm} \text{ ...(3.23)}
\[ I_0(t_2) = 0 \]

With the boundary condition \[ \frac{\partial I}{\partial t} = -a \ln(1 + \delta (T - t)) - \left( a + \delta (T - t) \right), \] one can get the inventory level as:

\[ I_0(t) = -\frac{a}{\delta} \ln(1 + \delta (T - t)) - \left( a + \delta (T - t) \right), \quad t_0 \leq t \leq T \quad \text{(3.30)} \]

The total cost per cycle consists of the following elements:

1. Ordering cost per cycle = \( C \)
2. Holding cost per cycle in RW
   \[ HO_{Rw} = \int_{t_0}^{t_1} I_0(t) dt = \frac{Fa}{(b + \beta)^2} e^{(b+\beta)t} - (b + \beta)t - 1 \]
3. Holding cost per cycle in OW
   \[ HO_{Ow} = H \left( \int_{0}^{t_0} I_0(t) dt + \int_{t_0}^{t_1} I_0(t) dt \right) \]
   \[ = H \left[ \frac{W}{\alpha} 1 - e^{-\alpha_0} \right] + \frac{a}{b + \alpha} (e^{(b+\alpha)(T-t)} - b + \alpha)(t_2) - 1 \]
4. Shortage cost per cycle:
   \[ SC = s \int_{t_2}^{T} I_0(t) dt = \frac{sa}{\delta^2} \delta (T - t_2) - \left( a + \delta (T - t_2) \right) \]
5. Opportunity cost due to lost sales per cycle:
   \[ OP = \pi a \left( 1 - \frac{1}{1 + \delta (T - t)} \right) dt = \frac{\pi a}{\delta} \delta (T - t_2) - \left( a + \delta (T - t_2) \right) \]

The number of deteriorated items in RW in \([0, t_1]\) is

\[ D_t = I_0(t) - \int_{0}^{t_0} a + bI(t) dt = \frac{ab}{(b + \beta)^2} (e^{(b+\beta)t} - 1 - b + \beta)(t) \]

and the number of deteriorated items in OW in \([0, t_2]\) is

\[ D_o = I_0(0) - \int_{t_0}^{t_2} a + bI(t) dt = W - a(t_2 - t_0) - \frac{ab}{(b + \alpha)^2} (e^{(b+\alpha)(T-t_1)} - 1 - b + \alpha)(t_2 - t_0) \]
6. Deterioration cost per cycle:

\[
DC = c_d \cdot (D_2 + D_0) = c_d \left\{ \frac{ab}{(b+\beta)} \left\{ e^{(b+\beta)t} - (b+\beta) \cdot t_1 \right\} + W - a \cdot t_2 - t_1 \right. \\
\left. - \frac{ab}{(b+\alpha)^2} \left\{ e^{(b+\alpha)(t_2-t_1)} - 1 - (b+\alpha) \cdot (t_2-t_1) \right\} \right\}
\]

The model is developed under two conditions depending on whether the permissible delay period is less than or equal to the inventory period and the permissible delay period is greater than the inventory period.

3.4.2 Case I : \( M \leq t_2 \) (i.e. when the permissible delay period is less than the inventory period)

In this situation, since the length of period with positive stock is larger than the credit period, the buyer can use the sale revenue to earn interest at an annual rate \( I_c \) in \( t_2 \) \((0, t_2)\). The interest earned \( IE_1 \) is

\[
IE_1 = PI_c \left( \int_0^{t_1} (a + bI(t)) \, dt + \int_{t_1}^{t_2} (a + bI(t)) \, dt \right)
\]

\[
= PI_c \left\{ \frac{a}{2(b+\beta)^2} \left[ 2b + \beta(b+\beta) \cdot t_1^2 + 2\beta e^{(b+\beta)t_1} \cdot (-1 + (b+\beta) \cdot t_1) \right] \\
+ \frac{a}{2(b+\alpha)^2} \left[ 2b + \alpha(b+\alpha)^2 \cdot (t_2 - t_1)^2 + 2\alpha e^{(b+\alpha)(t_2-t_1)} \cdot (-1 + (b+\alpha) \cdot (t_2 - t_1)) \right] \right\}
\]

(3.31)

\[
I_r
\]

However beyond the fixed credit period, an interest with annual interest rate \( I_r \) is to be paid on the unsold stock and the interest payable \( IP \) is given by:
\[ IP = PL \int_{M}^{t} I_{b}(t) dt = \frac{PLa}{(b+\alpha)^2} e^{(b+\alpha)((t-M)/2-M)} \left[ 1 - (b+\alpha)(t_2-M) \right] \]

