CHAPTER – 1

INTRODUCTION

1.1 HISTORICAL DEVELOPMENTS
The concept of Shannon entropy and mutual information developed by Claude E. Shannon [1] in 1948 has found applications not only in modeling communication channels [2] but also in the field of statistics [3-6], psychology [7], economics [8, 9], fault diagnosis [10], pattern recognition [11], image registration [12], biology [13, 14], coding theory [15-18], cybernetics [19-21] and many more [22]. The Shannon entropy [1] measures the uncertainty associated with random variables embedded in a random experiment. On the other hand, mutual information quantifies the dependence between two random variables (probability distributions). Divergence measure is another entity, which effectively measures the discrimination/dependence between two probability distributions and is closely related with the mutual information. Kullback and Leibler [23] proposed a measure of divergence, called the Kullback - Leibler (KL) divergence as a generalization of Shannon’s entropy [1]. Other important measures involving more than one probability distribution are the relative information generating function and the information improvement function introduced by Hooda and Bhaker [24] and Theil [8] respectively. Investigations in information theory developed under the aegis of these information measures. These information measures became popular due to their applicability in determining and reducing uncertainty in real time systems.
In real world, the complexity generally arises from uncertainty in the form of ambiguity. The probability theory has been an age old and effective tool to handle uncertainty, but it can be applied only to real systems modelled by crisp sets. Uncertainty due to its complexity should not only be analyzed from its objectivity but also from the subjective point of view because it may occur due to unreliable information, incomplete information about the occurrence of events.
Uncertainty is the major cause of complexity, such as incomplete, due to unreliable information, from different sources. As a result, quantifying uncertainty in many systems specially modelled by fuzzy sets and its generalizations is a difficult task. Fuzzy sets, a
generalization of crisp sets, introduced by Zadeh [25], is an excellent tool for measuring the uncertainty caused by vagueness. The theory of fuzzy sets has found broad applications in multiple fields such as fault diagnosis [26], portfolio selection [27], diversion of water [28], decision making [29-32], student evaluation [33], segmentation [34], medical diagnosis [35] etc.

Luca and Termini [36] proposed non-probabilistic entropy for fuzzy sets for measuring uncertainty associated with fuzzy sets. Analogous to divergence measure defined in information theory, Bhandari and Pal [37] introduced fuzzy divergence measure and studied its properties. Thereafter, Shang and Jiang [38] pointed out the shortcomings of the Bhandari and Pal [37] divergence measure and proposed the modified version of it. Later on, various divergence measures and their generalizations were proposed by various authors [39-41] and applied them to other areas.

Atanassov [42] introduced the concept of intuitionistic fuzzy sets, as a generalization of fuzzy sets characterized by membership function, non-membership and hesitation function and applied them to different disciplines such as pattern recognition [43-51], clustering [45], medical diagnosis [52-58], decision making [59-73] etc. This generalization has proved to be more effective in handling uncertainty in ambiguous situations. Using the concept of IFS, Burillo and Bustince [74] introduced intuitionistic fuzzy entropy for measuring uncertain information. Later, various researchers introduced a number of entropy measures for IFS and shown its applications in diverse areas [63, 65, 66, 68, 70]. Vlachos and Sergiadis [44] introduced the intuitionistic fuzzy divergence measure to represent how close two intuitionistic fuzzy sets are. Thereafter, various divergence measures under intuitionistic fuzzy environment have been proposed by numerous authors and applied them to multifarious domains such as pattern recognition [44, 45, 47, 50], decision making [61, 68, 69, 72], image processing [44], clustering [45], medical diagnosis [44, 58].

Various researchers and practitioners have shown that the uncertainty measures for both crisp sets, fuzzy sets and its generalizations either does not satisfy the basic axioms or are not effective in quantifying uncertainty in many real time systems. Therefore, the need to develop new measures of uncertainty is always there considering the dynamic and ever changing world we live in. Keeping this in mind, we have proposed several new uncertainty measures which over comes all limitations embedded in measures existing in the literature and successfully applied them in diverse areas.
We first give an overview of the existing uncertainty measures proposed by various authors for crisp sets, fuzzy sets and intuitionistic fuzzy sets.

### 1.2 INFORMATION THEORETIC MEASURES

#### 1.2.1. ENTROPY MEASURES

Let

\[ \mathcal{X}_n = \left\{ (p_1, p_2, \ldots, p_n) : p_i \geq 0, \ i = 1, 2, \ldots, n, \sum_{i=1}^{n} p_i = 1 \right\} \quad n = 2, 3, \ldots \]  

be the set of all finite discrete complete probability distributions.

The entropy function defined by Shannon \[1\] is given by

\[ H_S(P) = -\sum_{i=1}^{n} p_i \log p_i = \sum_{i=1}^{n} p_i (-\log p_i) = \sum_{i=1}^{n} p_i \Delta(p_i) \]  

for all \( P = (p_1, p_2, \ldots, p_n) \in \mathcal{X}_n \) with \( \Delta(p_i) = -\log p_i \), \( p_i \) denoting the probability of occurrence of \( i^{th} \) event. The entropy function \( H_S(P) \) given by equation (1.2) gives the average uncertainty related with (1.1) and can be characterized by considering appropriate axioms. In equation (1.2), \( \Delta(p_i) \) denotes the information gain function associated with the occurrence of \( i^{th} \) event.

