CHAPTER VI

M/E_k/1 queueing model

This chapter presents the construction of control chart for the random system length of the Erlangian service queueing model under non- fuzzy and fuzzy environments. Numerical illustrations are presented to analyse the behavior of the parameters of control chart.

Erlang service queueing model is applied in many real life situation where service is provided in phases such as collection of samples in a laboratory, repairing of certain types of machines, security check outs and so on. This necessitates the analysis of Erlang service queueing model.

6.1 M/E_k/1 queueing model

The assumptions underlying M/E_k/1 queueing model are

i. arrivals follow Poisson distribution with rate \( \lambda \).

ii. if the service times \( t_1, t_2, \ldots, t_k \) are independent and exponentially distributed with parameter \( \mu \) for \( k \) phases then for any customer the total service time is \( t = t_1 + t_2 + \ldots + t_k \). Then the service time distribution follows Erlang with parameters \( (\mu, k) \) having probability density function as \( f(t) = (\mu k)^k t^{k-1} \exp(-\mu t)/((k-1)!, 0 \leq t \leq \infty, k \geq 1). \)

iii. customers are served with first come, first served basis

iv. the waiting capacity of infinite size

Let \( p_n \) be the probability that there are \( n \) phases in the system, \( n \) be the total number of phases in the system and \( k \) be number of phases for a unit in which phases are served one by one.

Each arrival will increase the number of phases by \( k \) in the system and each time a phase is completed, the number of phases in the system is decreased
by unity. Thus if there are m units waiting in the queue and one unit is under service at the s\(^{\text{th}}\) phase then \(n = mk + s\).

If \(\mu\) is the number of units served per unit time, then \(\mu k\) will be the number of phases served per unit time and \(1/(\mu k)\) is the time taken for servicing one at each phase.

The steady state equations are

\[
\lambda p_0 = k\mu p_1
\]

\[
(\lambda + k\mu) p_n = \lambda p_{n-k} + k\mu p_{n+1}, \quad n \geq 1
\]

(6.1)

Let \(\rho = \lambda / \mu\) be the traffic intensity. The equation (6.1) yields

\[
k p_1 = \rho p_0,
\]

\[
(k+\rho) p_n = kp_{n+1} + p_{n-k}, \quad n \geq 1
\]

(6.2)

Equation (6.2) gives \(p_0 = 1 - \rho\)

and

\[
p_n = (1 - \rho) \sum_{i=0}^{\infty} \sum_{j=0}^{m} \sum_{i=0}^{m} (\rho/k)^m (-1)^i \binom{m}{i} \binom{m + j - 1}{j}
\]

where the summation is taken over all combinations of \(i, j\) and \(m\) satisfying \(n = m+ik+j\) for given \(k\).

### 6.2 Mean and variance of random system length

Let the random variable \(N_s\) denote the number of customers in the system of M/E\(_k\)/1 queueing model. The mean and variance of number of units in the system are

\[
E(N_s) = \rho + (k+1)\rho^2/(2k(1-\rho))
\]

(6.3)

\[
\text{Var}(N_s) = \rho(1+\rho k)(2\rho k^3(1-\rho) + 2\rho^2 k - 5\rho k + 6k + 3\rho p)/(6k(1-\rho)^2) - A^2
\]

(6.4)

where \(A = \rho + (k+1)\rho^2/(2k(1-\rho))\)
6.3 Parameters of $N_s$ control chart

Shewhart type control chart is constructed by approximating the statistic under consideration by a normal distribution. The parameters of the $N_s$ control chart are

\[
\begin{align*}
\text{CL} & = E(N_s) \\
\text{UCL} & = E(N_s) + 3\sqrt{\text{Var}(N_s)} \\
\text{LCL} & = E(N_s) - 3\sqrt{\text{Var}(N_s)}
\end{align*}
\] (6.5)

6.4 Numerical illustration

The parameters of control chart to the number of customers in system are calculated using equations (6.3), (6.4) in (6.5) by considering $\rho =0.05(0.05)0.95$ and $k=(1,2,3,4,8,10)$ and are presented in Table 6.1. LCL values are not given in the Table as they are negative and assumed as zero.

