Chapter 6
FREE VIBRATION AND TRANSIENT ANALYSIS OF PLATES

The piezoelectric composite laminate plates are frequently used in dynamic engineering application to replace the conventional structures due to their versatile benefits in the practical use. The natural frequencies of the dynamic structures play a vital role in the design of aircraft, aerospace and rocket and missile structures fabrication when they are subjected to dynamic fluctuating loads.

The induced vibrations are serious in the design of dynamic structures, if the natural frequency of the material is matches with the operating frequency of the acting load, then the structure may caused to damage due to resonance. The control of natural frequencies of dynamic structures with smart composite laminated plates is successfully possible under electromechanical loading. In this work an effort is focused to study the vibration and dynamic transient analysis of smart composite laminated plates using the higher order theory under electromechanical loading. The influence of number of layers, aspect ratios and different modulus ratios on the vibration, dynamic transient behavior of smart composite laminated plates for different voltages of both models 1&2 is studied.

6.1. FREE VIBRATION ANALYSIS FOR DISPLACEMENT MODEL – 1.

6.1.1. Free vibration analysis of simply supported anti-symmetric cross ply laminated plates.

For free vibration, set the thermal and mechanical loads to zero in Eq. (4.5) and assume periodic solutions of the form:

\[ u_0 = u_0(x, y) e^{-i\omega t} ; \quad v_0 = v_0(x, y) e^{-i\omega t} ; \quad w_0 = w_0(x, y) e^{-i\omega t} ; \]
\[ \theta_x = \theta_x(x, y) e^{-i\omega t} ; \quad \theta_y = \theta_y(x, y) e^{-i\omega t} ; \quad \theta_x^* = u_0(x, y) e^{-i\omega t} ; \]
\[ v_0^* = v_0^*(x, y) e^{-i\omega t} ; \quad \theta_x^* = \theta_x^*(x, y) e^{-i\omega t} ; \quad \theta_y^* = \theta_y^*(x, y) e^{-i\omega t} \]
Therefore

\[
\begin{bmatrix}
S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} & S_{17} & S_{18} & S_{19} \\
S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} & S_{27} & S_{28} & S_{29} \\
S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} & S_{37} & S_{38} & S_{39} \\
S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} & S_{47} & S_{48} & S_{49} \\
S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} & S_{57} & S_{58} & S_{59} \\
S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} & S_{67} & S_{68} & S_{69} \\
S_{71} & S_{72} & S_{73} & S_{74} & S_{75} & S_{76} & S_{77} & S_{78} & S_{79} \\
S_{81} & S_{82} & S_{83} & S_{84} & S_{85} & S_{86} & S_{87} & S_{88} & S_{89} \\
S_{91} & S_{92} & S_{93} & S_{94} & S_{95} & S_{96} & S_{97} & S_{98} & S_{99}
\end{bmatrix}
\begin{bmatrix}
U_{mn} \\
V_{mn} \\
W_{mn} \\
X_{mn} \\
Y_{mn}
\end{bmatrix} = 0
\]

For free vibration Eq. (6.1) reduces to the Eigen value problem as:

\[
[U, V, W, X, Y] + \lambda^n = 0
\]

\[
\text{For model calculations refer Appendix – B}
\]
6.1.2. Free vibration analysis of simply supported anti-symmetric angle ply laminated plates.

For free vibration, set the thermal and Mechanical loads to zero in Eq. (4.8) and assume periodic solutions of the form:

\[ u_0 = u_0(x, y) e^{-i\omega t}; \quad v_0 = v_0(x, y) e^{-i\omega t}; \quad w_0 = w_0(x, y) e^{-i\omega t}; \]
\[ \theta_x = \theta_x(x, y) e^{-i\omega t}; \quad \theta_y = \theta_y(x, y) e^{-i\omega t}; \quad u_0^* = u_0^*(x, y) e^{-i\omega t}; \]
\[ v_0^* = v_0^*(x, y) e^{-i\omega t}; \quad \theta_x^* = \theta_x^*(x, y) e^{-i\omega t}; \quad \theta_y^* = \theta_y^*(x, y) e^{-i\omega t}; \]

Therefore

\[
\begin{bmatrix}
S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} & S_{17} & S_{18} & S_{19} \\
S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} & S_{27} & S_{28} & S_{29} \\
S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} & S_{37} & S_{38} & S_{39} \\
S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} & S_{47} & S_{48} & S_{49} \\
S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} & S_{57} & S_{58} & S_{59} \\
S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} & S_{67} & S_{68} & S_{69} \\
S_{71} & S_{72} & S_{73} & S_{74} & S_{75} & S_{76} & S_{77} & S_{78} & S_{79} \\
S_{81} & S_{82} & S_{83} & S_{84} & S_{85} & S_{86} & S_{87} & S_{88} & S_{89} \\
S_{91} & S_{92} & S_{93} & S_{94} & S_{95} & S_{96} & S_{97} & S_{98} & S_{99} \\
\end{bmatrix}
\begin{bmatrix}
U_{mn} \\
V_{mn} \\
W_{mn} \\
X_{mn} \\
Y_{mn} \\
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

For free vibration Eq. (6.4) reduces to the Eigen value problem as:

\[
(\mathbf{S} - \omega^2 \mathbf{M}) \{ \Delta \} = \{ 0 \}
\]

Where \( \Delta = (U_{mn}, V_{mn}, W_{mn}, X_{mn}, Y_{mn})^T \)

For a non trivial solution \( \{ \Delta \} \neq 0 \) the determinant of the coefficient matrix in Eq. (6.5) should be zero, which yields the characteristic equation as:

\[
|\mathbf{S}| - \lambda |\mathbf{M}| = 0
\]

…..Eq. (6.6)
Where $\lambda = \omega^2$ is the Eigen value.