Shannon entropy [1] is fundamentally important from application point of view and also studied by various researchers to define its generalizations. Some significant generalizations of Shannon entropy are listed below.

Renyi [75] entropy of order- \( \alpha \)

\[ H_R(P) = \frac{1}{(1-\alpha)} \log \left( \sum_{i=1}^{n} p_i^\alpha \right), \quad \alpha \neq 1, \quad \alpha > 0, \]  

which reduces to Shannon’s entropy [1] for \( \alpha = 1 \). This was the first systematic attempt to develop a generalization of Shannon’s entropy [1]. Thereafter, many generalizations of Shannon [1] entropy and Renyi [75] entropy were reported as follows.
Havrda and Charvát [76] entropy of type-β

$$H_{HC}(P) = \frac{1}{2^{1-\beta} - 1} \left( \sum_{i=1}^{n} p_i^\beta - 1 \right), \quad \beta \neq 1, \beta > 0. \quad (1.4)$$

Kapur [77] entropy

$$H_K(P) = \frac{1}{1-\alpha} \log \left( \sum_{i=1}^{n} \frac{\sum_{i=1}^{n} p_i^{\alpha+\beta-1}}{\sum_{i=1}^{n} p_i^\beta} \right), \quad (1.5)$$

where $\alpha \neq 1, \alpha > 0, \beta > 0, \alpha + \beta - 1 > 0$.

Arimoto [78] entropy

$$H_A(P) = \frac{1}{2^{t-1} - 1} \left( \sum_{i=1}^{n} p_i^{\frac{1}{t}} \right)^t - 1, \quad t \neq 1, t > 0. \quad (1.6)$$

Sharma and Mittal [79] entropy

$$H_{SM}(P) = \frac{1}{2^{1-\beta} - 1} \left( \sum_{i=1}^{n} p_i^{\alpha-\beta-1} \right), \quad \alpha \neq 1, \beta \neq 1, \alpha, \beta > 0. \quad (1.7)$$

Sharma and Taneja [80] entropy of order-$(\alpha, \beta)$

$$H_{ST}(P) = \frac{1}{2^{1-\alpha-2^{1-\beta}}} \sum_{i=1}^{n} \left( p_i^\alpha - p_i^\beta \right), \quad \alpha \neq \beta, \alpha, \beta > 0. \quad (1.8)$$

Belis and Guiasu [81] entropy

$$H(P, U) = -\sum_{i=1}^{n} u iP_i \log p_i, \quad (1.9)$$

Here $U = (u_1, u_2, \ldots, u_n, \ldots)$ is the utility distribution associated with the random events.

Further, Pal and Pal [82, 83] examined Shannon entropy [1] in detail and observed that exponential function has an advantage over logarithmic function in some cases. Keeping this in mind, Pal and Pal [82] firstly defined a new exponential entropy measure as follows
The information gain function in (1.10) is exponential in nature which results in well defined lower and upper bounds for (1.10).

The concept of Shannon entropy [1] and its counterparts is very helpful for determination of uncertainty at the input and the output of noisy communication channels. The study of information transmission over such channels is a challenging task due to the presence of noise and other unwanted elements due to which the transmitted messages reaches the destination in distorted form. One probable way of achieving this is by attaching weights to transmitted messages which will alter the information both at the transmitting and receiving end. However one challenging question which still remains unanswered is how to attach these weights and minimize the information loss.

1.2.2. DIVERGENCE MEASURE

The divergence measures [23] determine the closeness of probability distributions to one another. They are also called dissimilarity measures or measures of discrimination between probability distributions.

Kullback and Leibler [23] proposed a measure of divergence between probability distributions \( P=(p_1, p_2, \ldots, p_n) \in \mathcal{P}_n \) and \( Q=(q_1, q_2, \ldots, q_n) \in \mathcal{P}_n \) given by

\[
D_{KL}(P \parallel Q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} \tag{1.11}
\]

The symmetric version of (1.11) is given by

\[
D_J(P \parallel Q) = D_{KL}(P \parallel Q) + D_{KL}(Q \parallel P) = \sum_{i=1}^{n} (p_i - q_i) \log \frac{p_i}{q_i} \tag{1.12}
\]

The divergence measure defined in (1.11) should satisfy the following properties:

(i) It should be non-negative.

(ii) It should be equal to zero if and only if two probability distributions \( P \) and \( Q \) are identical.

(iii) It should be a complex function of \( (p_1, p_2, \ldots, p_n) \).
However there exists divergence measures other than the Kullback and Leibler [23] divergence in literature, proposed by various authors in the past years and also found numerous applications in the field of signal selection [84], speaker recognition [85], econometric approximation [86], image registration [87, 88], statistical data processing [89], symbolic sequence analysis [90] etc. Thereafter many generalizations of divergence measure were reported, some of the significant ones are listed below:

Lin [91] observed that the measure given by (1.11) is undefined when \( q_i = 0 \) and \( p_i \neq 0 \) for any \( i \). To overcome this shortcoming, Lin [91] introduced a modified version of Kullback and Leibler [23] divergence, given by

\[
D_L (P|Q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{\left(p_i + q_i \right) / 2}. \tag{1.13}
\]

Lin [91] divergence is also called as Jensen Shannon divergence and applied them to variety of applications [92-98].