Therefore the total average cost per unit time is:

\[ TC_1(t_1, T) = \frac{1}{T} \left[ OC + HO_{Ow} + HO_{Op} + SC + OP + DC + IP - IE_1 \right] \]

\[ = \frac{1}{T} \left\{ A + \frac{Fa}{(b+\beta)^2} (e^{(b+\beta)t} - (b+\beta)t - 1 + H \left[ \frac{W}{\alpha} \left( 1 - e^{-\alpha t} \right) + \frac{a}{(b+\alpha)^2} (e^{(b+\alpha)t_2-t_1}) \right. \right. \\
\left. \left. - (b+\alpha)(t_2-t_1) \right) + \frac{a}{(b+\alpha)^2} (e^{(b+\alpha)t_2-t_1}) \left. - (b+\alpha)(t_2-t_1) \right] + \frac{s + \delta \pi}{\delta^2} \delta (T-t_2) - \ln \left[ 1 + \delta (T-t_2) \right] \right\} \]

\[ = \frac{PL}{2} \left[ 2b+\alpha \right] \left[ (2b+\alpha) (t_2-t_1)^2 + 2be^{(b+\alpha)t_1} - 1 + (b+\alpha)t_2 \right] + \frac{a}{2} \frac{(2b+\alpha)^3}{(b+\alpha)^2} \left[ 2b+\alpha \right] \left[ (t_2-t_1)^2 + 2be^{(b+\alpha)(t_2-t_1)} - 1 + (b+\alpha)(t_2-t_1) \right] \]

\[ + \frac{PLa}{(b+\alpha)^2} e^{(b+\alpha)((t-M)/2-M)} - (b+\alpha)(t_2-M) \]

\[ (3.33) \]

Now, for minimizing the total average cost per unit time, the optimal values of \( t_1^* \) and \( T^* \) can be obtained by solving the following equations simultaneously:

\[ \frac{dT_{C_1}(t_1, T)}{dt_1} = \frac{dT_{C_2}(t_1, T)}{dT} = 0 \]

\[ (3.34) \]

Provided they satisfy the sufficient conditions

\[ \frac{\partial^2 T_{C_1}(t_1, T)}{\partial \tau^2} \bigg|_{\tau = t_1^*} > 0, \quad \frac{\partial^2 T_{C_1}(t_1, T)}{\partial \tau^2} \bigg|_{\tau = T^*} > 0 \]
\[
\left( \frac{\partial^2 \text{TC}_1(t_1, T)}{\partial t_1^2} \right) \left| \frac{\partial^2 \text{TC}_1(t_1, T)}{\partial T^2} \right| + \frac{\partial^2 \text{TC}_1(t_b, T)}{\partial t_b \partial T} \bigg|_{t_1, T}^2 > 0
\]

and

\[
\frac{1}{T} \left[ \frac{a(F + c_1 b)}{b + \beta} e^{\beta + \text{Pr}_1} - 1 \right] + \frac{a H - c_1 b}{b + \alpha} \left( 1 - e^{\beta + \text{Pr}_1 t_1 - t_2} \right) + H W e^{-\text{Pr}_1} + c_2 a
\]

\[-P_{t_1} \left\{ \frac{a t_1}{b + \beta} \beta + b e^{\beta + \text{Pr}_1} \right\} - \frac{a(t_2 - t_1)}{b + \alpha} \left( \alpha + b e^{\beta + \text{Pr}_1 t_2 - t_2} \right) \bigg|_{t_1, T} = 0
\]

\[\text{(3.35)}\]

\[
\frac{a(s + \beta \pi(T - t_2)}{T(1 + \delta(T - t_1))} \left[ A + \frac{F a}{(b + \beta)^2} \left( e^{\beta + \text{Pr}_1} - (b + \beta) t_1 - 1 \right) \right]
\]

\[
H \left[ \frac{W}{\alpha} 1 - e^{-\alpha t_2} \right] + \frac{a}{b + \alpha} \left( e^{\beta + \text{Pr}_1 t_2 - t_2} - (b + \alpha) (t_2 - t_1) - 1 \right)
\]

\[
+ c_3 \left\{ \frac{a \beta}{b + \beta} \left( e^{\beta + \text{Pr}_1 t_2 - t_2} - (b + \beta) t_1 \right) + W - a t_2 - t_1 \right\} - \frac{a b}{b + \alpha} \frac{1}{2} \left( \alpha e^{\beta + \text{Pr}_1 t_2 - t_2} - (b + \alpha) t_2 - t_1 \right)
\]

\[
+ \frac{a(s + \beta \pi)}{\delta^2} \delta(T - t_2 - ln[1 + \delta(T - t_2)]) \bigg|_{t_1, T} = 0
\]

\[\text{(3.36)}\]

