Chapter 4

Hypersurface of a Finsler Space with Generalized Kropina Conformal Change Metric

4.1 Introduction

and P. N. Pandey[84] studied a hypersurface of a Finsler space with Randers change of Matsumoto metric.

The generalized Kropina metric[111] of a Finsler space is given by

\[ L = \frac{\alpha^{n+1}}{\beta^n} \quad (n \neq 0, -1), \]

where \( \alpha(x, y) = \sqrt{a_{ij}(x)y^iy^j} \) and \( \beta(x, y) = b_iy^i \).

Let \( F^*n = (M^n, L^*) \) be another Finsler space over the same manifold \( M^n \). If \( L^*(x, y) = e^{\sigma(x)}L(x, y) \), then the change of metric is a conformal change and the function \( \sigma(x) \) is conformal factor [114].

If the conformal change is given by

\[ (4.1.1) \quad L^*(x, y) = e^{\sigma(x)}L^{n+1}_{\beta^n}, \quad \text{where} \quad \beta = b_i(x)y^i, \]

then it is called a generalized Kropina conformal change[111].

If \( \sigma(x) \) is constant then (4.1.1) reduces to generalized Kropina homothetic change

\[ (4.1.2) \quad L^*(x, y) = kL^{n+1}_{\beta^n}, \]

where \( k \) is a constant.

The aim of the present chapter is to study a Finsler space whose metric is obtained from the metric of a Finsler space by generalized Kropina conformal change and to obtain a necessary and sufficient condition for these Finsler spaces to be projectively related. It is also planned to study the relation between the hypersurface of a Finsler space and the hypersurface of a Finsler space whose metric is obtained by the generalized Kropina conformal change.
4.2 Finsler Space with Generalized Kropina Conformal Change Metric

Let $F^* = (M^n, L^*)$ be an $n$-dimensional Finsler space on the differentiable manifold $M^n$ whose metric is obtained from the metric of the Finsler space $F^n$ by generalized Kropina conformal change (4.1.1).

Throughout the chapter, the geometric objects associated with $F^*$ will be asterisked *.

Differentiating (4.1.1) partially with respect to $y^i$, we get

\[ l_i^* = e^{\sigma(x)} \left\{ (n + 1) \frac{L^n}{\beta^n} l_i - n \frac{L^{n+1}}{\beta^{n+1}} b_i \right\}, \]

where $l_i = \dot{\partial}_i L$ and $l_i^* = \dot{\partial}_i L^*$.

Differentiating (4.2.1) partially with respect to $y^j$ and using (1.7.2), we have

\[ h_{ij}^* = (n + 1) e^{2\sigma(x)} \frac{L^{2n}}{\beta^{2n}} \left\{ g_{ij} - n \frac{L}{\beta} (l_i b_j + l_j b_i) + n \frac{L^2}{\beta^2} b_i b_j + (n - 1) l_i l_j \right\}, \]

where $h_{ij}^* = L^* \dot{\partial}_j l_i^*$.

Using (4.2.1), we find

\[ l_i^* l_j^* = e^{2\sigma(x)} \left\{ (n + 1)^2 \frac{L^{2n}}{\beta^{2n}} l_i l_j - n (n + 1) \frac{L^{2n+1}}{\beta^{2n+1}} (l_i b_j + l_j b_i) \right\} + n^2 \frac{L^{2(n+1)}}{\beta^{2(n+1)}} b_i b_j, \]

(4.2.3)
From (4.2.2), (4.2.3) and $g^*_{ij} = h^*_{ij} + l^*_il^*_j$, we have

$$g^*_{ij} = e^{2\sigma(x)}(n + 1)\frac{L^{2n}}{\beta^{2n}}g_{ij} + e^{2\sigma(x)}(2n + 1)\frac{L^{2(n+1)}}{\beta^{2(n+1)}}b_ib_j$$

(4.2.4)

$$- e^{2\sigma(x)}2n(n + 1)\frac{L^{2n+1}}{\beta^{2n+1}}(l_ib_j + l_jb_i)$$

$$+ e^{2\sigma(x)}2n(n + 1)\frac{L^{2n}}{\beta^{2n}}l_il_j.$$ 

Since $g^*_{ij}g^{*jk} = \delta^k_i$, the inverse metric tensor $g^{*ij}$ is given by