Theil [8] information improvement function

\[
D_T (P|Q|R) = \sum_{i \in N} p_i \log \frac{q_i}{r_i}. \tag{1.14}
\]

Here \( P = (p_1, p_2, \ldots, p_n) \in \mathfrak{I}_n \) is the posterior probability distribution associated with \( n \) events of a random experiment, \( Q = (q_1, q_2, \ldots, q_n) \in \mathfrak{I}_n \) is the original prediction and \( R = (r_1, r_2, \ldots, r_n) \in \mathfrak{I}_n \) is the revised probability distribution.

Taneja [99-101] has proposed a number of mean divergence measures based on some well known mean inequality. Some of the significant are listed below:

Arithmetic – Geometric mean divergence [100]

\[
D_T^{AG} (P \parallel Q) = 1 - \sum_{i=1}^{n} \sqrt{p_i q_i} \tag{1.15}
\]

Arithmetic – Harmonic mean divergence [100]

\[
D_T^{AH} (P \parallel Q) = 1 - \sum_{i=1}^{n} \frac{2 p_i q_i}{p_i + q_i} \tag{1.16}
\]
Geometric–Harmonic mean divergence \[100\]

\[
D_{T}^{GH}(P \mid Q) = \sum_{i=1}^{n} \left( \sqrt[2]{p_i q_i} - \frac{2p_i q_i}{p_i + q_i} \right).
\]

(1.17)

1.2.3. INFORMATION GENERATING FUNCTIONS

The Shannon entropy \([1] \) can be viewed as a first moment about origin associated with the random variable \(-\log p_i\), also known as the information gain function and \(i \in N\), where \(N\) being a discrete countable sample space. Since the first moment measures the uncertainty, second and the higher moments may be important for us from the statistical and information theoretic point of view. For generating these moments we need a generating function which generates the Shannon entropy and its counterparts. Keeping this in mind, Golomb \([102]\) introduced information generating function, given by

\[
I_{\alpha}(P) = \sum_{i \in N} p_i^\alpha,
\]

(1.18)

where \(P=(p_1, p_2, \ldots, p_i, \ldots)\) is a complete probability distribution, \(i \in N\), where \(N\) being a discrete countable sample space and \(\alpha\) is a real variable. Thereafter, different information generating functions have been introduced by authors, some of the significant ones are listed below:

Guisau and Reischer \([103]\) relative information generating function

\[
I_{\alpha}(P, Q) = \sum_{i \in N} p_i^\alpha q_i^{1-\alpha}
\]

(1.19)

Hooda and Singh \([104]\) information improvement generating function

\[
I_{\alpha}(P, Q, R) = \sum_{i \in N} p_i q_i^{\alpha-1} r_i^{1-\alpha}
\]

(1.20)

Hooda and Bhaker \([105]\) weighted entropy generating function

\[
I(P, U, \alpha) = \sum_{i \in N} u_i p_i^\alpha, \quad \alpha \geq 1
\]

(1.21)

Thereafter, several generating functions introduced by numerous researchers \([106-110]\) and utilized in proving the characterization theorems. Now in next section, we move towards the
extension of crisp sets, \textit{i.e.}, fuzzy sets and discussed in detail the information measures for fuzzy sets.

1.3 FUZZY SETS

Zadeh [25] extended the theory of crisp sets to fuzzy sets, which measures the degree of fuzziness. The elements of fuzzy sets are characterized by a membership function, which maps each element of a particular domain to a value between 0 and 1 known as membership value or the degree of membership. This generalization is essential in domains where information is incomplete or imprecise. Now, define fuzzy sets and some of its basic operations given as follows.

Definition 1.1 [25]: A fuzzy set \( \hat{A} \) in a discrete universe of discourse \( X = \{ x_1, x_2, \ldots, x_n \} \) is given by

\[
\hat{A} = \left\{ \langle x, \mu_{\hat{A}}(x) \rangle \left| x \in X \right. \right\},
\]

where \( \mu_{\hat{A}} : X \rightarrow [0,1] \) is the membership function of \( \hat{A} \). The number \( \mu_{\hat{A}}(x) \) describes the degree of membership of \( x \in X \) in \( \hat{A} \).

