CHAPTER 3

TWO NEW ENTROPY MEASURES AND THEIR APPLICATIONS IN DROUGHT RISK ASSESSMENT FOR BETTER WATER MANAGEMENT

3.1. INTRODUCTION

Over increasing demand and deficiency of water brings the situation of drought. Droughts result from severe water shortage due to lack of rains over long periods of time disturbing various human activities and lead to problems like crop production failure, depleted groundwater level, depletion in water bodies like lakes, rivers, reservoirs, shortage of drinking water, reduced feed availability for animals, famine etc. The main intent is how to examine water scarcity situation for the duration of drought so that we conserve the water during drought time [122-127]. It is considered as unexpected probabilistic attribute of drought leads several uncertainties, which cushion the impact of decision about water security during drought periods and risk regarding water security may be underrated. The analysis of risk of water security with reference to its uncertainty and complexity is essential during drought situations. Since entropy function are effective tools for determining uncertainty in real time systems, therefore two new parametric entropy measures are introduced to effectively deal with uncertainty in drought situations. Drought risk assessment involves three types of indices namely, reliability $R_e$, resiliency $R_s$ and vulnerability $V_u$ along with a drought risk index [125, 127, 131]. The weight determination of three indices is difficult due to uncertainty involved in determination of these indices. Many investigations have been done by practitioners in the direction of drought risk assessment in widespread influenced regions [122 - 131].

In this chapter, two new parametric entropy measures are defined, one is parametric generalization of Shannon entropy and the other one is parametric exponential entropy measure. Some elegant properties and bounds of the proposed parametric entropy functions
are also studied. An integrated approach of entropy weight technique for order preference by similarity to an ideal solution (EW-TOPSIS) [130] and the proposed parametric entropy measures are utilized in determining the water security risk for the case study of Haihe River Basin during drought period. The results demonstrate that the proposed entropy approach adequately takes into consideration the complexity and uncertainty information of the water resources, which makes the assessment results more appropriate and accurate.

3.2. NEW PARAMETRIC LOGARITHMIC ENTROPY MEASURE

There exist several parametric generalizations of Shannon entropy such as Renyi [75] entropy, Havrda and Charvat’s [76] entropy etc. is available in the literature. The parameter provides flexibility especially in situations where the uncertainty is to be evaluated in different context.

For all \( P = (p_1, p_2, \ldots, p_n) \in \mathcal{P}_n \), a new parametric logarithmic entropy measure is defined as

\[
H^k_i(P) = k - \sum_{i=1}^{n} p_i \log(p_i + 2^k - 1)
\]  

(3.1)

The equation (3.1) can be expressed as

\[
H^k_i(P) = k - \sum_{i=1}^{n} p_i \left(k - \log(p_i + 2^k - 1)\right) = \sum_{i=1}^{n} p_i \Omega(p_i),
\]  

(3.2)

where

\[
\Omega(p_i) = \left[k - \log(p_i + 2^k - 1)\right], \forall i
\]  

(3.3)

The equation (3.2) can be written as

\[
H^k_i(P) = -\sum_{i=1}^{n} p_i \left\{ \log \frac{p_i + 2^k - 1}{2^k} \right\}
\]  

(3.4)

where \( k = 1, 2, \ldots, m; m \in \mathbb{N} \), \( \mathbb{N} \) being the set of natural numbers and \( \Omega(p_i) \), \( \forall i \) denoting the information gain function associated with measure (3.2). For \( k = 0 \), the relation (3.3) reduces to \(-\log p_i\), which basically represents the gain in information from an event with probability \( p_i \); \( k = 1 \) gives \( 1 - \log (p_i + 1) = -\log \left( (p_i + 1)/2 \right) \), which basically represents gain in information from an event with probability \( (p_i + 1)/2 \); \( k = 2 \) gives
$2 - \log_2 (p_i + 3) = - \log_2 ((p_i + 3)/4)$, which basically represents the information gain from an event with probability $(p_i + 3)/4$ and so on. As we increase the value of $k$, uncertainty defined by measure (3.2) goes on increasing. The information gain function $\Omega(p_i), \forall i$ satisfies the following properties:

1. $\Omega(p_i) = \left[k - \log_2 (p_i + 2^k - 1)\right]$ is defined at all points in $[0, 1]$.
2. $\lim_{p_i \to 0} \Omega(p_i) = \Omega(p_i = 0) = l_1, l_1 \geq 0$ and finite.
3. $\Omega(p_i = 1) = l_2, l_2 \geq 0$ and finite.
4. $l_2 < l_1$.
5. $\Omega(p_i)$ and $H_k^l(P)$ are continuous for $0 \leq p_i \leq 1$.
6. With increase in $p_i$, $\Omega(p_i)$ decreases linearly.

### 3.2.1. Properties of Proposed Parametric Logarithmic Entropy Measure $H_k^l(P)$

**P1.** The proposed parametric logarithmic entropy function $H_k^l(P)$ defined by relation (3.1) is concave in nature.

**Proof.** Consider the relation (3.1), we have

$$f(p) = k - p \log_2 (p + 2^k - 1); 0 \leq p \leq 1; k = 1, 2, \ldots, m; m \in N,$$

Then, we have

$$f'(p) = - \log_2 (p + 2^k - 1) - \left(\frac{p}{p + 2^k - 1}\right),$$

and

$$f''(p) = 2 - p - 2^{k+1} \left(\frac{p}{p + 2^k - 1}\right)^2 < 0.$$

Hence, $f(p)$ is a concave function. Therefore the proposed entropy measure $H_k^l(P)$ is concave in nature.
The proposed entropy function \( H^k_l(P) \) given by (3.1) reaches the maximum when all the probabilities \( p_i \)'s are equal.