(4.2.5)

$$g^{*ij} = \frac{1}{p}g^{ij} - \frac{L^2}{p\beta^2\left\{\frac{L^2\beta^2}{\beta^2} + \frac{1-n}{n}\right\}}b^ib^j + \frac{2L}{p\beta\left\{\frac{L^2\beta^2}{\beta^2} + \frac{1-n}{n}\right\}}(l^ib^j + l^jb^i)$$

$$- \frac{2n}{p}\left\{\frac{\beta^2(n + 1) - nL^2b^2}{\beta^2(1-n) + nL^2b^2}\right\}l^jl^j,$$

where $p = e^{2\sigma(x)}(n + 1)\frac{L^{2n}}{\beta^{2n}}$, $b^i = g^{ij}b_j$ and $b^2 = b^ib_i$.

Differentiating (4.2.4) partially with respect to $y^k$, we find

(4.2.6)

$$C^*_{ijk} = p\left\{C_{ijk} - \frac{n}{\beta}(g_{jk}b_i + g_{ki}b_j + g_{ij}b_k) - \frac{n(2n + 1)L^2}{\beta^3}b_ib_jb_k$$

$$+ \frac{n}{L}(g_{jk}l_i + g_{ki}l_j + g_{ij}l_k) - \frac{2n^2}{\beta}(b_il_jl_k + b_jl_kl_i + b_kl_il_j)$$

$$+ \frac{n(2n + 1)L}{\beta^2}(b_ibjl_k + b_jbkl_i + b_kbil_j) + \frac{2n(n - 1)}{L}l_il_jk_k\right\}.$$ 

Transvecting (4.2.6) with $g^{*jh}$, we have
\[(4.2.7)\]
\[
 C_{ik}^{h} = C_{ik}^{h} + ALC_{ijk}b^{i}(2\beta l^{h} - Lb^{h}) - \frac{n}{\beta}(\delta_{k}^{h}b_{i} + \delta_{i}^{h}b_{k})
\]
\[
 + \frac{n}{L}(\delta_{k}^{h}l_{i} + \delta_{i}^{h}l_{k}) - A\beta g_{ik}b^{h} + A(\beta^{2} + n\beta^{2} - nL^{2}b^{2})g_{ik}l^{h}
\]
\[
 - AL^{2}\beta^{2}(4n^{2} + 2n + 1)b^{h}b_{i}b_{k} - 2n^{2}A(\beta^{2} + L^{2}b^{2})(b_{i}l_{k} + b_{k}l_{i})l^{h}
\]
\[
 - AL[3n\beta^{2} + 2n^{2}\beta^{2} + n(2n + 1)L^{2}b^{2}]l^{h}b_{i}b_{k}
\]
\[
 + AL\beta^{2}(4n - 4n^{2} + 1)b^{h}(b_{i}l_{k} + b_{k}l_{i})
\]
\[
 + 2A(2n^{2}\beta^{2} - n\beta^{2} - nL^{2}b^{2} - \beta^{2})l^{h}l_{i}l_{k},
\]

where \(A = \frac{n}{nL^{2}b^{2} + \beta^{2}(1-n)}\).

Thus, we have

**Theorem 4.2.1:** The components of the metric tensor, inverse metric tensor, Cartan tensor and associate Cartan tensor of a Finsler space with generalized Kropina conformal changed metric are given by \((4.2.4)\), \((4.2.5)\), \((4.2.6)\) and \((4.2.7)\) respectively.

Let us denote the difference of Cartan connection coefficients \(F_{jk}^{i}\) of the Finsler space \(F^{\alpha}\) and Cartan connection coefficients \(F_{jk}^{*i}\) of the Finsler space \(F^{*\alpha}\) by \(D_{jk}^{i}\). Thus, we have

\[(4.2.8)\]
\[
 F_{jk}^{*i} = F_{jk}^{i} + D_{jk}^{i}.
\]

Transvecting \((4.2.8)\) by \(y^{k}\) and using \(((1.8.4))(h)\), we get

\[(4.2.9)\]
\[
 G_{j}^{*i} = G_{j}^{i} + D_{0j}^{i},
\]

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where \( D_{0j}^i = D_{kj}^i y^k \).