The effect of $\rho$ and $k$ on the control chart parameters CL and UCL are also depicted in Fig. 6.1. This figure reveals that increase in $k$

i. decreases the CL values

ii. increases the UCL values

Numerical values presented for $N_s$ control chart in Table 6.1 reveal that

i. increase in $k$, decreases the CL values and increases UCL for fixed $\rho$

ii. increase in $\rho$, increase both CL and UCL values for fixed $k$
Fig. 6.1 Effect of \( \rho \) and \( k \) on control limits of \( N_s \)
Table 6.1 Parameters of control chart for $N_s$ of M/E$_k$/1 queueing model

<table>
<thead>
<tr>
<th></th>
<th>k = 1</th>
<th>k = 2</th>
<th>k = 3</th>
<th>k = 4</th>
<th>k = 8</th>
<th>k = 10</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CL</td>
<td>UCL</td>
<td>CL</td>
<td>UCL</td>
<td>CL</td>
<td>UCL</td>
</tr>
<tr>
<td>0.05</td>
<td>0.0526</td>
<td>0.7588</td>
<td>0.0520</td>
<td>0.7891</td>
<td>0.0518</td>
<td>0.8343</td>
</tr>
<tr>
<td>0.10</td>
<td>0.1111</td>
<td>1.1652</td>
<td>0.1083</td>
<td>1.2548</td>
<td>0.1074</td>
<td>1.3877</td>
</tr>
<tr>
<td>0.15</td>
<td>0.1764</td>
<td>1.5434</td>
<td>0.1699</td>
<td>1.7152</td>
<td>0.1676</td>
<td>1.9682</td>
</tr>
<tr>
<td>0.20</td>
<td>0.2500</td>
<td>1.9271</td>
<td>0.2375</td>
<td>2.2029</td>
<td>0.2333</td>
<td>2.6061</td>
</tr>
<tr>
<td>0.25</td>
<td>0.3333</td>
<td>2.3333</td>
<td>0.3125</td>
<td>2.7355</td>
<td>0.3056</td>
<td>3.3183</td>
</tr>
<tr>
<td>0.30</td>
<td>0.4285</td>
<td>2.7759</td>
<td>0.3964</td>
<td>3.3277</td>
<td>0.3857</td>
<td>4.1202</td>
</tr>
<tr>
<td>0.35</td>
<td>0.5384</td>
<td>3.2689</td>
<td>0.4913</td>
<td>3.9951</td>
<td>0.4756</td>
<td>5.0290</td>
</tr>
<tr>
<td>0.40</td>
<td>0.6667</td>
<td>3.8289</td>
<td>0.6000</td>
<td>4.7569</td>
<td>0.5778</td>
<td>6.0680</td>
</tr>
<tr>
<td>0.45</td>
<td>0.8182</td>
<td>4.4772</td>
<td>0.7261</td>
<td>5.6380</td>
<td>0.6955</td>
<td>7.2648</td>
</tr>
<tr>
<td>0.50</td>
<td>1.0000</td>
<td>5.2426</td>
<td>0.8750</td>
<td>6.6724</td>
<td>0.8333</td>
<td>8.6590</td>
</tr>
<tr>
<td>0.55</td>
<td>1.2222</td>
<td>6.1664</td>
<td>1.0542</td>
<td>7.9082</td>
<td>0.9981</td>
<td>10.3088</td>
</tr>
<tr>
<td>0.60</td>
<td>1.5000</td>
<td>7.3095</td>
<td>1.2750</td>
<td>9.4175</td>
<td>1.2000</td>
<td>12.2960</td>
</tr>
<tr>
<td>0.65</td>
<td>1.8571</td>
<td>8.7676</td>
<td>1.5554</td>
<td>11.3126</td>
<td>1.4548</td>
<td>14.7504</td>
</tr>
<tr>
<td>0.70</td>
<td>2.3333</td>
<td>10.6999</td>
<td>1.9250</td>
<td>13.7795</td>
<td>1.7889</td>
<td>17.8850</td>
</tr>
<tr>
<td>0.75</td>
<td>3.0000</td>
<td>13.9233</td>
<td>2.4375</td>
<td>17.1524</td>
<td>2.2500</td>
<td>22.0790</td>
</tr>
<tr>
<td>0.80</td>
<td>4.0000</td>
<td>17.4164</td>
<td>3.2000</td>
<td>22.0976</td>
<td>2.9333</td>
<td>28.0825</td>
</tr>
<tr>
<td>0.85</td>
<td>5.6667</td>
<td>24.1058</td>
<td>4.4625</td>
<td>30.1675</td>
<td>4.0611</td>
<td>37.6314</td>
</tr>
<tr>
<td>0.90</td>
<td>9.0000</td>
<td>37.4605</td>
<td>6.9750</td>
<td>46.0130</td>
<td>6.3000</td>
<td>55.8981</td>
</tr>
<tr>
<td>0.95</td>
<td>19.0000</td>
<td>77.4808</td>
<td>14.4875</td>
<td>92.8749</td>
<td>12.9830</td>
<td>108.6294</td>
</tr>
</tbody>
</table>
6.5 Fuzzy control chart of $N_s$ for $M/E_k/1$ Queueing model