The real positive roots of the Eq. (6.6) give the square of the natural frequency $\omega_{mn}$ associated with mode $(m, n)$. The smallest value of the characteristic equation is called the fundamental natural frequency.

### 6.2 FREE VIBRATION ANALYSIS FOR DISPLACEMENT MODEL-2.

#### 6.2.1. Free vibration analysis of simply supported anti-symmetric cross ply laminated plates.

For free vibration analysis of piezoelectric composite laminated plates set the thermal and mechanical loads to zero in Eq. (4.13) and assume periodic solution of the form:

\[
\begin{align*}
    u &= u^0(x, y) e^{-i\omega t}; \\
    v &= v^0(x, y) e^{-i\omega t}; \\
    w &= w^0(x, y) e^{-i\omega t}; \\
    \theta &= \theta^0(x, y) e^{-i\omega t}; \\
    \phi &= \phi^0(x, y) e^{-i\omega t}.
\end{align*}
\]

where:

\[
\begin{align*}
    w^0 &= w_0^0(x, y) e^{-i\omega t}; \\
    \theta^0 &= \theta^0(x, y) e^{-i\omega t}; \\
    \phi^0 &= \phi^0(x, y) e^{-i\omega t}.
\end{align*}
\]

And thus:

\[
\begin{pmatrix}
S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} & S_{17} & S_{18} & S_{19} & S_{110} & S_{111} & S_{112} \\
S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} & S_{27} & S_{28} & S_{29} & S_{210} & S_{211} & S_{212} \\
S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} & S_{37} & S_{38} & S_{39} & S_{310} & S_{311} & S_{312} \\
S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} & S_{47} & S_{48} & S_{49} & S_{410} & S_{411} & S_{412} \\
S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} & S_{57} & S_{58} & S_{59} & S_{510} & S_{511} & S_{512} \\
S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} & S_{67} & S_{68} & S_{69} & S_{610} & S_{611} & S_{612} \\
S_{71} & S_{72} & S_{73} & S_{74} & S_{75} & S_{76} & S_{77} & S_{78} & S_{79} & S_{710} & S_{711} & S_{712} \\
S_{81} & S_{82} & S_{83} & S_{84} & S_{85} & S_{86} & S_{87} & S_{88} & S_{89} & S_{810} & S_{811} & S_{812} \\
S_{91} & S_{92} & S_{93} & S_{94} & S_{95} & S_{96} & S_{97} & S_{98} & S_{99} & S_{910} & S_{911} & S_{912} \\
S_{101} & S_{102} & S_{103} & S_{104} & S_{105} & S_{106} & S_{107} & S_{108} & S_{109} & S_{1010} & S_{1011} & S_{1012} \\
S_{111} & S_{112} & S_{113} & S_{114} & S_{115} & S_{116} & S_{117} & S_{118} & S_{119} & S_{1110} & S_{1111} & S_{1112} \\
S_{121} & S_{122} & S_{123} & S_{124} & S_{125} & S_{126} & S_{127} & S_{128} & S_{129} & S_{1210} & S_{1211} & S_{1212} \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
U_{\text{mm}} \\
V_{\text{mm}} \\
W_{\text{mm}} \\
X_{\text{mm}} \\
Y_{\text{mm}} \\
Z_{\text{mm}} \\
U_{\text{mm}} \\
V_{\text{mm}} \\
W_{\text{mm}} \\
X_{\text{mm}} \\
Y_{\text{mm}} \\
Z_{\text{mm}} \\
\end{pmatrix}
\]
For free vibration Eq. (6.7) reduces to the Eigen values problem as:

\[
([S] - \omega^2 [M]) [\Delta] = [0]
\]

Where \( \Delta = (U_{mn}, V_{mn}, W_{mn}, X_{mn}, Y_{mn}, Z_{mn}, U^{*}_{mn}, V^{*}_{mn}, W^{*}_{mn}, X^{*}_{mn}, Y^{*}_{mn}, Z^{*}_{mn})^T \)

For a non trivial solution \( \{\Delta\} \neq 0 \) the determinant of the coefficient matrix in Eq. (6.8) should be zero, which yields the characteristic equation:

\[
|S| - \lambda [M] = 0
\]

Where \( \lambda = \omega^2 \) is the Eigen value.

The real positive roots of the Eq. (6.9) give the square of the natural frequency \( \omega_{mn} \) associated with mode \((m, n)\). The smallest value of the characteristic equation is called the fundamental natural frequency.