**Proof.** From equation (3.1), we have

\[
H^k_l(P) = k - \sum_{i=1}^{n} p_i \log(p_i + 2^k - 1)
\]

\[
= k - \sum_{i=1}^{n-1} p_i \log(p_i + 2^k - 1) - \left[ 1 - (p_1 + p_2 + \cdots + p_{n-1}) \right] \log \left[ 1 - (p_1 + p_2 + \cdots + p_{n-1}) + 2^k - 1 \right]
\]

Now, consider

\[
\frac{\partial}{\partial p_i} H^k_l(P) = 0; \quad i = 1, \ldots, n - 1,
\]

which gives

\[
\log \left( \frac{1 - (p_1 + p_2 + \cdots + p_{n-1}) + 2^k - 1}{p_i + 2^k - 1} \right) = \frac{p_i}{p_i + 2^k - 1} - \frac{\left[ 1 - (p_1 + p_2 + \cdots + p_{n-1}) \right]}{\left[ 1 - (p_1 + p_2 + \cdots + p_{n-1}) + 2^k - 1 \right]}
\]

Suppose \( p_i = y_i \) and \( 1 - (p_1 + p_2 + \cdots + p_n) = z_i \), then the above equality becomes

\[
\frac{z_i}{\left( z_i + 2^k - 1 \right)} + \log \left( z_i + 2^k - 1 \right) = \frac{y_i}{\left( y_i + 2^k - 1 \right)} + \log \left( y_i + 2^k - 1 \right)
\]

Now, we define a function

\[
f(y) = \frac{y}{\left( y + 2^k - 1 \right)} + \log \left( y + 2^k - 1 \right), \quad 0 \leq y \leq 1 \quad (3.5)
\]

Let \( y_1, y_2 \in [0,1] \). Then from equation (3.5), we have

\[
f(y_1) = \frac{y_1}{\left( y_1 + 2^k - 1 \right)} + \log \left( y_1 + 2^k - 1 \right),
\]

and
\[ f(y_2) = \frac{y_2}{(y_2 + 2^k - 1)} + \log\left( y_2 + 2^k - 1 \right). \]

Now, if \( y_1 > y_2 \) then

\[ \frac{y_1}{(y_1 + 2^k - 1)} + \log\left( y_1 + 2^k - 1 \right) > \frac{y_2}{(y_2 + 2^k - 1)} + \log\left( y_2 + 2^k - 1 \right) \]

which gives \( f(y_1) > f(y_2) \).

If \( y_1 < y_2 \), then

\[ \frac{y_1}{(y_1 + 2^k - 1)} + \log\left( y_1 + 2^k - 1 \right) < \frac{y_2}{(y_2 + 2^k - 1)} + \log\left( y_2 + 2^k - 1 \right) \]

which gives \( f(y_1) < f(y_2) \).

Therefore, \( f(y_1) = f(y_2) \) if and only if \( y_1 = y_2 \). So, we can write \( y_i = z_i \) for \( i = 1, 2, \ldots, n-1 \) and hence

\[ \sum_{i=1}^{n-1} p_i = \sum_{i=1}^{n-1} \left( 1 - (p_1 + p_2 + \cdots + p_{n-1}) \right) \]

or

\[ p_1 + p_2 + \cdots + p_{n-1} = (n-1) - (n-1)(1 - p_n) \]

\[ \Rightarrow p_n = 1/n. \]

Therefore, the proposed entropy function \( H_k^1(P) \) achieves the maximum when all the probabilities are equal.

**P3.** The function given by the equation (3.1) monotonically increases in \([0,0.5)\), monotonically decreases in \((0.5, 1]\) and attains the maximum at \( p = 1/2 \).

**Proof.** Consider the function defined by equation (3.1)
\[ H^k_i(P) = k - \left( p \log(p + 2^k - 1) + (1 - p) \log(2^k - p) \right), \quad 0 \leq p \leq 1, \]

\[ \Rightarrow \frac{dH^k_i(P)}{dp} = \log \left( \frac{2^k - p}{p + 2^k - 1} \right) - \frac{(2p - 1)(2^k - 1)}{(p + 2^k - 1)(2^k - p)} \]

For \( p \in (0, 0.5) \), then

\[ \log \left( \frac{2^k - p}{p + 2^k - 1} \right) > \frac{(2p - 1)(2^k - 1)}{(p + 2^k - 1)(2^k - p)} \]

and if \( p \in (0.5, 1] \), then

\[ \log \left( \frac{2^k - p}{p + 2^k - 1} \right) < \frac{(2p - 1)(2^k - 1)}{(p + 2^k - 1)(2^k - p)} \]

Therefore, we have

\[ \frac{dH^k_i(P)}{dp} > 0, \quad \text{if } p \in (0, 0.5); \]

\[ \frac{dH^k_i(P)}{dp} < 0, \quad \text{if } p \in (0.5, 1] \]

and

\[ \frac{dH^k_i(P)}{dp} = 0, \quad \text{if } p = 0.5, \]

which gives the required result.

P4. The proposed entropy measure \( H^k_i(P) \) given by (3.1) is non-negative.

Proof. We know for all \( i \), we have
Taking log on both sides in the above inequality, we have

\[-\sum_{i=1}^{n} \log \left( \frac{p_i + 2^k - 1}{2^k} \right) \geq -\log 1\]

\[\Rightarrow -\sum_{i=1}^{n} \log \left( \frac{p_i + 2^k - 1}{2^k} \right) \geq 0\]

\[\Rightarrow -\sum_{i=1}^{n} p_i \log \left( \frac{p_i + 2^k - 1}{2^k} \right) \geq 0\]

\[\Rightarrow k - \sum_{i=1}^{n} p_i \log \left( p_i + 2^k - 1 \right) \geq 0.\]

\[\Rightarrow H^k_i (P) \geq 0.\]

Therefore, proposed entropy measure \(H^k_i (P)\) is non-negative.