$M/E_k/1$ queueing model is analyzed by constructing fuzzy control chart for number of customers in the system in this section. Based on Zadeh’s extension principle, by employing $\alpha$-cut approach, the membership functions are formulated by assuming the arrival rate, service rate and number of phases as fuzzy numbers. Numerical results are given to assess further.

Construction of membership functions of the parameters of the control chart for number of customers in system of $M/E_k/1$ queueing model are considered by assuming that the arrival rate, the service rate and the number of phases as fuzzy numbers represented by

$$\tilde{\lambda} = \{ (x, \varphi_{\tilde{\lambda}}(x)) | x \in X \}$$
$$\tilde{\mu} = \{ (y, \varphi_{\tilde{\mu}}(y)) | y \in Y \}$$
$$\tilde{k} = \{ (s, \varphi_{\tilde{k}}(s)) | s \in S \}$$

Where $\varphi_{\tilde{\lambda}}(x)$, $\varphi_{\tilde{\mu}}(y)$ and $\varphi_{\tilde{k}}(s)$ denote the membership functions; $X$, $Y$ and $S$ are the supports of the fuzzy numbers $\tilde{\lambda}$, $\tilde{\mu}$ and $\tilde{k}$ respectively.

$P(x, y, s)$ and $\tilde{P}(\tilde{\lambda}, \tilde{\mu}, \tilde{k})$ denote the parameters of control chart of the number of customers in the system in the crisp and fuzzy environments respectively. $\tilde{P}(\tilde{\lambda}, \tilde{\mu}, \tilde{k})$ is a fuzzy number as $\tilde{\lambda}$, $\tilde{\mu}$ and $\tilde{k}$ are fuzzy numbers. Using Zadeh’s extension principle, the membership function of the parameters of control chart of the number of customers in the system is defined as

$$\varphi_{\tilde{P}(\tilde{\lambda}, \tilde{\mu}, \tilde{k})}(z) = \sup_{x \in X, y \in Y, s \in S} \min \{ \varphi_{\tilde{\lambda}}(x), \varphi_{\tilde{\mu}}(y), \varphi_{\tilde{k}}(s) | z = P(x, y, s) \}$$ (6.6)

6.6 Fuzzy control chart for number of customers in system

The parameters of the fuzzy control chart for number of customers in the system are
\[ \begin{align*}
CL(x, y, s) &= x(x + 2sy - sx)/(2sy(y - x)) \\
UCL(x, y, s) &= CL(x, y, s) + 3\sqrt{\text{Var}(x, y, s)} \\
LCL(x, y, s) &= CL(x, y, s) - 3\sqrt{\text{Var}(x, y, s)}
\end{align*} \] (6.7)

\[ \text{Var}(x, y, s) = (4s(\pm sA)(6s + 3A - 5sA + 2s^3A - 2s^2A^2 + 2sA^3) - A^2(A + 2s - sA)^2)/(12(1 - A)) \]

where \( A = x/y \)

A mathematical programming approach is developed for deriving the desired membership functions for CL, UCL and LCL on the basis of \( \alpha \)-cuts.

**The \( \alpha \)-cut approach based on the extension principle**

The \( \alpha \)-cuts of \( \tilde{\lambda} \), \( \tilde{\mu} \) and \( \tilde{k} \) as crisp sets are

\[ \begin{align*}
\lambda(\alpha) &= \{ x \in X \mid \varphi_{\tilde{\lambda}}(x) \geq \alpha \} \\
\mu(\alpha) &= \{ y \in Y \mid \varphi_{\tilde{\mu}}(y) \geq \alpha \} \\
k(\alpha) &= \{ s \in S \mid \varphi_{\tilde{k}}(s) \geq \alpha \}
\end{align*} \] (6.8)