### 6.2.2. Free vibration analysis of simply supported anti-symmetric angle ply laminated plates.

For free vibration analysis of piezoelectric composite laminated plates set the thermal and mechanical loads to zero in Eq. (4.17) and assume periodic solution of the form:
\[ u_0 = u_0(x, y) e^{-i \omega t}; \quad v_0 = v_0(x, y) e^{-i \omega t}; \quad w_0 = w_0(x, y) e^{-i \omega t}; \quad \theta_z = \theta_z(x, y) e^{-i \omega t}; \]
\[ \theta_y = \theta_y(x, y) e^{-i \omega t}; \quad u_0^* = u_0(x, y) e^{i \omega t}; \quad \theta_z^* = \theta_z(x, y) e^{i \omega t}; \quad v_0^* = v_0(x, y) e^{i \omega t}; \]
\[ w_0^* = w_0(x, y) e^{i \omega t}; \quad \theta_y^* = \theta_y(x, y) e^{i \omega t}; \quad \theta_z^* = \theta_z(x, y) e^{i \omega t}. \]

And thus:

\[
\begin{bmatrix}
S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} & S_{17} & S_{18} & S_{19} & S_{110} & S_{111} & S_{112} \\
S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} & S_{27} & S_{28} & S_{29} & S_{210} & S_{211} & S_{212} \\
S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} & S_{37} & S_{38} & S_{39} & S_{310} & S_{311} & S_{312} \\
S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} & S_{47} & S_{48} & S_{49} & S_{410} & S_{411} & S_{412} \\
S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} & S_{57} & S_{58} & S_{59} & S_{510} & S_{511} & S_{512} \\
S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} & S_{67} & S_{68} & S_{69} & S_{610} & S_{611} & S_{612} \\
S_{71} & S_{72} & S_{73} & S_{74} & S_{75} & S_{76} & S_{77} & S_{78} & S_{79} & S_{710} & S_{711} & S_{712} \\
S_{81} & S_{82} & S_{83} & S_{84} & S_{85} & S_{86} & S_{87} & S_{88} & S_{89} & S_{810} & S_{811} & S_{812} \\
S_{91} & S_{92} & S_{93} & S_{94} & S_{95} & S_{96} & S_{97} & S_{98} & S_{99} & S_{910} & S_{911} & S_{912} \\
S_{101} & S_{102} & S_{103} & S_{104} & S_{105} & S_{106} & S_{107} & S_{108} & S_{109} & S_{1010} & S_{1011} & S_{1012} \\
S_{111} & S_{112} & S_{113} & S_{114} & S_{115} & S_{116} & S_{117} & S_{118} & S_{119} & S_{1110} & S_{1111} & S_{1112} \\
S_{121} & S_{122} & S_{123} & S_{124} & S_{125} & S_{126} & S_{127} & S_{128} & S_{129} & S_{1210} & S_{1211} & S_{1212}
\end{bmatrix}
\begin{bmatrix}
U_m^* \\
V_m \\
W_m \\
X_m \\
Y_m \\
Z_m^*
\end{bmatrix}

\[ \begin{bmatrix}
U_m \\
V_m \\
W_m \\
X_m \\
Y_m \\
Z_m^*
\end{bmatrix}
\begin{bmatrix}
U_m^* \\
V_m \\
W_m \\
X_m \\
Y_m \\
Z_m^*
\end{bmatrix}
\begin{bmatrix}
S_{ij} \\
S_{ij} \\
S_{ij} \\
S_{ij} \\
S_{ij} \\
S_{ij}
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[ \text{.....Eq. (6.10)} \]

For free vibration Eq. (6.10) reduces to the Eigen values problem as:

\[ (S) - \omega^2 (M) \Delta = 0 \]

\[ \text{..... Eq. (6.11)} \]

Where \( \Delta = (U_m, V_m, W_m, X_m, Y_m, Z_m, U_m^*, V_m^*, W_m^*, X_m^*, Y_m^*, Z_m^*) \)

For a non trivial solution \( \{ \Delta \} \neq 0 \) the determinant of the coefficient matrix in Eq. (6.11) should be zero, which yields the characteristic equation:
\[ [S] - \lambda [M] = 0 \quad \text{…..Eq. (6.12)} \]

Where \( \lambda = \omega^2 \) is the Eigen value.

The real positive roots of the Eq. (6.12) give the square of the natural frequency \( \omega_{mn} \) associated with mode \((m, n)\). The smallest value of the characteristic equation is called the fundamental natural frequency.

6.3 RESULTS AND DISCUSSIONS

In view to know importance of the present HOSD theories to model both thick and thin piezoelectric laminated composite substrates, the solutions are developed for HOSD theories with nine DOF and twelve DOF individually for both types of laminated plates of respective displacement model-1 and 2.

Numerical calculations are worked out for the free undamped transverse vibration analysis of piezoelectric laminated composite substrates. The influence of side-to-thickness ratio, aspect ratio, coupling between the stretching and bending, modulus ratio and number of layers on the natural frequencies of smart composite plates are studied.

The following material properties for piezoelectric composite laminated plates are used:

**Composite material (Glass/Epoxy):**

- \( E_2 = E_3 = 10^6 \) Gpa
- \( E_1/E_2 = 40; \ G_{12} = 0.6 \ E_2; \ G_{23} = 0.5 \ E_2; \ G_{12} = G_{13} = 0.5; \)
- \( \nu_{12} = \nu_{23} = \nu_{13} = 0.25; \)

**PFRC material:**

- \( E_2 = E_3 = 10^6 \) Gpa
- \( E_1/E_2 = 25; \ G_{12} = 0.5 \ E_2; \ G_{23} = 0.2 \ E_2; \ G_{12} = G_{13} = 0.5; \)
- \( \nu_{12} = \nu_{23} = \nu_{13} = 0.25; \)

The natural frequencies of piezoelectric composite laminated plate are presented in non-dimensional form using the multiplier \((a^2/h)\sqrt{(\rho_c/E_c)}\).