Definition 1.2 Set Operations on FS [25]: Let \( FS(X) \) denote the family of all FS in \( X \) and consider \( \hat{A}, \hat{B} \in FS(X) \) given by

\[
\hat{A} = \left\{ \langle x, \mu_{\hat{A}}(x) \rangle \left| x \in X \right. \right\}, \quad \hat{B} = \left\{ \langle x, \mu_{\hat{B}}(x) \rangle \left| x \in X \right. \right\},
\]

then some set operations are defined as follows:

(i) Containment

\( \hat{A} \subseteq \hat{B} \) if and only if \( \mu_{\hat{A}}(x) \leq \mu_{\hat{B}}(x) \); \( \forall x \in X \)

(ii) Equality

\( \hat{A} = \hat{B} \) if and only if \( \hat{A} \subseteq \hat{B} \) and \( \hat{B} \subseteq \hat{A} \)

(iii) Complement

\( \hat{A}^c = \left\{ \langle x, 1 - \mu_{\hat{A}}(x) \rangle \left| x \in X \right. \right\} \)
(iv) Intersection
\[ \hat{A} \cap \hat{B} = \left\{ (x, \mu_\hat{A}(x) \land \mu_\hat{B}(x)) \mid x \in X \right\} \]

(v) Union
\[ \hat{A} \cup \hat{B} = \left\{ (x, \mu_\hat{A}(x) \lor \mu_\hat{B}(x)) \mid x \in X \right\}, \]

where \( \lor, \land \) respectively, represent the maximum and minimum operators.

1.3.1. FUZZY ENTROPY MEASURES

The fuzzy entropy is defined on the basis of membership function and it basically quantifies the amount of information gained from a fuzzy set. Luca and Termini [36] defined fuzzy entropy of a fuzzy set \( \hat{A} \) corresponding to Shannon entropy [1] and also proposed a set of axioms which a fuzzy entropy function should satisfy, given as follows

\[
H_{LT}(\hat{A}) = -\frac{1}{n} \sum_{i=1}^{n} \left[ \mu_\hat{A}(x_i) \log \mu_\hat{A}(x_i) + \left(1 - \mu_\hat{A}(x_i)\right) \log \left(1 - \mu_\hat{A}(x_i)\right) \right].
\]  

(1.23)

Kaufman and Swanson [111] fuzzy entropy

\[
H_{KS}(\hat{A}) = -\frac{1}{\log n} \sum_{i=1}^{n} \left( \Phi_\hat{A}(x_i) \log \Phi_\hat{A}(x_i) \right),
\]  

(1.24)

where

\[
\Phi_\hat{A}(x_i) = \frac{\mu_\hat{A}(x_i)}{\sum_{i=1}^{n} \mu_\hat{A}(x_i)}; \quad \forall \ x_i \in X.
\]  

(1.25)

Yager [112] fuzzy entropy

\[
H_Y(\hat{A}) = 1 - \frac{d_p(\hat{A}, \hat{A}^C)}{n^{V_p}},
\]  

(1.26)

where
\[ d_p \left( \hat{A}, \hat{B} \right) = \left( \sum_{i=1}^{n} \left| \mu_{\hat{A}}(x_i) - \mu_{\hat{B}}(x_i) \right|^p \right)^{1/p}, \quad p \geq 1. \] (1.27)

Bhandari and Pal [37] fuzzy entropy

\[ H_{BP} \left( \hat{A} \right) = \frac{1}{n(1-\alpha)} \sum_{i=1}^{n} \log \left[ \left( \mu_{\hat{A}}(x_i) \right)^\alpha + \left( 1 - \mu_{\hat{A}}(x_i) \right)^\alpha \right], \quad \alpha \neq 1, \alpha > 0. \] (1.28)

Kapur [113] defined the following measure of fuzzy entropy given by

\[ H_K \left( \hat{A} \right) = \frac{1}{n(1-\beta)} \sum_{i=1}^{n} \left[ \left( \mu_{\hat{A}}(x_i) \right)^\beta + \left( 1 - \mu_{\hat{A}}(x_i) \right)^\beta - 1 \right], \quad \beta \neq 1, \beta > 0. \] (1.29)

Hooda [114] fuzzy entropy

\[ H_H \left( \hat{A} \right) = \frac{1}{n(1-\beta)} \sum_{i=1}^{n} \left[ \left( \mu_{\hat{A}}(x_i) \right)^\alpha + \left( 1 - \mu_{\hat{A}}(x_i) \right)^\alpha \right]^{\frac{\beta-1}{\alpha-1}} - 1, \quad \alpha \neq 1, \beta \neq 1, \alpha, \beta > 0. \] (1.30)

1.3.2. FUZZY DIVERGENCE MEASURES

In earlier sections, we studied that a probabilistic divergence measure quantifies the dissimilarity between two probability distributions. In a similar manner, a fuzzy divergence measure quantifies the dissimilarity between two fuzzy sets. Considering \( \hat{A} \) and \( \hat{B} \) as two fuzzy sets defined on discrete universe of discourse \( X = \{x_1, x_2, \ldots, x_n\} \) having the membership values \( \mu_{\hat{A}}(x_i) \) and \( \mu_{\hat{B}}(x_i) \), \( i = 1, 2, \ldots, n \) respectively, Bhandari and Pal [37] proposed the fuzzy divergence measure of \( \hat{B} \) to \( \hat{A} \), given by

\[ D_{BP} \left( \hat{A} | \hat{B} \right) = \frac{1}{n} \sum_{i=1}^{n} \left[ \mu_{\hat{A}}(x_i) \log \frac{\mu_{\hat{A}}(x_i)}{\mu_{\hat{B}}(x_i)} + \left( 1 - \mu_{\hat{A}}(x_i) \right) \log \frac{1 - \mu_{\hat{A}}(x_i)}{1 - \mu_{\hat{B}}(x_i)} \right]. \] (1.31)