Transvecting (4.2.9) by \( y^j \) and using ((1.8.4))(i), we get

(4.2.10) \[ 2G^*i = 2G^i + D^i_{00}, \]

where \( D^i_{00} = D^i_{0j} y^0 \).

Differentiating (4.2.10) partially with respect to \( y^j \) and using (4.2.9), we have

(4.2.11) \[ \dot{\partial}_j D^i_{00} = 2D^i_{0j}, \]

The expressions for \( D^i_{00}, D^i_{0j} \) and \( D^i_{jk} \) are calculated as follows.

Differentiating (4.2.1) partially with respect to \( y^j \), we find

(4.2.12) \[ L^*_ij = (n+1)e^{\sigma(x)} \frac{L^{n-1}}{\beta^n} \left\{ LL_{ij} + n l_i l_j - \frac{nL}{\beta} (l_i b_j + l_j b_i) + \frac{nL^2}{\beta^2} b_i b_j \right\}, \]

where \( L^*_ij = \dot{\partial}_j l^*_i \) and \( L_{ij} = \dot{\partial}_j l_i \).

Differentiating (4.2.12) partially with respect to \( y^k \), we get

(4.2.13)

\[
L^*_{ijk} = (n+1)e^{\sigma(x)} \left\{ \frac{L^n}{\beta^n} L_{ijk} + n(n-1) \frac{L^{n-2}}{\beta^n} l_i l_j l_k 
+ \frac{nL^{n-1}}{\beta^n} (L_{ij} l_k + L_{jk} l_i + L_{ki} l_j) - \frac{nL^n}{\beta^{n+1}} (L_{ij} b_k + L_{jk} b_i + L_{ki} b_j) 
- \frac{n^2 L^{n-1}}{\beta^{n+1}} (l_i l_j b_k + l_j l_k b_i + l_k l_i b_j) 
+ \frac{n(n+1)L^n}{\beta^{n+2}} (l_i b_j b_k + l_j b_k b_i + l_k b_i b_j) - \frac{n(n+2)L^{n+1}}{\beta^{n+3}} b_i b_j b_k \right\},
\]

\[ \text{53} \]
where $L^*_{ijk} = \dot{\partial}_k L^*_{ij}$ and $L_{ijk} = \dot{\partial}_k L_{ij}$.

Differentiating (4.2.12) partially with respect to $x^k$, we get

(4.2.14)

$$
\partial_k L^*_{ij} = (n + 1)e^{\sigma(x)} \left\{ \begin{bmatrix}
\frac{L^n}{\beta^n} L_{ij} + \frac{nL^{n-1}}{\beta^n} l_i l_j - \frac{nL^n}{\beta^{n+1}} (l_i b_j + l_j b_i) \\
+ \frac{nL^{n+1}}{\beta^{n+2}} b_i b_j
\end{bmatrix} \sigma_k \\
+ \frac{L^n}{\beta^n} \partial_k L_{ij} + \left[ \begin{bmatrix}
\frac{nL^{n-1}}{\beta^n} L_{ij} + \frac{n(n-1)L^{n-2}}{\beta^n} l_i l_j \\
- \frac{n^2L^{n-1}}{\beta^{n+1}} (l_i b_j + l_j b_i) + \frac{n(n+1)L^n}{\beta^{n+2}} b_i b_j
\end{bmatrix} \partial_k L
\right]
\right. \\
\left. + \left[ \begin{bmatrix}
\frac{n(n+1)L^n}{\beta^{n+2}} (l_i b_j + l_j b_i) - \frac{n(n+2)L^{n+1}}{\beta^{n+3}} b_i b_j \\
- \frac{nL^n}{\beta^{n+1}} L_{ij} - \frac{n^2L^{n-1}}{\beta^{n+2}} l_i l_j
\end{bmatrix} \partial_k \beta
\right]
\right. \\
\left. + \left[ \begin{bmatrix}
\frac{nL^{n-1}}{\beta^n} l_i - \frac{nL^n}{\beta^{n+1}} b_i \end{bmatrix} \partial_k l_i + \left[ \begin{bmatrix}
\frac{nL^{n-1}}{\beta^n} l_i - \frac{nL^n}{\beta^{n+1}} b_i \end{bmatrix} \partial_k l_j
\right]
\right.
\left. + \left[ \begin{bmatrix}
\frac{nL^{n+1}}{\beta^{n+2}} b_j - \frac{nL^n}{\beta^{n+1}} l_j
\end{bmatrix} \partial_k b_i + \left[ \begin{bmatrix}
\frac{nL^{n+1}}{\beta^{n+2}} b_j - \frac{nL^n}{\beta^{n+1}} l_j
\end{bmatrix} \partial_k b_j
\right],
\right.
$$