These crisp sets may be expressed as

\[ \begin{align*}
\lambda(\alpha) &= [x^L_{\alpha}, x^U_{\alpha}] = [\min_{x \in X} \{ x \mid \varphi_{\tilde{\lambda}}(x) \geq \alpha \}, \max_{x \in X} \{ x \mid \varphi_{\tilde{\lambda}}(x) \geq \alpha \}] \\
\mu(\alpha) &= [y^L_{\alpha}, y^U_{\alpha}] = [\min_{y \in Y} \{ y \mid \varphi_{\tilde{\mu}}(y) \geq \alpha \}, \max_{y \in Y} \{ y \mid \varphi_{\tilde{\mu}}(y) \geq \alpha \}] \\
k(\alpha) &= [s^L_{\alpha}, s^U_{\alpha}] = [\min_{s \in S} \{ s \mid \varphi_{\tilde{k}}(s) \geq \alpha \}, \max_{s \in S} \{ s \mid \varphi_{\tilde{k}}(s) \geq \alpha \}]
\end{align*} \] (6.9)

The intervals defined above provide information on the arrival rate, the service rate and number of phases lie at possibility \( \alpha \)

The bounds of these intervals in (6.9) are functions of \( \alpha \) may be obtained as

\[ \begin{align*}
x^L_{\alpha} &= \min \varphi_{\tilde{k}}^{-1}(\alpha), \quad x^U_{\alpha} = \max \varphi_{\tilde{k}}^{-1}(\alpha) \\
y^L_{\alpha} &= \min \varphi_{\tilde{\mu}}^{-1}(\alpha), \quad y^U_{\alpha} = \max \varphi_{\tilde{\mu}}^{-1}(\alpha) \\
\text{and} \quad s^L_{\alpha} &= \min \varphi_{\tilde{\lambda}}^{-1}(\alpha), \quad s^U_{\alpha} = \max \varphi_{\tilde{\lambda}}^{-1}(\alpha)
\end{align*} \]

The membership function defined in (6.8) is parameterized by \( \alpha \).
Construction of membership function

Consider the membership functions of the parameters of the control chart for number of customers in system. As given in (6.6), $\varphi_{CL}(z)$ is the minimum of $\varphi_x(x)$, $\varphi_y(y)$ and $\varphi_s(s)$ such that $z = CL(x, y, s)$.

To deal with the value of the membership function, we need at least one of the following three cases to hold such that $z = CL(x, y, s)$ and $\varphi_{CL}(z) = \alpha$.

\begin{enumerate}
  \item $\varphi_x(x) = \alpha$, $\varphi_y(y) \geq \alpha$, $\varphi_s(s) \geq \alpha$
  \item $\varphi_x(x) \geq \alpha$, $\varphi_y(y) = \alpha$, $\varphi_s(s) \geq \alpha$
  \item $\varphi_x(x) \geq \alpha$, $\varphi_y(y) \geq \alpha$, $\varphi_s(s) = \alpha$
\end{enumerate}

The following are the formulated parametric non-linear programs for finding the lower and upper bounds of the $\alpha$-cut of $\varphi_{CL}(z)$ corresponding to the cases stated in (6.10).

**Case i**

\[
(CL)_a^{L_1} = \min(x(x + 2sy - sx)/(2sy(y - x)))
\]

subject to $x_a^L \leq x \leq x_a^U$, $y \in \mu(\alpha)$ and $s \in k(\alpha)$

\[
(CL)_a^{U_1} = \max(x(x + 2sy - sx)/(2sy(y - x)))
\]

subject to $x_a^L \leq x \leq x_a^U$, $y \in \mu(\alpha)$ and $s \in k(\alpha)$

**Case ii**

\[
(CL)_a^{L_2} = \min(x(x + 2sy - sx)/(2sy(y - x)))
\]

subject to $x \in \lambda(\alpha)$, $y_a^L \leq y \leq y_a^U$ and $s \in k(\alpha)$

\[
(CL)_b^{U_2} = \max(x(x + 2sy - sx)/(2sy(y - x)))
\]
subject to \( x \in \lambda(\alpha) \), \( y^L_{\alpha} \leq y \leq y^U_{\alpha} \) and \( s \in k(\alpha) \)