Shang and Jiang [38] observed that the divergence measure proposed in (1.31) is undefined when \( \mu_{\hat{B}}(x_i) \) tends to 0 or 1. Therefore, to overcome this shortcoming, Shang and Jiang [38] proposed a fuzzy divergence measure, which is a modified version of (1.31), given as follows
\[
D_{SJ}(\hat{A}|\hat{B}) = \frac{1}{n} \sum_{i=1}^{n} \left[ \mu_{\hat{A}}(x_i) \log \frac{\mu_{\hat{A}}(x_i)}{\mu_{\hat{A}}(x_i) + \mu_{\hat{B}}(x_i)} + (1 - \mu_{\hat{A}}(x_i)) \log \frac{1 - \mu_{\hat{A}}(x_i)}{2 - \mu_{\hat{A}}(x_i) - \mu_{\hat{B}}(x_i)} \right]
\]

(1.32)

A number of generalized fuzzy divergence measures exist in literatures which are either based on information theoretic concepts or are based on the properties of fuzzy sets and its analogues. Some of these measures, which are extensively utilized by researchers from the application point of view, are listed below.

Hooda [114] generalized fuzzy divergence

\[
D_H(\hat{A}|\hat{B}) = \frac{1}{n(\beta-1)} \sum_{i=1}^{n} \left[ \left( \mu_{\hat{A}}(x_i) \right)^{\beta} \left( \mu_{\hat{B}}(x_i) \right)^{1-\beta} + \left( 1 - \mu_{\hat{A}}(x_i) \right)^{\beta} \left( 1 - \mu_{\hat{B}}(x_i) \right)^{1-\beta} - 1 \right],
\]

(1.33)

where \( \beta > 1, \beta \neq 1 \).

Ohlan [41] generalized fuzzy divergence

\[
D_O(\hat{A}|\hat{B}) = \sum_{i=1}^{n} \frac{(\mu_{\hat{A}}(x_i) - \mu_{\hat{B}}(x_i))^2}{2^t} \left[ \left( \frac{\mu_{\hat{A}}(x_i) + \mu_{\hat{B}}(x_i)}{\sqrt{\mu_{\hat{A}}(x_i) \cdot \mu_{\hat{B}}(x_i)}} \right)^{t+1} + \left( \frac{2 - \mu_{\hat{A}}(x_i) - \mu_{\hat{B}}(x_i)}{\sqrt{(1 - \mu_{\hat{A}}(x_i)) \cdot (1 - \mu_{\hat{B}}(x_i))}} \right)^{t+1} \right],
\]

(1.34)

where \( t = 0, 1, 2, \ldots \).

In the next section, we move towards the extension of fuzzy sets, i.e., intuitionistic fuzzy sets and study some entropy and divergence measures for IFS.

1.4 INTUITIONISTIC FUZZY SETS

Atanassov [42] introduced the concept of intuitionistic fuzzy sets, characterized by a membership function and a non-membership function. This generalization of fuzzy sets has been popular and effective due to its simplicity and easy applicability in multiple domains.

We now present some basic preliminaries related to intuitionistic fuzzy sets.
**Definition 1.3 [42]:** An intuitionistic fuzzy set $A$ defined on universe of discourse $X$, introduced by Atanassov [42], given by the expression

\[ A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}, \]

where the functions $\mu_A(x): X \to [0, 1]$ and $\nu_A(x): X \to [0, 1]$ denote the membership degree and the non-membership degree to $A$ respectively. For every $x \in X$

\[ 0 \leq \mu_A(x) + \nu_A(x) \leq 1. \]

Further, $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ for all $x \in X$, denotes hesitation degree (or intuitionistic index) to $A$ which basically expresses lack of information of whether $x$ belongs to $A$ or not. It is obvious that $0 \leq \pi_A(x) \leq 1$ for every $x \in X$. In fact, when $\mu_A(x) = 1 - \nu_A(x)$ for all $x \in X$, an IFS is converted into a FS.

Let $AIFS(X)$ denote the family of all Atanassov intuitionistic fuzzy sets in the finite universe $X = \{x_1, x_2, \ldots, x_n\}$ and is used throughout in this thesis. Consider $A, B \in AIFS(X)$ given by $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}$ and $B = \{ \langle x, \mu_B(x), \nu_B(x) \rangle \mid x \in X \}$. Some set operations on $AIFS(X)$ are defined as follows:

- **Complement of $A$**
  \[ A^C = \{ \langle x, \nu_A(x), \mu_A(x) \rangle \mid x \in X \}. \]

- **Inclusion Relation**
  $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x), \forall x \in X$.