Hence, the non-dimensionalized fundamental natural frequency is represented as:

- \( \sigma = \omega (a^2/h) \sqrt{(\rho_c/E_c)} \)

In this chapter various numerical examples are presented and discussed for verifying the accuracy of the present higher order shear deformation theory in predicting the
fundamental frequencies of smart composite plates and the non-dimensionalized results are presented in Appendix-F in the form of Tables 6.1 to 6.4

6.3.1 The effect of side-to-thickness ratio, number of layers, aspect ratio, modulus ratio, shear and coupling on fundamental natural frequency

- Fig. 6.1 shows the variation of non-dimensionalized natural frequency for various modulus ratios and for various voltages with and without piezoelectric material (actuator) is investigated.
- From the Fig.6.1 it is found that the effect of coupling is more significant for all modular ratios in model 1 and model 2. It is observed that the influence of piezo effect is decreased for modular ratio at zero voltage.
- Fig. 6.2 shows the variation of non-dimensionalized natural frequency for various modulus ratios and for various voltages with and without piezoelectric material (actuator) for non-piezoelectric unidirectional laminate subjected to mechanical loading.
- From the Fig.6.2 it is found that the maximum frequencies are occurring at \( a/h = 40 \) at 100 V and drastically decreasing at \( a/h = 10 \). Also noticed that piezo effect is close that of without piezo effect.
- Fig. 6.3 & Fig.6.4 shows the variation of non-dimensionalized fundamental natural frequencies of the angle-ply laminates for various modulus ratios with and without piezoelectric materials for Zero voltage and 100 V. From the figures it is found that the frequencies are decreased at 100 V compared at Zero voltage and also the piezo effect in both cases are slightly decreased with respect to without piezo effect.
- As the number of layers increase without changing the total thickness, the extensional coupling effect is decreased and it leads to increase the fundamental frequencies.
Fig. 6.1. Non dimensional fundamental frequency Vs Modulus ratio for cross ply laminated composite plates (0°/90°/0°/90°) at Vt=0

Fig. 6.2. Non dimensional fundamental frequency Vs Modulus ratio for cross ply laminated composite plates (0°/90°/0°/90°) at Vt=100
Fig. 6.3. Non dimensional fundamental frequency Vs Modulus ratio for anti-symmetric angle ply square laminated plates (\(-45^\circ /45^\circ /-45^\circ /45^\circ\)) at \(V_t=0\)

V = 0

Fig. 6.4. Non dimensional fundamental frequency Vs Side to thickness ratio for cross ply square laminated plates (0/90/0/90) at \(V_t=0\)

V = 0
6.4. TRANSIENT ANALYSIS FOR DISPLACEMENT MODEL – 1.

Transient analysis of simply supported anti-symmetric cross-ply and angle-ply laminated plates using higher order displacement model-1 is presented in this section.

6.4.1. Equations of motion

For simply supported anti-symmetric cross-ply and angle-ply laminates, the Navier's solution method can be used to reduce the governing equations of motion to differential equations in time. These are given by equations 4.5 & 4.8 for anti-symmetric cross-ply and angle-ply laminates. For the Transient analysis, set the thermal and mechanical loads to zero in Eq. (4.5) & Eq. (4.8) and assume periodic solutions of the form:

\[ u_0 = u_0(x, y) e^{-i \omega t}; \quad v_0 = v_0(x, y) e^{-i \omega t}; \quad w_0 = w_0(x, y) e^{-i \omega t}; \]
\[ \theta_x = \theta_x(x, y) e^{-i \omega t}; \quad \theta_y = \theta_y(x, y) e^{-i \omega t}; \quad \theta_z^* = u_0^*(x, y) e^{-i \omega t}; \]
\[ v_0^* = v_0^*(x, y) e^{-i \omega t}; \quad \theta_x^* = \theta_x^*(x, y) e^{-i \omega t}; \quad \theta_y^* = \theta_y^*(x, y) e^{-i \omega t}; \]

Therefore

\[
\begin{bmatrix}
S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} & S_{17} & S_{18} & S_{19} \\
S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} & S_{27} & S_{28} & S_{29} \\
S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} & S_{37} & S_{38} & S_{39} \\
S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} & S_{47} & S_{48} & S_{49} \\
S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} & S_{57} & S_{58} & S_{59} \\
S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} & S_{67} & S_{68} & S_{69} \\
S_{71} & S_{72} & S_{73} & S_{74} & S_{75} & S_{76} & S_{77} & S_{78} & S_{79} \\
S_{81} & S_{82} & S_{83} & S_{84} & S_{85} & S_{86} & S_{87} & S_{88} & S_{89} \\
S_{91} & S_{92} & S_{93} & S_{94} & S_{95} & S_{96} & S_{97} & S_{98} & S_{99}
\end{bmatrix}
\begin{bmatrix}
U_{mn} \\
V_{mn} \\
W_{mn} \\
X_{mn} \\
Y_{mn}
\end{bmatrix}
= 0
\]