where $\sigma_k = \frac{\partial \sigma(x)}{\partial x^k}$.

Let us denote the symmetric and skew symmetric parts of the tensor $b_{i|j}$ by $r_{ij}$ and $s_{ij}$ respectively. Thus, we have

(4.2.15) \( a \) \) $2r_{ij} = b_{i|j} + b_{j|i}$, \( b \) \) $2s_{ij} = b_{i|j} - b_{j|i}$. 

From (4.2.15), we have

(4.2.16) \[ b_{i|j} = r_{ij} + s_{ij}, \]
which may be re-written as

\[(4.2.17) \quad \partial_j b_i = r_{ij} + s_{ij} + b_r F_{ij}^r.\]

Transvecting (4.2.17) with \(y^i\), we have

\[(4.2.18) \quad (\partial_j b_i)y^i = r_{0j} + s_{0j} + b_r G_{ij}^r,\]

where \(0\) stands for the contraction with respect to \(y^i\), i.e. \(r_{0j} = r_{ij}y^i\) and \(s_{0j} = s_{ij}y^i\).

Since the \(h\)-covariant derivative of \(L\) and \(l_i\) with respect to Cartan connection vanish identically, we have

\[(4.2.19) \quad \partial_k l_i = L_{ir} G_k^r + l_r F_{ik}^r.\]

and

\[(4.2.20) \quad \partial_k L = l_i G_k^r.\]

Differentiating \(\beta = b_i y^i\) with respect to \(x^k\) and using (4.2.18), we have

\[(4.2.21) \quad \partial_k \beta = r_{0k} + s_{0k} + b_r G_k^r.\]

Since the \(h\)-covariant derivative of the tensor \(L_{ij}^*\) with respect to Cartan connection vanishes identically, we have

\[(4.2.22) \quad \partial_k L_{ij}^* - L_{ijr}^* G_k^r + L_{ir}^* F_{jk}^r - L_{jr}^* F_{ik}^r = 0.\]

Using (4.2.1), (4.2.8), (4.2.9), (4.2.10), (4.2.12), (4.2.13), (4.2.14), (4.2.17), (4.2.19), (4.2.20) and (4.2.21) in (4.2.22) then transvecting the resulting equation with \(y^k\), we have
\[ L^*_i \sigma_0 + (n + 1)e^{\sigma(x)} \left\{ \begin{array}{c} -\frac{nL^n}{\beta^{n+1}} L_{ij} + \frac{n(n+1)L^n}{\beta^{n+2}} (l_ib_j + l_jb_i) \\ -\frac{n^2L^{n-1}}{\beta^{n+1}} l_il_j - \frac{n(n+2)L^{n+1}}{\beta^{n+3}} b_ib_j \end{array} \right\} r_{00} \]

\[(4.2.23)\]

\[ + \left[ \frac{nL^{n+1}}{\beta^{n+2}} b_j - \frac{nL^n}{\beta^{n+1}} l_j \right] (r_{i0} + s_{i0}) \]

\[ + \left[ \frac{nL^{n+1}}{\beta^{n+2}} b_i - \frac{nL^n}{\beta^{n+1}} l_i \right] (r_{j0} + s_{j0}) \]

\[ - L^*_{ij} D^r_{00} - L^*_i D^r_{0j} - L^*_j D^r_{0i} = 0, \]

where \( \sigma_0 = \sigma_k y^k \) and \( r_{00} = r_{0i} y^i \).