- **Union of $A$ and $B$**
  \[ A \cup B = \{ \langle x, \max \{ \mu_A(x), \mu_B(x) \}, \min \{ \nu_A(x), \nu_B(x) \} \rangle \mid x \in X \}. \]

- **Intersection of $A$ and $B$**
  \[ A \cap B = \{ \langle x, \min \{ \mu_A(x), \mu_B(x) \}, \max \{ \nu_A(x), \nu_B(x) \} \rangle \mid x \in X \}. \]
1.4.1. INTUITIONISTIC FUZZY ENTROPY MEASURES

An intuitionistic fuzzy entropy measure measures the uncertainty associated with IFS which intuitively should be function of membership degree, non membership degree and the hesitation index. The first initiative in this direction was taken by by Burillo and Bustince [74] and defined the intuitionistic fuzzy entropy given as follows

\[ E_{BB}(A) = \frac{1}{n} \sum_{i=1}^{n} \pi_{A}(x_i). \]  

(1.37)

However, the above measures considered only the membership and the non membership functions and not the hesitation index. Some researchers realized that omitting one of the functions may lead to the incorrect result. Therefore, keeping this in mind, several authors and researchers incorporate the hesitation index and as a result some new IFE measures are introduced.

Szmidt and Kacprzyk [115] intuitionistic fuzzy entropy

\[ E_{SK}(A) = \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{\mu_{A}(x_i) \wedge \nu_{A}(x_i) + \pi_{A}(x_i)}{\mu_{A}(x_i) \vee \nu_{A}(x_i) + \pi_{A}(x_i)} \right]. \]  

(1.38)

Zeng and Li [116] intuitionistic fuzzy entropy

\[ E_{ZL}(A) = 1 - \frac{1}{n} \sum_{i=1}^{n} \left[ \mu_{A}(x_i) - \nu_{A}(x_i) \right]. \]  

(1.39)

Hung and Yang [117] intuitionistic fuzzy entropy

\[ E_{HY}(A) = -\frac{1}{n} \sum_{i=1}^{n} \left[ \mu_{A}(x_i) \log \mu_{A}(x_i) + \nu_{A}(x_i) \log \nu_{A}(x_i) + \pi_{A}(x_i) \log \pi_{A}(x_i) \right]. \]  

(1.40)

Zhang and Jiang [118] intuitionistic fuzzy entropy

\[ E_{ZJ1}(A) = -\frac{1}{n} \sum_{i=1}^{n} \left[ \left( \frac{\mu_{A}(x_i) + 1 - \nu_{A}(x_i)}{2} \right) \log \left( \frac{\mu_{A}(x_i) + 1 - \nu_{A}(x_i)}{2} \right) \right] \]
\[ + \left( \frac{\nu_{A}(x_i) + 1 - \mu_{A}(x_i)}{2} \right) \log \left( \frac{\nu_{A}(x_i) + 1 - \mu_{A}(x_i)}{2} \right), \]  

(1.41)

\[ E_{ZJ2}(A) = \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{\mu_{A}(x_i) \wedge \nu_{A}(x_i)}{\mu_{A}(x_i) \vee \nu_{A}(x_i)} \right]. \]  

(1.42)
Vlachos and Sergiadis [44] intuitionistic fuzzy entropy

\[ E_{VS}(A) = -\frac{1}{n} \sum_{i=1}^{n} \left[ \mu_A(x_i) \log \mu_A(x_i) + \nu_A(x_i) \log \nu_A(x_i) - (1 - \pi_A(x_i)) \log (1 - \pi_A(x_i)) - \pi_A(x_i) \right]. \]  

(1.43)

Ye [119] intuitionistic fuzzy entropy

\[ E_{Y1}(A) = \frac{1}{n} \sum_{i=1}^{n} \left[ \sin \left( \frac{\pi \times (\mu_A(x_i) + 1 - \nu_A(x_i))}{4} \right) - 1 \right] \times \frac{1}{\sqrt{2} - 1}, \]  

(1.44)

\[ E_{Y2}(A) = \frac{1}{n} \sum_{i=1}^{n} \left[ \cos \left( \frac{\pi \times (\mu_A(x_i) + 1 - \nu_A(x_i))}{4} \right) - 1 \right] \times \frac{1}{\sqrt{2} - 1}. \]  

(1.45)

Wei et al. [120] intuitionistic fuzzy entropy

\[ E_{W}(A) = \frac{1}{n} \sum_{i=1}^{n} \left[ \sqrt{2} \cos \left( \frac{\mu_A(x_i) - \nu_A(x_i)}{4} \right) - 1 \right] \times \frac{1}{\sqrt{2} - 1}. \]  

(1.46)

Li et al. [121] intuitionistic fuzzy entropy

\[ E_{L}(A) = 1 - \frac{1}{2n} \sum_{i=1}^{n} \left( |\mu_A(x_i) - \nu_A(x_i)| + |\mu_A(x_i) - \nu_B(x_i)| \right). \]  

(1.47)

Guo and Song [70] intuitionistic fuzzy entropy

\[ E_{G}(A) = (1 - |\mu_A(x_i) - \nu_A(x_i)|) \frac{1 + \pi_A(x_i)}{2}. \]  

(1.48)

1.4.2. INTUITIONISTIC FUZZY DIVERGENCE MEASURES

An intuitionistic fuzzy divergence measures [44] represent the dissimilarity between two intuitionistic fuzzy sets and it should be a function of membership function, non membership function and hesitation index in order that it takes all the uncertainty paradigms into account.