\[
\begin{bmatrix}
m_{11} & 0 & 0 & m_{14} & 0 & m_{16} & 0 & m_{18} & 0 \\
0 & m_{22} & 0 & 0 & m_{25} & 0 & m_{27} & 0 & m_{29} \\
0 & 0 & m_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\
m_{41} & 0 & 0 & m_{44} & 0 & m_{46} & 0 & m_{48} & 0 \\
0 & m_{52} & 0 & 0 & m_{55} & 0 & m_{57} & 0 & m_{59} \\
m_{61} & 0 & 0 & m_{64} & 0 & m_{66} & 0 & m_{68} & 0 \\
0 & m_{72} & 0 & 0 & m_{75} & 0 & m_{77} & 0 & m_{79} \\
m_{81} & 0 & 0 & m_{84} & 0 & m_{86} & 0 & m_{88} & 0 \\
0 & m_{92} & 0 & 0 & m_{95} & 0 & m_{97} & 0 & m_{99}
\end{bmatrix}
\begin{bmatrix}
\dot{U}_{mn} \\
\dot{V}_{mn} \\
\dot{W}_{mn} \\
\dot{X}_{mn} \\
\dot{Y}_{mn}
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}

\Rightarrow \text{Eq. (6.13)}

Therefore the transient analysis in the absence of thermal effects the Eq. (6.13) reduces to the following form as:
6.4.2. Newmark’s Time Integration

The set of equations in 6.14, for any fixed m and n can be solved exactly using either the Laplace transform method or the model analysis method. Both methods are algebraically complicated and require the determination of Eigen values and Eigen functions as in the state-space method. Alternatively numerical solutions to the equation 6.14 by using the well known group of Newmark’s integration scheme for second order differential equations are presented. In this numerical integration method, the time derivatives are approximated, using difference approximations, and therefore solution is obtained only for discrete times and not as a continuous function of time.

In Newmark’s method, the function of time and its first derivatives are approximated using Taylor’s series and only terms up to the second derivative are included. Therefore the acceleration, velocity and displacements are expressed as:

\[ \{u_t + \delta t\} = \{u_t\} + \{u_t\} \Delta t + \left[ \left( \frac{1}{2} - \gamma \right) \{\ddot{u}_t\} + \gamma \{\ddot{u}_t + \delta t\} \right] \quad \text{……Eq. (6.15a)} \]

\[ \{\ddot{u}_t + \delta t\} = \{\ddot{u}_t\} + \{(1 - \alpha)\{\ddot{u}_t\} + \alpha \{\ddot{u}_t + \delta t\}\} \Delta t \quad \text{……Eq. (6.15b)} \]

\[ \{\dddot{u}_t + \delta t\} = \{(1 - \alpha)\{\dddot{u}_t\} + \alpha \{\dddot{u}_t + \delta t\}\} \quad \text{……Eq. (6.15c)} \]

Where \( \Delta t \) is time increment

\( t \) is the current time

\( t + \Delta t \) is the next time at which the solution is sought.

Substituting the equation (6.15 c) in to the equations (6.15a) & (6.15b) and solving

\[ \{\dddot{u}_t + \delta t\} = a_3 \{\dddot{u}_t + \delta t\} - a_4 \{\dddot{u}_t\} - a_5 \{\dddot{u}_t + \delta t\} \quad \text{……Eq. (6.16a)} \]

\[ \{\ddot{u}_t + \delta t\} = \{\ddot{u}_t\} + a_1 \{\dddot{u}_t\} + a_2 \{\dddot{u}_t + \delta t\} \quad \text{……Eq. (6.16b)} \]

Where \( a_1 = (1 - \alpha)\Delta t \), \( a_2 = \alpha \Delta t \), \( a_3 = -\frac{2}{\gamma(\Delta t)^2} \), \( a_4 = a_3 \Delta t \), \( a_5 = \frac{1 - \gamma}{\gamma} \)

The parameters \( \alpha \) and \( \gamma \) are selected such that the error induced in the approximation of (Eq.6.15) does not grow unboundedly as the scheme is applied at each time step to determine the solution at the next time.

When the error induced is bounded such schemes set to be numerically stable schemes. Sometimes there is a restriction on the size of the time step that would make the error remain bounded. In such cases, the scheme is said to be conditionally stable. Schemes for which \( \gamma < \alpha \) and \( \alpha \geq 0.5 \) are conditionally stable.

Pre-multiplying the equation (6.15a) with \( [M] \) and using the equation (6.14) at \( t = t + \Delta t \) to replace \( [M] \{\dddot{u}_t + \delta t\} \), is obtained as
\( \{ \bar{R} \} \{ u_{t+\delta t} \} = \{ \bar{F} \} \)                ……Eq. (6.17)  

Where \( \{ \bar{R} \} = [K] + a_3 [M] \)                  ……Eq. (6.17a)  

\( \{ \bar{F} \} = [F]_{t+\Delta t} + [M](a_3 \{ u_t \} + a_4 \{ \dot{u}_t \} + a_5 \{ \ddot{u} \}) \)   …… Eq. (6.17b)  

The equation is (6.17b) representing, a system of algebraic equations among the discrete values of \( \{ u_t \} \) at time \( t = t + \Delta t \) in terms of known values at time \( t \). Thus the values \( u_1 = U_{mn}(t), u_2 = V_{mn}(t) \) and \( u_3 = W_{mn}(t) \) …… are determined at time \( t = t_1, t_2, \ldots, t_n \) by repeated solutions of equation (6.17). At the first time step (i.e. \( t=0 \)), the values \( u_0, \dot{u}_0, \ddot{u}_0 \) are the initial known quantities. For Model 1 the transient response for both anti-symmetric cross-ply and angle-ply laminates are estimated by adopting the Eq. (4.3) & (4.5). The procedure is repeated for every time step to compute transient characteristics.