To acquire the optimal values of \( t_1 \) and \( T \) that minimizes \( \text{TC}_1(\cdot, \cdot) \); the following algorithm was developed to find the optimal values of \( t_1 \) and \( T \) say \( t_1^*, T^* \):
Step 1. Perform (i) – (iv)

(i) Start with \( t_{1,(1)} = t_2 \).

(ii) Substituting \( t_{1,(1)} \) into equation (3.35) evaluate \( T_{(1)} \).

(iii) Using \( T_{(1)} \) determine \( T_{1,(2)} \) from equation (3.36) (using MATHEMATICA – 5.2).

(iv) Repeat (ii) and (iii) until no change occurs in the values of \( t_1 \) and \( T \).

\[
t_1 \quad t_2
\]

Step 2. Compare \( t_1 \) and \( t_2 \).

(i) If \( t_1 \leq t_2 \), is feasible, then go to Step 3.

\[
t_2 < t_1 \quad t_1 \quad t_2 = t_1
\]

(ii) If \( t_2 < t_1 \), is not feasible. Set \( t_1^* = t_1 \) and evaluate the corresponding values of \( T \) from equation (3.36), then go to step 3.

\[
t_1^*, T^*
\]

Step 3. Compute the corresponding TC ( \( I_c \)).

3.4.3 Case 2: \( M > t_2 \) (i.e. when the permissible delay period is greater than the inventory period)

\[
M > t_2
\]

Since \( M > t_2 \), the buyer pays no interest and earns interest at an annual rate during the period \((0, M)\). Interest earned in this case, denoted by \( I_{E_2} \), is given by:

\[
I_{E_2} = PI_c \left[ \int_0^1 (t_1 - t) (a + b l(t)) \, dt + \int_{t_1}^1 t_2 - t \, a + b l(t) \, dt \right]
\]

\[
+ M - t_2 \int_0^1 a + b l(t) \, dt + \int_{t_1}^1 a + b l(t) \, dt + \int_{t_1}^1 \frac{a}{2(b + \alpha)} \left[ 2b + b \alpha \right] \left[ t_2 - t_1 \right]^2 + 2be^{\beta t_{(2)}} - 1 + (b + \beta) t_1 \right] + \frac{a}{2(b + \alpha)^2} \left[ 2b + b \alpha \right] \left( t_2 - t_1 \right) + 2be^{b \alpha t_{(2)}} - 1 + (b + \alpha) \left( t_2 - t_1 \right) + M - t_2 \right] \left\{ \left[ a \left( t_2 + \frac{ab}{b} \right) e^{b \alpha t_{(2)}} - 1 + (b + \beta) t_1 \right] + \frac{ab}{b + \alpha} - \left( e^{b \alpha t_{(2)}} - 1 + (b + \alpha) \left( t_2 - t_1 \right) \right) \right\}
\]
(3.37)
Then the total average cost per unit time is

$$TC_2(t_i, T) = \left( \frac{1}{T} \right) OC + HO_{kg} + HO_{on} + SC + OP + DC - IE_2$$

$$= \frac{1}{T} \left[ A + \frac{F a}{(b+\beta)^2} \left( e^{(b+\beta)t_i} - (b+\beta)t_i - 1 \right) \right. + H \left[ \frac{W}{\alpha} \left( 1 - e^{-\alpha t_1} \right) + \frac{a}{b+\alpha} \left( e^{(b+\alpha)t_1} - 1 \right) \right]$$

$$- \left( b + \alpha \right) (t_2 - t_i) - 1 + c_d \left[ \frac{ab}{(b+\alpha)^2} \left( e^{(b+\alpha)(t_2 - t_i)} - 1 - (b+\alpha) (t_2 - t_i) \right) \right] + \frac{a s + \delta t}{\delta} \left( \delta T - t_2 \right) - \ln \left[ 1 + \delta \left( T - t_2 \right) \right]$$

$$- P I \left\{ \frac{a}{2(b+\beta)^2} \left[ 2b + \beta \left( b + \beta \right) \right] t_i^2 + 2b \left( e^{(b+\beta)t_i} - 1 - (b+\beta) t_i \right) \right\}$$

$$+ \frac{a}{2(b+\alpha)^2} \left[ 2b + \alpha \left( b + \alpha \right) \right] t_i^2 + 2b \left( e^{(b+\alpha)t_i} - 1 - (b+\alpha) t_i \right) \right\} \right\}$$