Let \( A \) and \( B \) be two intuitionistic fuzzy sets fuzzy sets defined in discrete universe of discourse \( X = \{x_1, x_2, \ldots, x_n\} \) having the membership values \( \mu_A(x_i) \) and \( \mu_B(x_i) \), \( i = 1, 2, \ldots, n \).
and non-membership values $\nu_A(x_i)$ and $\nu_B(x_i)$, $i=1,2,...,n$ respectively. Vlachos and Sergiadis [44] defined the intuitionistic fuzzy divergence measure of intuitionistic fuzzy set $B$ relative to intuitionistic fuzzy set $A$ by

$$D_{VS}(A \mid B) = \sum_{i=1}^{n} \left( \mu_A(x_i) \ln \left( \frac{\mu_A(x_i)}{(1/2)(\mu_A(x_i) + \mu_B(x_i))} \right) + \nu_A(x_i) \ln \left( \frac{\nu_A(x_i)}{(1/2)(\nu_A(x_i) + \nu_B(x_i))} \right) \right).$$

The symmetric version of the measure (1.49) is given by

$$D_{VS}(A \parallel B) = D_{VS}(A \mid B) + D_{VS}(B \mid A).$$

Let $A = \{(x_i, \mu_A(x_i), \nu_A(x_i), \pi_A(x_i)) | x_i \in X \}$ and $B = \{(x_i, \mu_B(x_i), \nu_B(x_i), \pi_B(x_i)) | x_i \in X \}$ be two intuitionistic fuzzy sets defined in $X = \{x_1, x_2, ..., x_n \}$. From the definition of intuitionistic fuzzy set, we have:

$$\mu_A(x_i) + \nu_A(x_i) + \pi_A(x_i) = 1, \quad \mu_B(x_i) + \nu_B(x_i) + \pi_B(x_i) = 1,$$

$$0 \leq \mu_A(x_i), \nu_A(x_i), \pi_A(x_i) \leq 1, \quad 0 \leq \mu_B(x_i), \nu_B(x_i), \pi_B(x_i) \leq 1; \quad x_i \in X.$$

This suggests that $(\mu_A(x_i), \nu_A(x_i), \pi_A(x_i))$ and $(\mu_B(x_i), \nu_B(x_i), \pi_B(x_i))$ may be considered as probability distributions for an element $x_i$. Using this probabilistic approach, Wei and Ye [47] introduced the modified version of (1.49), given by

$$D_{WY}(A \mid B) = \sum_{i=1}^{n} \left( \mu_A(x_i) \ln \left( \frac{\mu_A(x_i)}{(1/2)(\mu_A(x_i) + \mu_B(x_i))} \right) \right) + \nu_A(x_i) \ln \left( \frac{\nu_A(x_i)}{(1/2)(\nu_A(x_i) + \nu_B(x_i))} \right) + \pi_A(x_i) \ln \left( \frac{\pi_A(x_i)}{(1/2)(\pi_A(x_i) + \pi_B(x_i))} \right).$$

The symmetric version of the measure (1.50) is given by

$$D_{WY}(A \parallel B) = D_{WY}(A \mid B) + D_{WY}(B \mid A).$$
Hung and Yang [45] defined another divergence measure called ‘J-divergence’ for measuring the difference between two IFS and then applied it to application of clustering analysis and pattern recognition.

\[
D_{HY}(A|B) = \sum_{i=1}^{n} \left[ - \frac{\mu_A(x_i) + \mu_B(x_i)}{2} \log \left( \frac{\mu_A(x_i) + \mu_B(x_i)}{2} \right) + \frac{\nu_A(x_i) + \nu_B(x_i)}{2} \log \left( \frac{\nu_A(x_i) + \nu_B(x_i)}{2} \right) + \frac{\pi_A(x_i) + \pi_B(x_i)}{2} \log \left( \frac{\pi_A(x_i) + \pi_B(x_i)}{2} \right) \right] + \left[ \frac{\mu_A(x_i) \log \mu_A(x_i)}{2} + \frac{\nu_A(x_i) \log \nu_A(x_i)}{2} + \frac{\pi_A(x_i) \log \pi_A(x_i)}{2} \right] \log x_i B_i A_i A_k B_i \right)
\]

Mao et al. [50] introduced intuitionistic fuzzy divergence

\[
D_M(A|B) = \sum_{i=1}^{n} \left[ \pi_A(x_i) \ln \left( \frac{\pi_A(x_i)}{(1/2)(\pi_A(x_i) + \pi_B(x_i))} \right) + \Delta_A(x_i) \ln \left( \frac{\Delta_A(x_i)}{(1/2)(\Delta_A(x_i) + \Delta_B(x_i))} \right) \right],
\]

where \( \Delta_A(x_i) = |\mu_A(x_i) - \nu_A(x_i)| \), denotes that how close the membership and non membership degrees are. The symmetric divergence measure of (1.52) is defined as follows

\[
D_M(A||B) = D_M(A|B) + D_M(B|A).
\]