6.5. TRANSIENT ANALYSIS FOR DISPLACEMENT MODEL – 2.

Transient analysis of simply supported anti-symmetric cross ply and angle – ply laminated plates using higher order displacement model-2 is presented in this section.

6.5.1. Equations of motion

For simply supported anti-symmetric cross-ply and angle-ply laminates, the Navier’s solution method can be used to reduce the governing equations of motion to differential equations in time. These are given by equations 4.13 & 4.17 for anti-symmetric cross ply and angle-ply laminates. For the Transient analysis, set the thermal and Mechanical loads to zero in Eq. (4.13) & Eq. (4.17) and assume periodic solutions of the form:

\[
\begin{align*}
    u_0 &= u_0(x, y) \, e^{-i\omega t}; \\
    v_0 &= v_0(x, y) \, e^{-i\omega t}; \\
    w_0 &= w_0(x, y) \, e^{-i\omega t}; \\
    \theta_x &= \theta_x(x, y) \, e^{-i\omega t}; \\
    \theta_y &= \theta_y(x, y) \, e^{-i\omega t}; \\
    \theta_0 &= \theta_0(x, y) \, e^{-i\omega t}; \\
    v_0' &= v_0'(x, y) \, e^{-i\omega t}; \\
    \theta_x' &= \theta_x'(x, y) \, e^{-i\omega t}; \\
    \theta_y' &= \theta_y'(x, y) \, e^{-i\omega t}.
\end{align*}
\]
Therefore

\[
\begin{bmatrix}
S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} & S_{17} & S_{18} & S_{19} & S_{110} & S_{111} & S_{112} \\
S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} & S_{27} & S_{28} & S_{29} & S_{210} & S_{211} & S_{212} \\
S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} & S_{37} & S_{38} & S_{39} & S_{310} & S_{311} & S_{312} \\
S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} & S_{47} & S_{48} & S_{49} & S_{410} & S_{411} & S_{412} \\
S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} & S_{57} & S_{58} & S_{59} & S_{510} & S_{511} & S_{512} \\
S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} & S_{67} & S_{68} & S_{69} & S_{610} & S_{611} & S_{612} \\
S_{71} & S_{72} & S_{73} & S_{74} & S_{75} & S_{76} & S_{77} & S_{78} & S_{79} & S_{710} & S_{711} & S_{712} \\
S_{81} & S_{82} & S_{83} & S_{84} & S_{85} & S_{86} & S_{87} & S_{88} & S_{89} & S_{810} & S_{811} & S_{812} \\
S_{91} & S_{92} & S_{93} & S_{94} & S_{95} & S_{96} & S_{97} & S_{98} & S_{99} & S_{910} & S_{911} & S_{912} \\
S_{101} & S_{102} & S_{103} & S_{104} & S_{105} & S_{106} & S_{107} & S_{108} & S_{109} & S_{1010} & S_{1011} & S_{1012} \\
S_{111} & S_{112} & S_{113} & S_{114} & S_{115} & S_{116} & S_{117} & S_{118} & S_{119} & S_{1110} & S_{1111} & S_{1112} \\
S_{121} & S_{122} & S_{123} & S_{124} & S_{125} & S_{126} & S_{127} & S_{128} & S_{129} & S_{1210} & S_{1211} & S_{1212}
\end{bmatrix}
\begin{bmatrix}
U_{\text{num}} \\
V_{\text{num}} \\
W_{\text{num}} \\
X_{\text{num}} \\
Y_{\text{num}} \\
Z_{\text{num}} \\
U_{\text{ex}} \\
V_{\text{ex}} \\
W_{\text{ex}} \\
X_{\text{ex}} \\
Y_{\text{ex}} \\
Z_{\text{ex}}
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
U_{\text{ex}} \\
V_{\text{ex}} \\
W_{\text{ex}} \\
X_{\text{ex}} \\
Y_{\text{ex}} \\
Z_{\text{ex}}
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

Therefore the transient analysis in the absence of thermal effects the Eq. (6.18) reduces to the following form as:

\[
[M]\{\ddot{u}\} + [K]\{u\} = \{F\}
\]

\[\text{.....Eq. (6.19)}\]

6.5.2. Newmark’s Time Integration

The set of equations in 6.19, for any fixed \(m\) and \(n\) can be solved exactly using either the Laplace transform method or the model analysis method. Both methods are algebraically complicated and require the determination of Eigen values and Eigen functions as in the state-space method. Alternatively numerical solutions to the equation 6.19 by using the well known group of Newmark’s integration scheme for second order differential equations are presented. In this numerical integration
method, the time derivatives are approximated, using difference approximations, and therefore solution is obtained only for discrete times and not as a continuous function of time.