Differentiating (4.2.1) partially with respect to \( x^j \), we have

\[ \partial_j l^*_i = l^*_i \sigma_j + e^{\sigma(x)} \left\{ \frac{n(n+1)L^{n-1}}{\beta^n} l_i - \frac{n(n+1)L^n}{\beta^{n+1}} b_i \right\} \partial_j L \]

\[ + e^{\sigma(x)} \left\{ \frac{n(n+1)L^{n+1}}{\beta^{n+2}} b_i - \frac{n(n+1)L^n}{\beta^{n+1}} l_i \right\} \partial_j \beta \]

\[ + e^{\sigma(x)} \left\{ \frac{(n+1)L^n}{\beta^n} \partial_j l_i - \frac{nL^{n+1}}{\beta^{n+1}} \partial_j b_i \right\}. \]

Since the \( h \)-covariant derivative of the vector \( l^*_i \) with respect to Cartan connection vanishes identically, we have

\[(4.2.25) \quad \partial_j l^*_i - L^*_i G^*_j - l^*_i F^*_ij = 0. \]

Using (4.2.1), (4.2.8), (4.2.10), (4.2.12) and (4.2.24) in (4.2.25), we have

\[ l^*_i \sigma_j + e^{\sigma(x)} \left\{ \begin{array}{c} -\frac{n(n+1)L^n}{\beta^{n+1}} l_i + \frac{n(n+1)L^{n+1}}{\beta^{n+2}} b_i \end{array} \right\} (r_{0j} + s_{0j}) \]

\[ - \frac{nL^{n+1}}{\beta^{n+1}} b_i |j \right\} - L^*_i D^r_{0j} - l^*_i D^r_{ij} = 0, \]

\[(4.2.26)\]

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which implies

\[(4.2.27)\]
\[
e^{\sigma(x)} \frac{nL^{n+1}}{\beta^{n+1}} b_{ij} = l_i^* \sigma_j - L_{ir}^* D_{0j}^r - l_r^* D_{ij}^r
\]
\[+ e^{\sigma(x)} \left\{ - \frac{n(n+1) L^n}{\beta^{n+1}} l_i + \frac{n(n+1) L^{n+1}}{\beta^{n+2}} b_i \right\} (r_{0j} + s_{0j}).\]

From (4.2.15)(a) and (4.2.27), we have

\[(4.2.28)\]
\[
2e^{\sigma(x)} \frac{nL^{n+1}}{\beta^{n+1}} r_{ij} = e^{\sigma(x)} \left\{ \left[ (n+1) \frac{L^n}{\beta^n} l_i - n \frac{L^{n+1}}{\beta^{n+1}} b_i \right] \sigma_j
\]
\[+ \left[ (n+1) \frac{L^n}{\beta^n} l_j - n \frac{L^{n+1}}{\beta^{n+1}} b_j \right] \sigma_i \right\}
\[+ n(n+1) \left[ \frac{L^{n+1}}{\beta^{n+2}} b_i - \frac{L^n}{\beta^{n+1}} l_i \right] (r_{0j} + s_{0j})
\[+ n(n+1) \left[ \frac{L^{n+1}}{\beta^{n+2}} b_j - \frac{L^n}{\beta^{n+1}} l_j \right] (r_{0i} + s_{0i}) \right\}
\[- L_{ir}^* D_{0j}^r - L_{jr}^* D_{0i}^r - 2l_i^* D_{ij}^r.
\]

Subtracting (4.2.28) from (4.2.23) and contracting with \(y^i y^j\), we have

\[(4.2.29)\]
\[
(n+1) \beta l_r D_{00}^r - n Lb_r D_{00}^r = -nLr_{00} + L\beta \sigma_0.
\]