Now, for minimizing the total average cost per unit time, the optimal values of

$$t_i$$

and $$T$$ can be obtained by solving the following equations simultaneously:

$$\frac{\partial TC_2}{\partial t_i} = 0 \quad \text{and} \quad \frac{\partial TC_2}{\partial T} = 0$$

Provided they satisfy the sufficient conditions:

$$\left. \frac{\partial^2 TC_2}{\partial t_i^2} \right|_{t_i = t_1^*} > 0, \quad \left. \frac{\partial^2 TC_2}{\partial T^2} \right|_{T = T^*} > 0$$
\[
\left( \frac{\partial^2 T_C}{\partial t^2} \right)_{T=t} + \left( \frac{\partial^2 T_C}{\partial T^2} \right)_{t=t} + \left( \frac{\partial T_C}{\partial t} \right)_{T=t} \right) > 0
\]

and

Equation (3.39) is equivalent to

\[
\frac{1}{T} \left\{ a_t \left( \frac{F + c_t b}{b + \beta} - 1 \right) - a_t \left( \frac{a}{b + \alpha} - 1 - e^{(b+\alpha)(t_2 - t_1)} \right) + H \alpha \right\} = 0
\]

\[
+ \left( \frac{a}{b + \alpha} - 1 - e^{(b+\alpha)(t_2 - t_1)} \right) = 0
\]

... (3.40)

and

\[
\frac{a_t}{T} \left\{ \frac{a_t}{b + \beta} - \frac{a}{b + \alpha} - \frac{a}{b + \alpha} \right\} = \frac{a_t}{b + \beta} \left\{ \frac{a}{b + \alpha} - (b + \beta)(t_2 - t_1) - 1 \right\}
\]

\[
+ \frac{a}{b + \alpha} \left\{ \frac{a}{b + \alpha} - (b + \beta)(t_2 - t_1) + W - a \right\} - \frac{a}{b + \alpha} \left\{ e^{(b+\alpha)(t_2 - t_1)} - 1 - (b + \alpha)^2 \right\}
\]

\[
+ \frac{a}{b + \alpha} \left\{ \frac{a}{b + \alpha} - (b + \beta)(t_2 - t_1) \right\}
\]

... (3.41)
To acquire the optimal values of \( t_1 \) and \( T \) that minimizes \( TC_1(t_1,T) \); the following algorithm was developed to find the optimal values of \( t_1 \) and \( T \) say \( (t_1^*, T^*) \):

### Algorithm- 2

**Step 1.** Perform (i) – (iv)

(i) Start with \( t_{i,(1)} = t_2 \).

(ii) Substituting \( t_{i,(1)} \) into equation (3.40) evaluate \( T_{i,(1)} \).

(iii) Using \( T_{i,(1)} \) determines \( t_{i,(2)} \) from equation (3.41) (using MATHEMATICA – 5.2).

(iv) Repeat (ii) and (iii) until no change occurs in the values of \( t_1 \) and \( T \).

**Step 2.** Compare \( t_1 \) and \( t_2 \).

(i) If \( t_1 \leq t_2 \), \( \left( t_1, T \right) \) is feasible, then go to Step 3.

\[
\begin{align*}
\frac{t_2}{t_1} & < 1 \\
\frac{t_1}{t_2} & < 1 \\
\end{align*}
\]

(ii) If \( \left( t_1, T \right) \) is not feasible. Set \( t_1 = t_2 \) and evaluate the corresponding values of \( T \) from equation (3.41), then go to step 3.

**Step 3.** Compute the corresponding \( TC \left( t_1^*, T^* \right) \).

### 3.5 Particular Cases

**Case-I: Inventory Model with Constant Demand Rate and Partial Backlogging**

If \( b = 0 \), then the discussed models reduce to inventory models in which demand rate is constant and shortages are partially backlogged. In this case, the total average cost per unit time when \( \left( t_1 \right) \) (i.e. when the permissible delay period is less than the inventory period) is

\[
TC_1(t_1, T) = \frac{1}{T} \left[ OC + \frac{HOM}{\overline{W}} + \frac{HOM}{C} + SC + OP + DC + IP - IE \right]
\]

\[
= \frac{T}{T} \left[ A + \frac{FD}{\beta^2} e^{\beta t_1 - \beta t_2 - \delta} \right] + \frac{H}{\alpha} \left[ W - D \right] t_2 - t_1 + \frac{t + \delta \pi}{\delta^2} \frac{D}{\delta} \delta(T - t_2)
\]
\[- \ln(1 + \delta T - t_i) + c_d \left( \frac{D}{B} e^{b_i} - 1 \right) - D t_i - W \right] + \frac{DPI_t}{2} \left( t_2 - M \right) - DPI_t \frac{t_2^2}{2} \right) \]