In Newmark’s method, the function of time and its first derivatives are approximated using Taylor’s series and only terms up to the second derivative are included. Therefore the acceleration, velocity and displacements are expressed as:

\[ u_{t+\delta t} = u_t + \dot{u}_t \Delta t + \left[ \frac{1}{2} - \gamma \right] \ddot{u}_t + \gamma \dddot{u}_t \Delta t + \ldots \] ...Eq. (6.20a)

\[ \ddot{u}_{t+\delta t} = \ddot{u}_t + [(1-\alpha)\dddot{u}_t + \alpha \dddot{u}_{t+\delta t}] \Delta t \] ...Eq. (6.20b)

\[ \dddot{u}_{t+\delta t} = [(1-\alpha)\dddot{u}_t + \alpha \dddot{u}_{t+\delta t}] \] ...Eq. (6.20c)

Where \( \Delta t \) is time increment

\( t \) is the current time

\( t + \Delta t \) is the next time at which the solution is sought.

Substituting the equation (6.20c) in to the equations (6.20a) & (6.20b) and solving

\[ \dddot{u}_{t+\delta t} = a_3 (u_{t+\delta t} - u_t) - a_4 \dddot{u}_t - a_5 (\dddot{u}_{t+\delta t}) \] ...Eq. (6.21a)

\[ \dddot{u}_{t+\delta t} = \dot{u}_t + a_1 \dddot{u}_t + a_2 (\dddot{u}_{t+\delta t}) \] ...Eq. (6.21b)

Where \( a_1 = (1-\alpha)\Delta t \), \( a_2 = \alpha \Delta t \), \( a_3 = \frac{2}{\gamma (\Delta t)^2} \), \( a_4 = a_3 \Delta t \), \( a_5 = \left( \frac{1-\gamma}{\gamma} \right) \)

The parameters \( \alpha \) and \( \gamma \) are selected such that the error induced in the approximation of (Eq.6.20) does not grow unboundedly as the scheme is applied at each time step to determine the solution at the next time.

When the error induced is bounded such schemes set to be numerically stable schemes. Sometimes there is a restriction on the size of the time step that would make the error remain bounded. In such cases, the scheme is said to be conditionally stable. Schemes for which \( \gamma < \alpha \) and \( \alpha \geq 0.5 \) are conditionally stable.

Pre-multiplying the equation (6.20a) with \([M]\) and using the equation (6.19) at \( t = t+\Delta t \) to replace \([M]\) \{\dddot{u}_{t+\delta t}\}, is obtained as

\[ \{R\} \{u_{t+\delta t}\} = \{F\} \] ...Eq. (6.22)

Where \( \{R\} = [K] + a_3 [M] \) ...Eq. (6.22a)

\[ \{\dddot{F}\} = [F]_{t+\Delta t} + [M](a_3 \{u_t\} + a_4 \dddot{u}_t + a_5 \dddot{u}) \] ...Eq. (6.22b)

The equation is (6.22b) representing, a system of algebraic equations among the discrete values of \( \{u_t\} \) at time \( t - t+\Delta t \) in terms of known values at time \( t \). Thus the values \( u_1 = U_{mn}(t) \), \( u_2 = V_{mn}(t) \) and \( u_3 = W_{mn}(t) \) ... are determined at time
t =\textit{t}_1, \textit{t}_2, \ldots \ldots, \textit{t}_n \text{ by repeated solutions of equation (6.22). At the first time step (i.e. } \textit{t}=0), \text{ the values } u_0, \dot{u}_0, \ddot{u}_0 \text{ are the initial known quantities. For Model 1 the transient response for both anti-symmetric cross-ply and angle-ply laminates are estimated by adopting the Eq. (4.11) & (4.13). The procedure is repeated for every time step to compute transient characteristics.}

\textbf{6.6 RESULTS AND DISCUSSIONS}

In respect to investigate the numerical approach and exactness of transient behavior of anti-symmetric cross-ply and angle-ply composite laminate plates with simple support boundary conditions and subjected to a step loading of higher order theories, a iso-parametric finite element formulation of a predicted higher order displacement model is considered. An analytical solution are developed for higher-order shear deformation theories with 9 DOF and 12 DOF individually for both anti-symmetric cross-ply and angle-ply laminated plates of respective displacement model-1 and 2.

To show the differences in to FOSD theory, the present higher order theory cannot require a shear correction coefficient due to more realistic representation of the cross sectional deformation. However on the basis of good agreement of the current results with the 3D elastic results and other theories, it is better to say that the theory is predicting the transient response of composite plates. In respect to test the correctness and efficiency of the produced theories and to investigate the effects of shear deformation, the given below material properties for piezoelectric composite laminated plates are used to achieve the numerical results:

\textbf{Elastic Layer (Graphite/Epoxy):}

Material properties of Graphite/Epoxy are taken as:
\begin{align*}
a &= b = 25 \text{ cm}; & h &= 1 \text{ cm}; & E_2 &= 2.1 \times 10^6 \text{ N} \cdot \text{s}^2/\text{cm}^4; & \rho &= 8 \times 10^{-6} \text{ N}/\text{cm}^2; \\
E_1/E_2 &= 25; & G_{12} &= 0.6 \ E_2; & G_{12} &= G_{13} = 0.5 \ E_2; & \nu_{12} &= 0.25;
\end{align*}

\textbf{PFRC material:}

Material properties of PFRC layer are taken as:
\begin{align*}
C_{11} &= 32.6 \text{ Gpa}; C_{12} = C_{21} = 4.3 \text{ Gpa}; & C_{13} &= C_{31} = 4.76 \text{ Gpa};
C_{22} &= C_{33} = 7.2 \text{ Gpa}; & C_{23} &= 3.85 \text{ Gpa};
C_{44} &= 1.05 \text{ Gpa}; & C_{55} &= C_{66} = 1.29 \text{ Gpa}; & \varepsilon_{31} &= -6.76 \text{ C/m}^2;
\end{align*}
\begin{align*}
g_{11} &= g_{22} = 0.037 \text{ E-9 C/Vm}; & g_{33} &= 10.64 \text{ E-9 C/Vm}
\end{align*}
The values of ‘α’ and ‘γ’ in the Newmark’s integration method are taken as 0.5, and it is related to a constant average acceleration method.