Let us denote \(l_r D_{00}^r\) by \(R\) and \(b_r D_{00}^r\) by \(S\). Thus, we have

\[(4.2.30)\]
\[
(n+1) \beta R - nLS = -nLr_{00} + L\beta \sigma_0,
\]
From (4.2.15)(b) and (4.2.27), we have

\[ 2e^{\sigma(x)} \frac{nL^{n+1}}{\beta_{n+1}} s_{ij} = e^{\sigma(x)} \left\{ \left[ (n + 1)\frac{L^n}{\beta^n} l_i - n\frac{L^{n+1}}{\beta^{n+1}} b_i \right] \sigma_j \right. \]

\[ - \left. \left[ (n + 1)\frac{L^n}{\beta^n} l_j - n\frac{L^{n+1}}{\beta^{n+1}} b_j \right] \sigma_i \right\} \]

(4.2.31)

\[ + n(n + 1) \left[ \frac{L^{n+1}}{\beta^{n+2}} b_i - \frac{L^n}{\beta^{n+1}} l_i \right] \left( r_{0j} + s_{0j} \right) \]

\[ - n(n + 1) \left[ \frac{L^{n+1}}{\beta^{n+2}} b_j - \frac{L^n}{\beta^{n+1}} l_j \right] \left( r_{0i} + s_{0i} \right) \]

\[ - L^i_{ir} D^r_{0j} + L^r_{jr} D^r_{0i} \]

Adding (4.2.23) and (4.2.31), using \( LL_{ir} = g_{ir} - l_i l_r \) and transvecting with \( b^i y^j \), we have

(4.2.32)

\[ n(n + 1)L(2L^2 b^2 - \beta^2)r_{00} + \left\{ (n + 1)\beta^2 - nL^2 b^2 \right\} L\beta \sigma_0 - 2nL^3 \beta s_{i0} b^i \]

\[ + L^3 \beta^2 \sigma_i b^i = (n + 1) \left\{ (1 - n)\beta^2 + nL^2 b^2 \right\} \left\{ LS - \beta R \right\} . \]

(4.2.30) and (4.2.32) constitute the system of algebraic equations in \( R \) and \( S \). Solving these equations, we have

(4.2.33)

\[ S = \frac{n \left[ L^2 b^2 - 2\beta^2 \right] r_{00} - 2nL^2 \beta s_{i0} b^i + L^2 \beta^2 \sigma_i b^i + 2\beta^3 \sigma_0}{\left[ (1 - n)\beta^2 + nL^2 b^2 \right]} \]

and
\[ R = \frac{-n(n+1)L\beta r_{00} - 2n^2L^3s_{i0}b^i + nL^3\beta\sigma_i b^i + L[(n+1)\beta^2 + nL^2b^2]\sigma_0}{(n+1)[(1-n)\beta^2 + nL^2b^2]} . \]

Transvecting (4.2.31) with \( y^j \) and using \( LL_{ir} = g_{ir} - l_i l_r \), we have

\[ 2n\frac{L^{n+1}}{\beta^{n+1}}s_{i0} = \left[ (n+1)\frac{L^n}{\beta^n}l_i - n\frac{L^{n+1}}{\beta^{n+1}}b_i \right] \sigma_0 - \frac{L^{n+1}}{\beta^n}\sigma_i 
+ (n+1)\left[ \frac{L^{n+1}}{\beta^{n+2}}b_i - \frac{L^n}{\beta^{n+1}}l_i \right] r_{00} - \left\{ (n+1)\frac{L^{n-1}}{\beta^n}g_{ir} \right. \\
+ (n^2 - 1)\frac{L^{n-1}}{\beta^n}l_i l_r - n(n+1)\frac{L^n}{\beta^{n+1}}b_i b_r \right. \\
- n(n+1)\frac{L^n}{\beta^{n+1}}l_r b_i + n(n+1)\frac{L^{n+1}}{\beta^{n+2}}b_i b_r \right\} D_{00} . \]