**P5.** The proposed entropy measure \(H^k_i (P)\) given by (3.1) is a decreasing function of \(k\).

**Proof.** Differentiating the equation (3.1) with respect to ‘\(k\)’, we have

\[\frac{\partial H^k_i (P)}{\partial k} = 1 - \frac{p}{p + 2^k - 1} \left( 2^k \log 2 \right) \leq 0.\]

Therefore, \(H^k_i (P)\) is a decreasing function of \(k\).

From the above properties, it is clear that \(H^k_i (P)\) given by equation (3.1) is itself a parametric entropy function for different values of \(k\). This yields the required result.

### 3.2.2. Bounds of \(H^k_i (P)\)

**Theorem 3.1** For all \(P = (p_1, p_2, \ldots, p_n) \in \mathcal{P}_n\), we have

\[0 \leq p_i \leq 1\]

\[0 \leq \frac{p_i + 1}{2} \leq 1\]

\[0 \leq \frac{p_i + 3}{4} \leq 1\]

\[\vdots\]

\[0 \leq \frac{p_i + 2^k - 1}{2^k} \leq 1\]
i. \[ H_k^l(P) \leq \frac{1}{2^{k-1}} H_1^l(P). \]

ii. \[ H_k^l(P) \leq \left( \frac{1}{2^k} \right) H_S(P). \]

**Proof:** Let \( f : X \to \mathbb{R} \) be a concave function, if for every \( x, y \in X \) and \( \lambda_1, \lambda_2 \in [0,1] \), we have

\[ f(\lambda_1 x + (1 - \lambda_2) y) \geq \lambda_1 f(x) + (1 - \lambda_2) f(y) \]  \hspace{1cm} (3.6)

Substituting \( x_1 = p_1, x_2 = 1, f(x) = \log x \) and \( \lambda_1 = \lambda_2 = 1/2 \) in (3.6), we obtain

\[ \log \left( \frac{p_1 + 1}{2} \right) \geq \frac{1}{2} (\log(p_1) + \log(1)) \]

\[ \Rightarrow -\sum_{i=1}^{n} p_i \log \left( \frac{p_1 + 1}{2} \right) \leq -\frac{1}{2} \sum_{i=1}^{n} p_i \log(p_i) \]

\[ \Rightarrow H_1^l(P) \leq \frac{1}{2} H_S(P), \]

where \( H_S(P) \) is the Shannon entropy [1].

Again substituting, \( x_1 = \frac{D_1 + 1}{2}, x_2 = 1, f(x) = \log x \) and \( \lambda_1 = \lambda_2 = 1/2 \) in (3.6), we get

\[ H_2^l(P) \leq \frac{1}{2} H_1^l(P) \]

Similarly,

\[ H_3^l(P) \leq \frac{1}{2} H_2^l(P) \leq \frac{1}{2^2} H_1^l(P) \]

\[ H_4^l(P) \leq \frac{1}{2} H_3^l(P) \leq \frac{1}{2^3} H_2^l(P) \]

\[ \vdots \]

\[ H_k^l(P) \leq \frac{1}{2} H_{k-1}^l(P) \leq \frac{1}{2^{k-1}} H_1^l(P) \]  \hspace{1cm} (3.7)

where \( k = 1, 2, \ldots, m; \ m \in \mathbb{N} \). This shows that as the value of \( k \) increases the uncertainty defined by measure (3.1) increases at a linear rate.

**ii.** Consider the following inequality [132]
\[
\left( p^{m-1} - q \right)^{1/m} \leq \frac{(m-1)p + q}{m} \tag{3.8}
\]

for \( p > 0, q > 0, m \geq 2 \) with equality if and only if \( p = q \).

Let us consider \( p = 1, m = 2^k \) and \( q = p_i \) in relation (3.8), we obtain

\[
p^{1/2^k} \leq \left( \frac{2^k - 1 + p}{2^k} \right)
\]

\[
\Rightarrow \frac{1}{2^k} \log p \leq \log \left( \frac{2^k - 1 + p}{2^k} \right)
\]

\[
\Rightarrow \sum_{i=1}^{n} p_i^{1/2^k} \log p_i \leq \sum_{i=1}^{n} p_i \log \left( \frac{p_i + 2^k - 1}{2^k} \right)
\]

\[
\Rightarrow H^{k}_l(P) \leq \left( \frac{1}{2^k} \right) H_S(P), \tag{3.9}
\]

where \( H_S(P) \) is the Shannon entropy [1]. This gives an upper bound for \( H^{k}_l(P) \).

This completes the proof. \( \square \)

In the next section, we define a new parametric entropy function, which is exponential in nature and study its properties.

### 3.3. NEW PARAMETRIC EXPONENTIAL ENTROPY MEASURE

In this section, a new parametric gain function whose expected value yields new entropy function called parametric exponential entropy function is defined as

\[
H^{k}_e(P) = e - \sum_{i=1}^{n} p_i \epsilon^{\left( \left( p_i + 2^k - 1 \right)/2^k \right)} \tag{3.10}
\]