Ohlan [72] introduced intuitionistic fuzzy divergence

\[
D_O(A|B) = \sum_{i=1}^{n} \left[ \frac{1}{2} - \left( 1 - \frac{\mu_A(x_i) + \nu_A(x_i)}{2} \right) \left( \frac{\mu_A(x_i) + 1 - \nu_A(x_i)}{2} - \frac{\mu_B(x_i) + 1 - \nu_B(x_i)}{2} \right) \right].
\]

Next, we discuss the objectives and motivations of the work done in this thesis.
1.5 OBJECTIVES AND MOTIVATIONS

Measuring uncertainty in today’s environment which is dynamic, complex and vague is a difficult task. The uncertainty measures proposed by various authors for both crisp and fuzzy sets possess some serious limitations either due to the violation of basic axioms or in the manner in which they are defined.

The Shannon entropy [1] satisfies the basic axioms of an uncertainty function, however it is not defined at extreme points plus it does not possess any parameter. The presence of parameter provides flexibility especially from the application point of view. Moreover, we are able to measure uncertainty for a wide range of values. Keeping this in mind, we have proposed new parametric entropy functions which are generalizations of Shannon entropy [1].

We have proposed a new parametric entropy function based on exponential information gain function. We have also defined a new weighted information generating function, which is a generalization of Golomb [102] function.

Various authors have defined the uncertainty measures for fuzzy sets and intuitionistic fuzzy sets analogous to the measures defined in information theory. The information theoretic measures are function of probabilities associated with the random experiment, whereas the fuzzy information measures and their generalizations are functions of membership degree, non-membership and hesitation degree. The probability of a random event and the membership degree of elements of fuzzy sets are two different concepts and as a result, the uncertainty measures defined in these two domains may not necessarily possess identical axioms or identical properties.

For example, the divergence measures proposed by Vlachos & Sergiadis [44] and Mao et. al. [50] for intuitionistic fuzzy sets is analogous to the divergence measure proposed by Lin [91] in information theory. However, Lin [91] divergence measure is a valid divergence measure satisfying all basic axioms but its fuzzy counterparts does not satisfy the basic axiom of non-negativity. Also the measure proposed by Mao et. al [50] does not vanish whenever two intuitionistic fuzzy sets are equal and vice versa. Keeping these limitations in mind, we have proposed new divergence measures for fuzzy sets and intuitionistic fuzzy sets, which satisfy all basic axioms to be a valid divergence measure. Moreover, we have also defined its parametric analogous in order to improve its applicability in applications such as medical diagnosis and decision making. The properties and the numerical results discussed in the
above mentioned applications shows that the proposed measures are well defined and applicable in real time systems.

1.6 OUTLINE OF THESIS

The thesis is divided into eight subsequent chapters and a brief description of chapters is as follows

1.6.1 CHAPTER 2

This chapter introduces a new weighted information generating functions and discuss the case for discrete probability distributions such as uniform probability distribution, geometric probability distribution and \( \beta \) power probability distribution. The relative information generating function and the information improvement function are discussed and utilized in obtaining the characterization theorems based on a set of axioms.

1.6.2 CHAPTER 3

The intent of this chapter is to define two new entropy measures, one of which is a generalization of Shannon entropy and the other one is achieved by taking the expectation of parametric exponential information gain function. The properties and bounds of the proposed entropy measures are discussed and proved. The findings of the proposed entropy measures are applied to drought risk assessment problem for water management.

1.6.3 CHAPTER 4

The purpose of this chapter is to define a new generalized fuzzy divergence measure based on exponential function by using the concept of Jensen Shannon divergence, called generalized fuzzy exponential divergence measure for measuring the amount of discrimination and its interesting properties are stated and proved. The proposed divergence measure is utilized to solve the problem of multi-criteria decision making under fuzzy settings and results show the applicability of the proposed measure.

The analysis of fuzzy sets is extended to intuitionistic fuzzy sets in subsequent chapters.
1.6.4 CHAPTER 5
This chapter proposes a new divergence measure for intuitionistic fuzzy sets, called intuitionistic fuzzy divergence measure. Some ingenious properties apart from the axioms are also studied and proved. The counter-intuitive cases of the existing divergence measures are also discussed through a numerical example. The significance of the proposed measure is shown through a numerical example application of medical diagnosis.

1.6.5 CHAPTER 6
This chapter extends the concept of intuitionistic fuzzy divergence measures to its parametric analogues and its application in medical diagnosis is illustrated through a numerical example. The importance of the parameter of the proposed measure is explained by the comparison of results with the existing results studied in the literature.

1.6.6 CHAPTER 7
In this chapter, a new entropy measure for intuitionistic fuzzy sets is proposed and studied its properties. Two numerical examples are given to illustrate the validity of the proposed entropy measure. The proposed measure plays a crucial role in decision making problem of mobile phone selection.

1.6.7 CHAPTER 8
This chapter summarizes the work done in the thesis and also discusses the scope of the research work in the future.