...(3.42)

\[ M > t_2 \]

The total average cost per unit time when \( M > t_2 \) (i.e. when the permissible delay period is greater than the inventory period) is

\[ TC_i(t_1, T) = \frac{1}{T} \left[ A + \frac{F + \beta c_d}{\beta^2} D e^{b_i} - \beta t_i - 1 \right] + \frac{H}{\alpha} \left( W - D t_i - t_1 \right) + \left( s + \delta t_i \right) \frac{D}{B} \delta T - t_i \]

\[- \ln(1 + \delta(T - t_2)) + c_d \left( \frac{D}{B} e^{b_i} - 1 \right) - D t_2 + W \right] - PI_t D t_2 \left( M - \frac{t_2}{2} \right) \]

...(3.43)

**Case-II: Inventory Model with Constant Demand Rate and Backorder**

If \( b = 0 \) and \( \beta = 0 \), then the discussed models reduce to inventory models in which demand rate is constant and shortages are fully backlogged. In this case, the total average cost per unit time when \( M > t_2 \) (i.e. when the permissible delay period is less than the inventory period) is

\[ TC_i(1/T) = (OC + HO_{RW} + HO_{OW} + SC + DC + IP - IE_1) \]

\[ = \frac{1}{T} \left[ A + \frac{F + \beta c_d}{\beta^2} D e^{b_i} - \beta t_i - 1 \right] + \frac{H}{\alpha} \left( W e^{a_i} - e^{a_i} - \alpha t_i - t_1 \right) + \]

\[ + \frac{D s}{2} (T - t_2)^2 + \frac{DPI_t}{2} \left( t_2 - M \right)^2 - DPI_t \frac{t_2^2}{2} \]

...(3.44)

\[ M > t_2 \]

The total average cost per unit time when \( M > t_2 \) (i.e. when the permissible delay period is greater than the inventory period) is

\[ TC = \frac{1}{T} [OC + HO_{RW} + HO_{OW} + SC + DC + IE] \]
\[
\begin{align*}
= \frac{1}{T} \left[ A + \frac{F + \beta c_2}{\beta^2} D(e^{R_t} - \beta t_1 - 1) + \frac{H + \alpha c_1}{\alpha^2} D(e^{\alpha u_t} - e^{\alpha u_1} - \alpha t_2 - t_1) \right] \\
+ \frac{D_s}{2} (T - t_2)^2 - D P_t d_s \left( M - \frac{t_2}{2} \right)
\end{align*}
\]
(3.44)

3.6 Numerical Example

To illustrate the results, we applied the proposed method to solve the following numerical example with the help of the software MATHEMATICA-5.2:

Let \( a = 570, b = 22.8, A = 200, F = 12, H = 10, W = 200, \alpha = 0.06, \beta = 0.08, s = 30, \)

\( I_r, I_c, t_2, t_1, T, \)

\( = 0.15, = 0.12, P = 200, = 0.2878, = 0.08, = 15. \) The optimal values of \( T, T, TC_1 \) and \( TC_2 \) have been computed. Computed results are displayed in Table 3.1.

<table>
<thead>
<tr>
<th>( M \leq t_2 )</th>
<th>( M &gt; t_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_1 = 0.2228 )</td>
<td>( t_1 = 0.2215 )</td>
</tr>
<tr>
<td>( T = 1.1046 )</td>
<td>( T = 0.6195 )</td>
</tr>
<tr>
<td>( TC_1 = 13636.3 )</td>
<td>( TC_2 = 5746.59 )</td>
</tr>
</tbody>
</table>