Case iii

\[
(CL)_{L_3}^k = \min(x(x + 2sy - sx))/((2sy(y - x))) \quad (6.15)
\]

subject to \( x \in \lambda(\alpha), y \in \mu(\alpha) \) and \( s^L_{\alpha} \leq s \leq s^U_{\alpha} \)

\[
(CL)_{L_3}^U = \max(x(x + 2sy - sx))/((2sy(y - x))) \quad (6.16)
\]

subject to \( x \in \lambda(\alpha), y \in \mu(\alpha) \) and \( s^L_{\alpha} \leq s \leq s^U_{\alpha} \)

From the definitions of \( \lambda(\alpha), \mu(\alpha) \) and \( \beta(\alpha) \) in equation (6.9), \( x \in \lambda(\alpha), y \in \mu(\alpha) \) and \( s \in k(\alpha) \) may be replaced by \( x \in [x^L_{\alpha}, x^U_{\alpha}], y \in [y^L_{\alpha}, y^U_{\alpha}] \) and \( s \in [s^L_{\alpha}, s^U_{\alpha}] \) respectively. As \( \alpha \)-cuts form a nested structure with respect to \( \alpha \), the values \( (CL)_{L_1}^k, (CL)_{L_2}^k \) and \( (CL)_{L_3}^k \) in equations (6.11), (6.13) and (6.15) are identical and \( (CL)_{U_1}^k, (CL)_{U_2}^k \) and \( (CL)_{U_3}^k \) in equations (6.12), (6.14) and (6.16) are identical. Therefore, determination of the lower bound \( (CL)_{L_1}^k \) and the upper bound \( (CL)_{U_1}^k \) of the \( \alpha \)-cuts of \( \tilde{L}_C \) enables to the construction of \( \phi_{\tilde{L}_C}^k \). The bounds may be rewritten as

\[
(CL)_{L_1}^k = \min_{x \in X, y \in Y, s \in S} x \frac{(x + 2sy - sx)}{(2sy(y - x))}
\]

subject to \( x^L_{\alpha} \leq x \leq x^U_{\alpha}, y^L_{\alpha} \leq y \leq y^U_{\alpha} \) and \( s^L_{\alpha} \leq s \leq s^U_{\alpha} \)

\[
(CL)_{U_3}^k = \max_{x \in X, y \in Y, s \in S} x \frac{(x + 2sy - sx)}{(2sy(y - x))}
\]

subject to \( x^L_{\alpha} \leq x \leq x^U_{\alpha}, y^L_{\alpha} \leq y \leq y^U_{\alpha} \) and \( s^L_{\alpha} \leq s \leq s^U_{\alpha} \)

At least one of \( x, y \) and \( s \) must be on the boundary of the constraints in equation (6.17) to satisfy \( \phi_{\tilde{L}_C}^k \) = \( \alpha \). According to Gal(1979), this pair of mathematical programs falls into the category of parametric non-linear programs. The programs facilitate the systematic study of how the optimal solutions change.
for different values of $x_\alpha^L, x_\alpha^U, y_\alpha^L, y_\alpha^U, s_\alpha^L$ and $s_\alpha^U$ as $\alpha$ ranges over the interval $[0,1]$. Based on the extension principle and the convexity property of fuzzy numbers we obtain increase in $(CL)_\alpha^L$ and decrease in $(CL)_\alpha^U$ for increase in $\alpha$.

If both the lower bound $(CL)_\alpha^L$ and the upper bound $(CL)_\alpha^U$ of the $\alpha$-cuts of $\tilde{CL}$ are invertible with respect to $\alpha$, then a left shape function $L(z)$ and a right shape function $R(z)$ may be obtained as $L(z) = [(CL)_\alpha^L]^{-1}$ and $R(z) = [(CL)_\alpha^U]^{-1}$. Then the membership function $\varphi_{CL}(z)$ can be expressed as

$$
\varphi_{CL}(z) = \begin{cases} 
L(z), & (CL)^L_{\alpha-0} \leq z \leq (CL)^L_{\alpha-1} \\
1, & (CL)^L_{\alpha-1} \leq z \leq (CL)^L_{\alpha-1} \\
R(z), & (CL)^U_{\alpha-1} \leq z \leq (CL)^U_{\alpha-0}
\end{cases}
$$

Yager ranking index method based on area compensation is used to defuzzify $\tilde{CL}$ of the number of customers in the queue into a crisp one for practical use. The Yager ranking index for centre line is

$$
\gamma(CL) = \frac{1}{2} \int_0^1 [(CL)^L_{\alpha} + (CL)^U_{\alpha}] \ d\alpha
$$

By following the similar procedure, the membership functions $\varphi_{UCL}(z)$ and $\varphi_{LCL}(z)$ and the Yager ranking indices relating to the parameters of the control chart namely UCL and LCL may be derived.