The equation (3.10) can be written as

\[
H^{k}_e(P) = \sum_{i=1}^{n} p_i \left( e^{-\epsilon \left( \left( p_i + 2^k - 1 \right)/2^k \right)} \right) = \sum_{i=1}^{n} p_i \zeta\left( p_i \right) \tag{3.11}
\]

where
\[ \zeta(p_i) = \begin{cases} e - e^{(p_i + 2^k - 1)/2^k} & \text{for } k = 1,2,\ldots, m, m \in \mathbb{N}, \text{ where } N \text{ is a natural number and } p_i \text{ represent the probability of occurrence of } i^{th} \text{ state of the } n \text{ state system, } 0 \leq p_i \leq 1, \forall i \text{ and } \sum_{i=1}^{n} p_i = 1. \end{cases} \] (3.12)

where \( k = 1,2,\ldots, m, m \in \mathbb{N}, \) and \( p_i \) represent the probability of occurrence of \( i^{th} \) state of the \( n \) state system, \( 0 \leq p_i \leq 1, \forall i \) and \( \sum_{i=1}^{n} p_i = 1. \) For \( k = 1, \) the function (3.12) gives \( e - e^{(p_i + 1)/2}, \) which basically represents the information gain from an event with probability \( (p_i + 1)/2, \) \( k = 2 \) gives \( e - e^{(p_i + 3)/4}, \) which basically represents the information gain from an event with probability \( (p_i + 3)/4 \) and so on. As a result, as the value of \( k \) increases, uncertainty goes on decreasing at an exponential rate. The information gain function \( \zeta(p_i), \forall i \) satisfies the following properties:

**E1.** \( \zeta(p_i) = e - e^{(p_i + 2^k - 1)/2^k} \) is defined at all points in \([0, 1] \).

**E2.** \( \lim_{p_i \to 0} \zeta(p_i) = \zeta(p_i = 0) = l_1, l_1 \geq 0 \) and finite.

**E3.** \( \zeta(p_i = 1) = l_2, l_2 \geq 0 \) and finite.

**E4.** \( l_2 < l_1. \)

**E5.** \( \zeta(p_i) \) and \( H^k_e(P) \) are continuous for \( 0 \leq p_i \leq 1. \)

**E6.** With increase in \( p_i, \zeta(p_i) \) decreases linearly.

### 3.3.1. PROPERTIES OF PARAMETRIC EXPONENTIAL ENTROPY MEASURE \( H^k_e(P) \)

**A1.** The function defined by (3.10) is concave in nature.

**Proof:** From equation (3.10), we have

\[ f(p) = e - pe^{(p+2^k - 1)/2^k} \]
Differentiating the above relation with respect to $p$, we get

$$f'(p) = -\left(1 + \frac{p}{2^k}\right) e^{\frac{(p+2^k - 1)/2^k}{k}}$$

and

$$f''(p) = -\frac{1}{2^k} \left(2 + \frac{p}{2^k}\right) e^{\frac{(p + 2^k - 1)/2^k}{k}} < 0.$$ 

Since $f''(p) < 0$ for $0 \leq p \leq 1$. Hence $f(p)$ is a concave function.

**A2.** The function $H^k_e(P)$ attains the maximum when all the probabilities $p_i$'s are equal.

**Proof:** We have

$$H^k_e(P) = e - \sum_{i=1}^{n} p_i e^{\frac{(p_i+2^k - 1)/2^k}{k}}$$

$$= e - \sum_{i=1}^{n} p_i e^{\frac{(p_i+2^k - 1)/2^k}{k}} - \left[1 - \left(p_1 + p_2 + \ldots + p_{n-1}\right)\right] e^{\frac{2^k - \left(p_1 + p_2 + \ldots + p_{n-1}\right)}{2^k}}$$

Now consider

$$\frac{\partial H^k_e(P)}{\partial p_i} = 0; \ i = 1,\ldots, n-1,$$

which gives

$$\left(1 + \frac{p_i}{2^k}\right) \cdot e^{\frac{(p_i+2^k - 1)/2^k}{k}} = \left(1 + \frac{1 - \left(p_1 + p_2 + \ldots + p_{n-1}\right)}{2^k}\right) \cdot e^{\frac{2^k - \left(p_1 + p_2 + \ldots + p_{n-1}\right)}{2^k}}$$

Let $p_i = y_i$ and $1 - \left(p_1 + p_2 + \ldots + p_n\right) = z_i$, then the above equality becomes

$$\left(1 + \frac{y_i}{2^k}\right) \cdot e^{\frac{(y_i+2^k - 1)/2^k}{k}} = \left(1 + \frac{z_i}{2^k}\right) \cdot e^{\frac{z_i+2^k - 1}{2^k}}$$

Now, define a function
\[
f(y) = \left(1 + \frac{y}{2^k}\right) e^{\left((y+2^k - 1)/2^k\right)}, \quad 0 \leq y \leq 1, k \geq 0
\]

So, \(f(y)\) is a bijection. Let \(y_1, y_2 \in [0,1]\). Then

\[
f(y_1) = \left(1 + \frac{y_1}{2^k}\right) e^{\left((y_1+2^k - 1)/2^k\right)}
\]

and

\[
f(y_2) = \left(1 + \frac{y_2}{2^k}\right) e^{\left((y_2+2^k - 1)/2^k\right)}.
\]

If \(y_1 > y_2\) then

\[
\left(1 + \frac{y_1}{2^k}\right) e^{\left((y_1+2^k - 1)/2^k\right)} > \left(1 + \frac{y_2}{2^k}\right) e^{\left((y_2+2^k - 1)/2^k\right)}.
\]

\[
\Rightarrow f(y_1) > f(y_2)
\]

If \(y_1 < y_2\) then

\[
\left(1 + \frac{y_1}{2^k}\right) e^{\left((y_1+2^k - 1)/2^k\right)} < \left(1 + \frac{y_2}{2^k}\right) e^{\left((y_2+2^k - 1)/2^k\right)}
\]

\[
\Rightarrow f(y_1) < f(y_2)
\]

Therefore, \(f(y_1) = f(y_2)\) if and only if \(y_1 = y_2\). So, we can write \(y_i = z_i\) for \(i = 1, 2, \ldots, n-1\)

\[
\sum_{i=1}^{n-1} p_i = \sum_{i=1}^{n-1} (1 - (p_1 + p_2 + \ldots + p_{n-1}))
\]

or

\[
p_1 + p_2 + \ldots + p_{n-1} = (n-1) - (n-1) (1 - p_n)
\]

which finally gives \(p_n = 1/n\).
A3. The function defined by (3.10) monotonically increases and monotonically decreases, respectively in $(0.5,1]$ and $[0,0.5)$ and attains the maximum at $p = 1/2$.