3.7 Sensitivity analysis

The sensitivity analysis of the case-1 and case-2 was made with respect to various parameters using MATHEMATICA-5.2. The results are presented in table 3.2.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>( M \leq t_2 )</th>
<th>( M &gt; t_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( % ) change in parameter ( s )</td>
<td>( TC_1 )</td>
<td>( TC_1 )</td>
</tr>
<tr>
<td>( H )</td>
<td>( -20% )</td>
<td>13555.2</td>
</tr>
<tr>
<td>( -10% )</td>
<td>13594.3</td>
<td>-0.3080</td>
</tr>
<tr>
<td>( 10% )</td>
<td>13678.3</td>
<td>0.3076</td>
</tr>
<tr>
<td>( 20% )</td>
<td>13720.1</td>
<td>0.6143</td>
</tr>
<tr>
<td>( -20% )</td>
<td>13639.2</td>
<td>0.0215</td>
</tr>
<tr>
<td>$\delta$</td>
<td>-10</td>
<td>13637.8</td>
</tr>
<tr>
<td>--------</td>
<td>-----</td>
<td>---------</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>13634.8</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>13633.3</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>-20</td>
<td>13612.6</td>
</tr>
<tr>
<td></td>
<td>-10</td>
<td>13624.5</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>13648.1</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>13659.9</td>
</tr>
<tr>
<td>$P$</td>
<td>-20</td>
<td>12421.0</td>
</tr>
<tr>
<td></td>
<td>-10</td>
<td>13037.4</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>14213.2</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>14778.2</td>
</tr>
<tr>
<td>$F$</td>
<td>-20</td>
<td>13231.1</td>
</tr>
<tr>
<td></td>
<td>-10</td>
<td>13434.5</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>13836.8</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>14035.8</td>
</tr>
</tbody>
</table>

![Percentage change in the total cost TC1 and TC2](image)

**Fig. 3.3** Effect of the holding cost of own warehouse
Fig. 3.4 Effect of the backlogging rate

Fig. 3.5 Effect of the deterioration rate in rented warehouse

Fig. 3.6 Effect of the purchasing cost
Fig. 3.7 Effect of the holding cost of rented warehouse

**Observations:**

1. From table 3.1 and 3.2, it is observed that TC₂ is always less than TC₁ with respect to the change in every parameter. This is due to the second case M > t₀. So, not interest was paid and some interest was earned.

2. From fig. 3.3, as the holding cost (H) of own warehouse increases, the total inventory cost is increased in both cases.

3. From fig. 3.4, as the backlogging rate (δ) increases, the total cost TC₃ decreases whereas TC₂ increases.

4. From fig. 3.5, as the deterioration rate (α) of own warehouse increases, the total inventory cost increases in both cases.

5. From fig. 3.6, as the purchasing cost (P) increases, the total inventory cost TC₁ increases whereas TC₂ decreases.

6. From fig. 3.7, as the holding cost (F) of rented warehouse increases, the total inventory cost is also increase in both cases.

### 3.8 Conclusion

In this chapter, inventory models were developed for deteriorating items with two-warehouses, permitting shortages under the conditions of permissible delay in payments. Holding costs and deterioration costs are different in OW and RW due to different preservation environments. The inventory costs (including holding cost and deterioration cost) in RW are assumed to be higher than those in OW. The model suggests that to reduce the inventory costs, it will be economical for firms to store the goods in OW to the maximum level and after that the remaining goods stored in RW, but clear the stocks in
RW before OW. So that rent of rented warehouse is minimum. From the viewpoint of the costs, decisions rules to find the optimal order cycle time $t_2$ contains two cases: (i) when the permissible delay period is less than the inventory period, (ii) when the permissible delay period is greater than the inventory period. Finally, a numerical example in Table 3.1 is studied to illustrate the theoretical results. From the Tables 3.1 and 3.2, it is observed that the total inventory cost $TC_2$ is always less then $TC_1$ with respect to the change in every parameter. Because the permissible delay period is greater than the inventory period. So, not interest was paid and can enable earning interest. Thus it can be concluded that the effect of permissible delay can not be ignored.

When the space of OW is abundant then there is no need to use RW; in this situation the above model reduces to the model given by Dye (2002). For the conditions with $\beta = 0$ and $\delta = 0$, the model reduces to the model given by Aggarwal and Jaggi (1995), they however, did not allow shortages.

Thus, this model incorporates some realistic features that are likely to be associated with some kinds of inventory. The model is very useful in the retail business. It can also be used for electronic components, fashionable clothes, domestic goods and other products which have the likely characteristics discussed above.

In the future research on this problem, it would be of interest to add effect of more realistic demand rate in the model (e.g. time-varying and stochastic demand patterns) and the effect of inflation in fuzzy environment. On the contrary, the possible extension of this work may relax the assumption of variable deterioration rates.
References