6.7 Numerical example

The Grabeur-Money savings and loan has a drive-up window. It is seen that during the busy periods for drive up service, customers arrive according to Poisson distribution and teller’s service time according to Erlang distribution. Each customer has to undergo 4 phases in the course of getting service and therefore the distribution of service time of the teller is according to Erlang 4 distribution. The building is located in a large shopping Centre, there is virtually no limit on the
number of customers wait. The company officials want to know, on an average, how many customers are waiting for service in the system along with its range namely maximum number of customers in the system and minimum number of customers in the system.

Assume that $\lambda = 1.5$, $\mu = 15$, $k = 4$ for the queueing situation described in the example.

By using the method described in Gross and Harris (1997) the expected system length is 0.1069

By using traditional Shewhart method the parameters of control chart using (6.5) for the described problem is

$$CL = 0.1069, \ UCL = 1.5541, \ LCL = -1.3196$$

This control chart method provides lower and upper control limits system length in addition to the expected system length.

The parameters of fuzzy control chart for number of customers in system are obtained to find the solution for the above situation.

Assume that the arrival rate and service rate are triangular fuzzy numbers such that

$$\tilde{\lambda} = [1, 1.5, 2] \text{ and } \tilde{\mu} = [10, 15, 20] \text{ and } k = 4.$$ Then we get

$$[x_\alpha^L, x_\alpha^U] = [0.5\alpha + 1.2 - 0.5\alpha] \text{ and } [y_\alpha^L, y_\alpha^U] = [5\alpha + 10, 20 - 5\alpha].$$

Bounds of parameters of fuzzy control chart for number of customers in system using equation (6.7) are

$$\left(\text{CL}_\alpha^L\right) = \left(-20.75\alpha^2 + 37\alpha + 157\right)/(220\alpha^2 - 1640\alpha + 3040)$$

$$\left(\text{CL}_\alpha^U\right) = \left(-20.75\alpha^2 + 46\alpha + 148\right)/(220\alpha^2 + 760\alpha + 640)$$

$$\left(\text{UCL}_\alpha^L\right) = \left(\text{CL}_\alpha^L\right) + 3\sqrt{\text{Var}_\alpha^L}; \quad \left(\text{UCL}_\alpha^U\right) = \left(\text{CL}_\alpha^U\right) + 3\sqrt{\text{Var}_\alpha^U}$$

(6.18)
\[(LCL)_\alpha^L = (CL)_\alpha^L - 3\sqrt{\text{Var}}_\alpha^L; \quad (LCL)_\alpha^U = (CL)_\alpha^U - 3\sqrt{\text{Var}}_\alpha^U\]

\[(\text{Var})_\alpha^L = \frac{-438.75\alpha^4 + 9180\alpha^3 - 49815\alpha^2 + 540\alpha + 280800}{18150\alpha^4 - 270600\alpha^3 + 1510200\alpha^2 - 3739200\alpha + 3465600} - \left(\frac{-20.75\alpha^2 + 37\alpha + 157}{220\alpha^2 - 1640\alpha + 3040}\right)^2\]

\[(\text{Var})_\alpha^U = \frac{-438.75\alpha^4 - 5670\alpha^3 - 5265\alpha^2 + 102600\alpha + 1490400}{18150\alpha^4 + 125400\alpha^3 + 322200\alpha^2 + 364800\alpha + 153600} - \left(\frac{-20.75\alpha^2 + 46\alpha + 148}{220\alpha^2 - 760\alpha + 640}\right)^2\]

The \(\alpha\) - cuts of CL and UCL for selected values of \(\alpha (= 0(.1)1)\) using equation (6.18) are calculated as presented in Table 6.2.