**Proof.** We have

\[ H^k_e(P) = e - p \left( \frac{(p+2^k -1)/2^k}{(2^k - p)/2^k} \right) - (1-p) \left( \frac{(2^k - p)/2^k}{1-p} \right), \quad 0 \leq p \leq 1 \]

\[ \Rightarrow \frac{dH^k_e(P)}{dp} = - \left( 1 + \frac{p}{2^k} \right) e^{(p+2^k -1)/2^k} + \left( 1 + \frac{(1-p)}{2^k} \right) e^{(2^k - p)/2^k} \]

For $p \in [0,0.5)$, then

\[ e^{(1-2p)/2^k} > \frac{(p+2^k)}{(1-p+2^k)} \]

and if $p \in (0.5,1]$, then

\[ e^{(1-2p)/2^k} < \frac{(p+2^k)}{(1-p+2^k)} \]

Therefore, $\frac{dH^k_e(P)}{dp} > 0$, if $p \in [0,0.5)$; $\frac{dH^k_e(P)}{dp} < 0$, if $p \in (0.5,1]$ and $\frac{dH^k_e(P)}{dp} = 0$, if $p = 0.5$.

A4. The proposed entropy measure $H^k_e(P)$ given by (3.10) is non-negative.

**Proof.** We know for all $i$, we have

\[ 0 \leq p_i \leq 1 \]
\[ 0 \leq \frac{p_i+1}{2} \leq 1 \]
\[ 0 \leq \frac{p_i+3}{4} \leq 1 \]

\[ \cdots \]
\[ 0 \leq \frac{p_i+2^k-1}{2^k} \leq 1 \]

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\[ e^0 \leq \sum_{i=1}^{n} e^{\left( \frac{p_i + 2^k - 1}{2^k} \right)} \leq e^1 \]
\[ e - \sum_{i=1}^{n} e^{\left( \frac{p_i + 2^k - 1}{2^k} \right)} \geq 0 \]
\[ \sum_{i=1}^{n} p_i \left( e - \sum_{i=1}^{n} e^{\left( \frac{p_i + 2^k - 1}{2^k} \right)} \right) \geq 0 \]
\[ H_{e}^{k}(P) \geq 0. \]

Therefore, proposed entropy measure \( H_{e}^{k}(P) \) is non-negative.

**A5.** The function defined by (3.10) is a decreasing function of \( k \).

**Proof.** Differentiating the function given by equation (3.10) with respect to ‘\( k \)’, we have

\[
\frac{\partial H_{e}^{k}(P)}{\partial k} = -pe^{\left( \frac{(p+2^k-1)/2^k}{2^k} \right)}(1-p) < 0.
\]

This yields the required result.

### 3.3.1. SIMILARITY INDEX AND RATE OF SIMILARITY

**Definition 3.1.** Two statistical parameters \( a \) and \( b \) are called similar with a similarity index of order \( p \) if there exists a sequence \( x_n \) in \([a, b]\) such that \( |x_{n+1} - l| \leq c|x_n - l|^p, n > 0 \).

Here \( n > 0 \) is a constant known as rate of similarity and \( l = \lim_{n \to \infty} x_n \). If \( p = 1 \) and \( 0 < c < 1 \), then parameters \( a \) and \( b \) have linear similarity or similarity index of first order. Similarity is *rapid* or *slow* according as \( c \) is near 0 or 1. Using induction, it may be observed that the condition for linear similarity for the sequence \( x_n \) such that \( 0 < c < 1, \ a < x_n < b \) can be simplified to the form \( |x_n - l| \leq c^n |b - a|^p, n > 0, 0 < c < 1 \).
**Theorem 3.2** - Two parameters $a$ and $b$ for the sequence $x_n = a + \frac{(2^n - 1)b}{2^n}, a \in A, b \in B$, have rate of similarity $1/2$ with a similarity index of order 1.

**Proof:** Clearly for $x_i = \frac{a + b}{2}$, we have

$$|x_i - b| \leq \frac{1}{2}|b - a|.$$

Similarly,

$$|x_2 - b| \leq \frac{1}{2^2}|b - a|.$$

Proceeding inductively, we can show that

$$|x_n - b| \leq \frac{1}{2^n}|b - a|.$$

This yields that two parameters $a$ and $b$ have rate of similarity $1/2$ and similarity if we take index of order 1.

**Remark 3.1:** If we take $a = p_i, b = 1$, then the dichotomous sequence $(p_i + 2^k - 1)/2^k$ lies in $[0, 1]$ for all $0 \leq p_i \leq 1$ decreases with a linear rate and rate of similarity $1/2^k$. As a result, from relation (4.1), it is easily observed that uncertainty decreases linearly with a rate $1/2^k$.