The effect of time step on the accuracy of the solution is investigated using a simply supported anti-symmetric cross-ply and angle-ply laminated plates under the uniformly distributed at selective times for three different time steps $\Delta t = 5, 20$ and $50 \mu s$ for $V_t = 0, 100, -100V$. The effect of larger time step is to reduce the amplitude and increase the time period. The above set of material properties are used and the non-dimensionalized results are presented in Appendix-F in the form of Table 6.5 to 6.10 and the tables shows the non-dimensionalized transverse shear stresses at selective time steps, for $\Delta t = 50 \mu s$.

6.6.1 The effect of side-to-thickness ratio, degree of orthotropy, number of layers, aspect ratio, modulus ratio, shear and coupling on displacements and stresses.

- The non-dimensionalized centre deflection Vs time of the cross-ply square laminate and for various voltages with and without piezoelectric actuator is noticed for model-1 and model-2 as shown in fig.6.6 and fig.6.7 respectively. For all the time steps the difference is not noticeable on the graphs for 100V and it is constant for zero voltage, then as the amplitude is increasing the variation in the deflections is negligible.

- The non-dimensionalized centre deflection Vs time of the angle-ply square laminate and for various voltages with and without piezoelectric actuator is noticed for model-1 as shown in fig.6.8. For all the time steps the effect of coupling between the layers is very close and then the effect of amplitude is increases as well as the periods of oscillations increased.

- The non-dimensionalized centre deflection Vs time of the angle-ply square laminate and for various voltages with and without piezoelectric actuator is noticed for model-2 as shown in fig.6.9. For all the time steps the effect of coupling between the layers is very close with the piezo effect and the small variation is noticed without piezo effect from $t= 300 \mu s$ onwards. Also it is noticed the magnitude of amplitude is little less for without piezo effect.

- Fig.6.10 & Fig.6.11 shows the non-dimensionalized maximum transverse shear stress Vs time for the cross-ply square laminate and for various voltages with
and without piezoelectric actuator is noticed for model-1 and model-2. The effect of the coupling on transient response is observed, it is very close for both models. Also from the figures it is observed that the effect of increasing the amplitude as well as the periods of oscillations.

Fig.6.12 & Fig.6.13 shows the non-dimentionalized maximum transverse shear stress Vs time for the angle-ply square laminate and for various voltages with and without piezoelectric actuator is noticed for model-1 and model-2. The effect of the coupling on transient response is observed, it is very close for model-1 and a little variation is observed for model-2. Also from the figures it is observed that the effect of increasing the amplitude as well as the periods of oscillations for the both models. The transverse shear stresses in the angle-ply laminates are less compared with the cross-ply laminates for the same testing conditions.
Fig. 6.5 Non dimensionalized central deflection ($\omega$) Vs Time (t) for simply supported (SS1) cross-ply (0/90) square laminated plate (Model-1)

- - - - without piezo

- --- with piezo

- Delta t=5
- Delta t=20
- Delta t=50
- Delta t=5
- Delta t=20
- Delta t=50

Fig. 6.6 Non dimensionalized central deflection ($\omega$) Vs Time (t) for simply supported (SS1) cross-ply (0/90) square laminated plate (Model-2)

- - - - without piezo

- --- with piezo

- Delta t=5
- Delta t=20
- Delta t=50
- Delta t=5
- Delta t=20
- Delta t=50
Fig. 6.7. Non dimensionalized central deflection ($\omega$) Vs Time (t) for SS1 angle-ply (-45/45) square laminated plate (Model-1)

--- without piezo

--- with piezo

Fig. 6.8. Non dimensionalized central deflection ($\omega$) Vs Time (t) for SS1 angle-ply (-45/45) square laminated plate (Model-2)

--- without piezo

--- with piezo
Fig. 6.9. Non dimensionalized maximum transverse shear stress ($\tau_{xz}$) Vs Time (t) for SS1 cross-ply (90/90) square laminated plate (Model-1)

- - - without piezo

Model-1, SSI with piezo
Model-2, SSI with piezo
Model-1, SSI without piezo
Model-2, SSI without piezo

Fig. 6.10. Non dimensionalized maximum transverse shear stress ($\tau_{yz}$) Vs Time (t) for SS1 cross-ply (90/90) square laminated plate

Model-1, SSI with piezo
Model-2, SSI with piezo
Model-1, SSI without piezo
Model-2, SSI without piezo
Fig. 6.11. Non dimensionalized maximum transverse shear stress ($\tau_{xz}$) Vs Time (t) for SS1 cross-ply (-45/45) square laminated plate

Fig. 6.12. Non dimensionalized maximum transverse shear stress ($\tau_{yz}$) Vs Time (t) for SS1 cross-ply (-45/45) square laminated plate