<table>
<thead>
<tr>
<th>(\alpha)</th>
<th>(x_a^L)</th>
<th>(x_a^U)</th>
<th>(y_a^L)</th>
<th>(y_a^U)</th>
<th>(\text{(CL)}_\alpha^L)</th>
<th>(\text{(CL)}_\alpha^U)</th>
<th>(\text{(UCL)}_\alpha^L)</th>
<th>(\text{(UCL)}_\alpha^U)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.00</td>
<td>2.00</td>
<td>10.0</td>
<td>20.0</td>
<td>0.0516</td>
<td>0.2313</td>
<td>0.8914</td>
<td>3.1038</td>
</tr>
<tr>
<td>0.1</td>
<td>1.05</td>
<td>1.95</td>
<td>10.5</td>
<td>19.5</td>
<td>0.0558</td>
<td>0.2122</td>
<td>0.9414</td>
<td>2.8587</td>
</tr>
<tr>
<td>0.2</td>
<td>1.10</td>
<td>1.90</td>
<td>11.0</td>
<td>19.0</td>
<td>0.0601</td>
<td>0.1953</td>
<td>0.9939</td>
<td>2.6433</td>
</tr>
<tr>
<td>0.3</td>
<td>1.15</td>
<td>1.85</td>
<td>11.5</td>
<td>18.5</td>
<td>0.0647</td>
<td>0.1801</td>
<td>1.0494</td>
<td>2.4526</td>
</tr>
<tr>
<td>0.4</td>
<td>1.20</td>
<td>1.80</td>
<td>12.0</td>
<td>18.0</td>
<td>0.0696</td>
<td>0.1665</td>
<td>1.1081</td>
<td>2.2826</td>
</tr>
<tr>
<td>0.5</td>
<td>1.25</td>
<td>1.75</td>
<td>12.5</td>
<td>17.5</td>
<td>0.0749</td>
<td>0.1542</td>
<td>1.1705</td>
<td>2.1301</td>
</tr>
<tr>
<td>0.6</td>
<td>1.30</td>
<td>1.70</td>
<td>13.0</td>
<td>17.0</td>
<td>0.0804</td>
<td>0.1431</td>
<td>1.2369</td>
<td>1.9925</td>
</tr>
<tr>
<td>0.7</td>
<td>1.35</td>
<td>1.65</td>
<td>13.5</td>
<td>16.5</td>
<td>0.0864</td>
<td>0.1329</td>
<td>1.3079</td>
<td>1.8677</td>
</tr>
<tr>
<td>0.8</td>
<td>1.40</td>
<td>1.60</td>
<td>14.0</td>
<td>16.0</td>
<td>0.0927</td>
<td>0.1235</td>
<td>1.3839</td>
<td>1.7539</td>
</tr>
<tr>
<td>0.9</td>
<td>1.45</td>
<td>1.55</td>
<td>14.5</td>
<td>15.5</td>
<td>0.0996</td>
<td>0.1149</td>
<td>1.4658</td>
<td>1.6498</td>
</tr>
<tr>
<td>1.0</td>
<td>1.50</td>
<td>1.50</td>
<td>15.0</td>
<td>15.0</td>
<td>0.1069</td>
<td>0.1069</td>
<td>1.5541</td>
<td>1.5541</td>
</tr>
</tbody>
</table>

The inverse functions \(L(z)\) and \(R(z)\) of \((CL)_\alpha^L, (CL)_\alpha^U, (UCL)_\alpha^L\) and \((UCL)_\alpha^U\) are obtained. The corresponding membership functions \(\varphi_{CL}(z)\) and \(\varphi_{UCL}(z)\) are expressed as

\[\varphi_{CL}(z) = \begin{cases} L(z), & 0.0516 \leq z \leq 0.1069 \\ R(z), & 0.1069 \leq z \leq 0.2313 \end{cases}\quad \text{and} \quad \varphi_{UCL}(z) = \begin{cases} L(z), & 0.8914 \leq z \leq 1.5541 \\ R(z), & 1.5541 \leq z \leq 3.1038 \end{cases}\]

The shapes of \(\varphi_{CL}(z)\) and \(\varphi_{UCL}(z)\) are displayed in Fig. 6.2.
The CL and UCL of $N_s$ at $\alpha = 0$ lies in $[0.0516, 0.2313]$ and $[0.8914, 3.1038]$ respectively. The values of the CL and UCL at $\alpha = 1$ are $0.1069$ and $1.5541$ respectively.

By using Yager ranking index method the crisp CL and UCL values of the number of customers in the system are obtained by

$$Y(\tilde{CL}) = 0.1177; \quad Y(UCL) = 1.9876$$

These values suggest the mean number and the maximum number of customers waiting in the system as crisp values for practical use.

**Conclusion**

In this chapter control charts are constructed to the number of customers in the $M/E_k/1$ queueing model in non-fuzzy and fuzzy environments. Numerical illustration is given to highlight the application.