### 3.4. APPLICATION OF RISK EVALUATION OF WATER SECURITY DURING DROUGHT PERIOD BY EW-TOPSIS APPROACH

Drought risk assessment plays a vital role in evaluating the risk of water security affected by drought. Therefore, it is required to examine the risk level of drought for diminishing the danger of scarcity of water so that drought management efficiently tackles with the drought situations. Low value of drought risk indicates that the water security must be decreased and vice versa. Since entropy functions quantify uncertainty associated with real time systems, therefore, we calculate the weight of three indices reliability $R_r$, resiliency $R_s$ and vulnerability $V_u$ by utilizing the proposed entropy measures given by (3.1) and (3.10). To
evaluate the drought risk of water security, the integrate approach of TOPSIS (Technique for order preference by similarity to an ideal solution) introduced by Hwang and Yoon [133] and entropy weight method called, EW-TOPSIS [130] are developed. The approach of TOPSIS is utilized to determine the performance of each alternative. The results demonstrate the status regarding security and scarcity of water during drought period by using the proposed entropy measures.

3.4.1. ALGORITHM FOR DETERMINING THE WEIGHT

Let $n$ and $m$ be the assessment objectives and valuation indices respectively. Let $Z_{n \times m}$ be the original matrix, given by

$$Z_{rs} = \begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1m} \\ Z_{21} & Z_{22} & \cdots & Z_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{n1} & Z_{n2} & \cdots & Z_{nm} \end{bmatrix}. \quad (3.13)$$

where $r = 1, 2, \ldots, n$ ; $s = 1, 2, \ldots, m$.

The steps for computing the weights by using the proposed parametric entropy measures given by (3.1) and (3.10) are as follows

i. Obtain the standardize matrix $(Q_{rs})_{n \times m}$ from the original matrix $Z = (Z_{rs})_{n \times m}$ by the given equation

$$Q_{rs} = \begin{cases} \frac{\log(Z_{rs})}{\log}\left(\prod_{r=1}^{n}Z_{rs}\right), & \text{for benefit index} \\ \frac{\log(z) - \log(z)}{1 - \log(\prod_{r=1}^{n}Z_{rs})}, & \text{for cost index} \end{cases}. \quad (3.14)$$

ii. Calculate the weights for the above matrix $Q_{rs}$ by using the proposed entropy measure (3.1) and (3.10) are defined as follows
\[
\left( w^k_l \right)_s = \frac{1 - \left( H^k_l (P) \right)_s}{\sum_{s=1}^{m} \left( 1 - \left( H^k_l (P) \right)_s \right)}
\]  
\( (3.15) \)

\[
\left( w^k_e \right)_s = \frac{1 - \left( H^k_e (P) \right)_s}{\sum_{s=1}^{m} \left( 1 - \left( H^k_e (P) \right)_s \right)},
\]  
\( (3.16) \)

where \( w^k_l \) and \( w^k_e \) are the weight of parametric logarithmic entropy measure and parametric exponential entropy measure, respectively. Considering equations (3.1) and (3.10), the entropy functions for the equation (3.15) and (3.16) are as follows

\[
\left( H^k_l (P) \right)_s = \left( k - \sum_{r=1}^{n} p_{rs} \log(p_{rs} + 2^k - 1) \right)
\]  
\( (3.17) \)

\[
\left( H^k_e (P) \right)_s = e - \sum_{r=1}^{n} p_{rs} e^{\left( \frac{(p_{rs} + 2^k - 1) / 2^k}{e} \right)}
\]  
\( (3.18) \)

where

\[
p_{rs} = \frac{Q_{rs}}{\sum_{s=1}^{m} Q_{rs}},
\]  
\( (3.19) \)

\( 0 \leq \left( w^k_l \right)_s, \left( w^k_e \right)_s \leq 1 \) and \( \sum_{s=1}^{m} \left( w^k_l \right)_s = 1 = \sum_{s=1}^{m} \left( w^k_e \right)_s \).

3.4.2. RISK EVALUATION FOR WATER SECURITY IN A DROUGHT REGION BY THE EW-TOPSIS APPROACH

The approach of EW-TOPSIS is utilized to determine the ranking of water security risk in the region that accompanied by drought and obtains the values in the interval [0, 1]. The value of drought risk reveals the status of security and scarcity of water. The steps are as follows:

**Step 1** - The decision matrix \( Z_{rs} \) be the \( n \times m \) matrix is normalized as
\[
X_{rs} = \frac{Z_{rs}}{\sqrt{\sum_{r=1}^{n} (Z_{rs})^2}} \tag{3.20}
\]

where \( r = 1, 2, \ldots, n \) and \( s = 1, 2, \ldots, m \).

**Step 2** - The normalized weighted decision matrix for the equations (3.1) and (3.10) are evaluated as

\[
(Y^k_i)_{rs} = \sum_{r=1}^{n} (w^k_i)_{rs} X'_rs, \tag{3.21}
\]

\[
(Y^k_e)_{rs} = \sum_{r=1}^{n} (w^k_e)_{rs} X'_rs \tag{3.22}
\]

**Step 3** - Let \( S, S' \) be the set of benefit objectives and cost objectives respectively. Construct the positive ideal solution \((Y^k_i)_s^+, (Y^k_e)_s^+\) and negative ideal solution \((Y^k_i)_s^-, (Y^k_e)_s^-\) for each objective by employing equation (3.21) and (3.22), calculated as follows

\[
(Y^k_i)_s^+ = \left\{ \left( \max_{1 \leq r \leq n} (Y^k_i)_{rs} \right) \left| s \in S \right. \right\}, \left( \min_{1 \leq r \leq n} (Y^k_i)_{rs} \right) \left| s \in S' \right. \right\}, \ s = 1, 2, \ldots, m \right\}. \tag{3.23}
\]

\[
(Y^k_e)_s^+ = \left\{ \left( \max_{1 \leq r \leq n} (Y^k_e)_{rs} \right) \left| s \in S \right. \right\}, \left( \min_{1 \leq r \leq n} (Y^k_e)_{rs} \right) \left| s \in S' \right. \right\}, \ s = 1, 2, \ldots, m \right\}. \tag{3.24}
\]

\[
(Y^k_i)_s^- = \left\{ \left( \min_{1 \leq r \leq n} (Y^k_i)_{rs} \right) \left| s \in S \right. \right\}, \left( \max_{1 \leq r \leq n} (Y^k_i)_{rs} \right) \left| s \in S' \right. \right\}, \ s = 1, 2, \ldots, m \right\}. \tag{3.25}
\]

\[
(Y^k_e)_s^- = \left\{ \left( \min_{1 \leq r \leq n} (Y^k_e)_{rs} \right) \left| s \in S \right. \right\}, \left( \max_{1 \leq r \leq n} (Y^k_e)_{rs} \right) \left| s \in S' \right. \right\}, \ s = 1, 2, \ldots, m \right\}. \tag{3.26}
\]

**Step 4** - Evaluate the separation measures for each index by using the Euclidean distance measure

\[
(D^k_i)_r^+ = \sqrt{\sum_{s=1}^{m} ((Y^k_i)_{rs} - (Y^k_i)_s^+)^2}; r = 1, 2, \ldots, n \tag{3.27}
\]

\[
(D^k_e)_r^+ = \sqrt{\sum_{s=1}^{m} ((Y^k_e)_{rs} - (Y^k_e)_s^+)^2}; r = 1, 2, \ldots, n \tag{3.28}
\]
\[ (D_l^k)^-_r = \sqrt{\sum_{s=1}^{m} \left( (y_{ls}^k)^- - (y_{ls}^k)^- \right)}; r = 1,2,\ldots,n \]  
(3.29)

\[ (D_e^k)^-_r = \sqrt{\sum_{s=1}^{m} \left( (y_{es}^k)^- - (y_{es}^k)^- \right)}; r = 1,2,\ldots,n \]  
(3.30)

**Step 5** – The value of drought risk of water security for each index is determined as follows

\[ (DRI_l^k)^*_r = \frac{(D_l^k)^-_r}{(D_l^k)^-_r + (D_l^k)^+_r} \]  
(3.31)

\[ (DRI_e^k)^*_r = \frac{(D_e^k)^-_r}{(D_e^k)^-_r + (D_e^k)^+_r} , \]  
(3.32)

where \((DRI_l^k)^*_r\) and \((DRI_e^k)^*_r\) are respectively the drought risk index for the proposed parametric entropy measures given by equation (3.1) and (3.10).

**Step 6** - Rank all the indices according to the values of \((DRI_l^k)^*_r\) and \((DRI_e^k)^*_r\). Higher value of \((DRI_l^k)^*_r\) and \((DRI_e^k)^*_r\) shows the worsen condition of water security and maximum scarcity of water in that region for the drought duration.

### 3.4.3. CASE STUDY OF HAIHE RIVER BASIN

We have taken the data of Haihe river basin on reliability, resiliency, and vulnerability as considered by Zhang et al. [125, 131], presented in Table 3.1. The normalized decision matrix is given in Table 3.2. The weights of the proposed parametric entropy measures \(H_l^k(P)\) and \(H_e^k(P)\) are calculated in Table 3.3. The normalized weighted decision matrix is shown in Table 3.4. We restrict ourselves to the value of \(k = 1\) and \(k = 2\). The values of \((D_l^k)^+_r\), \((D_l^k)^-_r\) and \((DRI_l^k)^*_r\) and \((D_e^k)^+_r\), \((D_l^k)^-_r\) and \((DRI_e^k)^*_r\), respectively for the parametric logarithmic measure and for the parametric exponential entropy measure are given in Table 3.5 and Table 3.6. Comparisons are also made in Table 3.6. So, it can be easily seen from Table 3.6, Tuhaimajia River and north of Haihe River has respectively the highest and
lowest value of drought risk index, \textit{i.e}, Tuhaimajia River needs maximum water security for diminishing the problem of water scarcity. The results are similar with the ones obtained in [125, 131].

**Table 3.1:** The original evaluation matrix of water system

<table>
<thead>
<tr>
<th>Water Name</th>
<th>$R_e$</th>
<th>$R_s$</th>
<th>$V_u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Luanhe River</td>
<td>0.43</td>
<td>0.42</td>
<td>0.36</td>
</tr>
<tr>
<td>North of Haihe River</td>
<td>0.41</td>
<td>0.41</td>
<td>0.16</td>
</tr>
<tr>
<td>South of Haihe River</td>
<td>0.36</td>
<td>0.36</td>
<td>0.42</td>
</tr>
<tr>
<td>Tuhaimajia River</td>
<td>0.37</td>
<td>0.35</td>
<td>0.59</td>
</tr>
</tbody>
</table>

**Table 3.2:** The standardized decision matrix by using (3.14)

<table>
<thead>
<tr>
<th>Water Name</th>
<th>$R_e$</th>
<th>$R_s$</th>
<th>$V_u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Luanhe River</td>
<td>0.32</td>
<td>0.321</td>
<td>0.359</td>
</tr>
<tr>
<td>North of Haihe River</td>
<td>0.360</td>
<td>0.352</td>
<td>0.287</td>
</tr>
<tr>
<td>South of Haihe River</td>
<td>0.338</td>
<td>0.332</td>
<td>0.330</td>
</tr>
<tr>
<td>Tuhaimajia River</td>
<td>0.319</td>
<td>0.330</td>
<td>0.351</td>
</tr>
</tbody>
</table>

**Table 3.3:** Weight $\left( w^k_i \right)_s$ and $\left( w^k_e \right)_s$ for the proposed parametric entropy measures by using equations (3.15) and (3.16)

<table>
<thead>
<tr>
<th>Weight $\left( w^k_i \right)_s$ for the proposed parametric logarithmic entropy measure</th>
<th>Weight $\left( w^k_e \right)_s$ for the proposed parametric exponential entropy measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>For $k = 1$</td>
<td>For $k = 2$</td>
</tr>
<tr>
<td>(( w^1_i ) )_1</td>
<td>(( w^1_i ) )_2</td>
</tr>
<tr>
<td>(( w^2_i ) )_1</td>
<td>(( w^3_i ) )_3</td>
</tr>
<tr>
<td>(( w^3_i ) )_1</td>
<td>(( w^2_i ) )_1</td>
</tr>
<tr>
<td>(( w^2_i ) )_3</td>
<td>(( w^2_i ) )_2</td>
</tr>
</tbody>
</table>
Table 3. 4: The normalized weighted decision matrix by using equation (3.21) and (3.22)

<table>
<thead>
<tr>
<th>Water System Name</th>
<th>((Y^k_{rs})_{rs}) For (k = 1)</th>
<th>((Y^k_{rs})_{rs}) For (k = 2)</th>
<th>((Y^k_{rs})_{rs}) For (k = 1)</th>
<th>((Y^k_{rs})_{rs}) For (k = 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Luanhe River</td>
<td>0.181 0.180 0.181 0.182 0.181 0.146</td>
<td>0.198 0.196 0.121 0.181 0.181 0.146</td>
<td>0.172 0.176 0.172 0.173 0.177 0.065</td>
<td>0.189 0.191 0.054 0.173 0.177 0.065</td>
</tr>
<tr>
<td>North of Haihe River</td>
<td>0.151 0.154 0.151 0.152 0.155 0.170</td>
<td>0.166 0.168 0.141 0.152 0.155 0.171</td>
<td>0.156 0.150 0.156 0.157 0.151 0.239</td>
<td>0.171 0.163 0.198 0.156 0.151 0.240</td>
</tr>
<tr>
<td>South of Haihe River</td>
<td>0.156 0.150 0.156 0.157 0.151 0.239</td>
<td>0.171 0.163 0.198 0.156 0.151 0.240</td>
<td>0.156 0.150 0.156 0.157 0.151 0.239</td>
<td>0.171 0.163 0.198 0.156 0.151 0.240</td>
</tr>
</tbody>
</table>

Table 3. 5: Separation measures by utilizing equation (3.27), (3.28), (3.29) and (3.30)

<table>
<thead>
<tr>
<th>Water System Name</th>
<th>((D^k_1)_r)</th>
<th>((D^k_2)_r)</th>
<th>((D^k_e)_r)</th>
<th>((D^k_1)_r)</th>
<th>((D^k_2)_r)</th>
<th>((D^k_e)_r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Luanhe River</td>
<td>0.094</td>
<td>0.092</td>
<td>0.077</td>
<td>0.081</td>
<td>0.093</td>
<td>0.091</td>
</tr>
<tr>
<td>North of Haihe River</td>
<td>0.177</td>
<td>0.033</td>
<td>0.145</td>
<td>0.036</td>
<td>0.174</td>
<td>0.033</td>
</tr>
<tr>
<td>South of Haihe River</td>
<td>0.08</td>
<td>0.107</td>
<td>0.071</td>
<td>0.087</td>
<td>0.079</td>
<td>0.105</td>
</tr>
<tr>
<td>Tuhaimajia River</td>
<td>0.039</td>
<td>0.176</td>
<td>0.043</td>
<td>0.145</td>
<td>0.039</td>
<td>0.174</td>
</tr>
</tbody>
</table>
Table 3.6: Comparison of results of drought risk of Haihe river basin

<table>
<thead>
<tr>
<th>Water System Name</th>
<th>For $k = 1$</th>
<th>For $k = 2$</th>
<th>Drought risk index values obtained by</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>($DRI^k_r$)$^<em>$, ($DRI^k_c$)$^</em>$</td>
<td>($DRI^k_r$)$^<em>$, ($DRI^k_c$)$^</em>$</td>
<td>Zhang et al. [125], Dong &amp; Liu [131], Dong &amp; Liu [131]</td>
<td></td>
</tr>
<tr>
<td>Luanhe River</td>
<td>0.4942</td>
<td>0.5129</td>
<td>0.4952, 0.4949</td>
<td>0.5000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.5208</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.5058</td>
</tr>
<tr>
<td>North of Haihe River</td>
<td>0.1584</td>
<td>0.2002</td>
<td>0.1607, 0.1602</td>
<td>0.4500</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.4884</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.4560</td>
</tr>
<tr>
<td>South of Haihe River</td>
<td>0.5717</td>
<td>0.5509</td>
<td>0.5707, 0.5710</td>
<td>0.5700</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.5833</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.5667</td>
</tr>
<tr>
<td>Tuhaimajia River</td>
<td>0.8181</td>
<td>0.7716</td>
<td>0.8155, 0.8161</td>
<td>0.6200</td>
</tr>
<tr>
<td></td>
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<td>0.6281</td>
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<tr>
<td></td>
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<td>0.6229</td>
</tr>
</tbody>
</table>

3.5. CONCLUSION

The drought risk valuation of water security during drought time is incomplete without consideration of uncertainties. Therefore, to assess the drought risk, the EW-TOPSIS approach is introduced which pay more attention towards complexities and uncertainties during drought time and demonstrates the status of drought risk for Haihe River Basin, which are in a grip of drought. Two new parametric entropy measures based on logarithmic gain function and exponential gain function are defined for evaluating the weight of indices namely, reliability, resiliency and vulnerability. The results are shown that the EW-TOPSIS approach is very effective and helpful in analyzing the risk assessment regarding water security and water scarcity in the drought region. In the next chapter, extend our work from probabilistic to fuzzy environment and introduce a new fuzzy divergence measure.