Transvecting (4.2.35) with \( g^{ij} \), we have

\[ 2n\frac{L^{n+1}}{\beta^{n+1}}s_{j0} = \left[ (n+1)\frac{L^n}{\beta^n}l^j - n\frac{L^{n+1}}{\beta^{n+1}}b^j \right] \sigma_0 - \frac{L^{n+1}}{\beta^n}\sigma^j 
+ (n+1)\left[ \frac{L^{n+1}}{\beta^{n+2}}b^j - \frac{L^n}{\beta^{n+1}}l^j \right] r_{00} - \left\{ (n+1)\frac{L^{n-1}}{\beta^n}\delta^j_r \right. \\
+ (n^2 - 1)\frac{L^{n-1}}{\beta^n}l^j l_r - n(n+1)\frac{L^n}{\beta^{n+1}}l^j b_r \right. \\
- n(n+1)\frac{L^n}{\beta^{n+1}}l_r b^j + n(n+1)\frac{L^{n+1}}{\beta^{n+2}}b^j b_r \right\} D_{00} , \]

where \( s^j_0 = s_{i0} g^{ij} \) and \( \sigma^j = \sigma_i g^{ij} \).
From (4.2.36), we have

\[
D_{00}^j = \frac{y^j}{L\beta} \left\{ -nLr_{00} - (n - 1)\beta R + nLS + L\beta \sigma_0 \right\}
\]

(4.2.37)

\[
+ \frac{nLb^j}{\beta^2} \left\{ Lr_{00} + \beta R - LS - \frac{L\beta}{(n + 1)} \sigma_0 \right\}
\]

\[- \frac{L^2}{(n + 1)} \sigma^j - \frac{2nL^2}{(n + 1)\beta} s_0^j.\]

Differentiating (4.2.37) partially with respect to \(y^k\) and using (4.2.11), we have

(4.2.38)

\[
D_{0k}^j = \delta_k^j \left\{ \frac{-nr_{00}}{2\beta} - \frac{(n - 1)R}{2L} + \frac{nS}{2\beta} + \sigma_0 \right\} + y^j \left\{ \frac{-nr_{0k}}{\beta} \right\}
\]

\[
+ \frac{nr_{00}b_k}{2\beta^2} + \frac{(n - 1)R}{2L^2} l_k - \frac{(n - 1)}{2L} R_k - \frac{nS}{2\beta^2} b_k + \frac{n}{2\beta} S_k + \frac{\sigma_k}{2}
\]

\[
+ nb^j \left\{ \frac{Lr_{00}l_k}{\beta^2} + \frac{L^2 r_{0k}}{\beta^2} b_k - \frac{L^2 r_{00}}{\beta^3} b_k + \frac{R}{2\beta} l_k + \frac{L}{2\beta} R_k - \frac{LR}{2\beta^2} b_k
\]

\[
- \frac{LS}{\beta^2} l_k + \frac{L^2 S}{\beta^3} b_k - \frac{L^2}{\beta^2} S_k - \frac{L}{\beta(n + 1)} \sigma_0 l_k + \frac{L^2}{2\beta^2} \frac{\sigma_0 b_k}{(n + 1)} \}
\]

\[- \frac{L^2}{\beta^2} \frac{\sigma_k}{(n + 1)} \} - \frac{Ll_k}{(n + 1)} \sigma^j + \frac{L^2}{(n + 1)} \sigma^t C_{tk}^j - \frac{2nL}{(n + 1)\beta} l_k s_0^j
\]

\[
+ \frac{nL^2}{(n + 1)\beta^2} b_k s_0^j - \frac{nL^2}{(n + 1)\beta} \left( s_k^j - 2s_0^t C_{tk}^j \right),
\]

where \(s_k^j = s_{ik}g^{ij}\), \(R_k = \dot{\sigma}_k R\) and \(S_k = \dot{\sigma}_k S\).
Differentiating (4.2.38) partially with respect to \( y^h \), we have

\[
\delta_h D_{0k}^j = \delta_k^j \left\{ \frac{-n r_0 h}{\beta} + \frac{n r_{00}}{2 \beta^2} b_h + \frac{(n-1) R}{2 L^2} l_h - \frac{(n-1)}{2 L} R_h - \frac{n S}{2 \beta^2} b_h + \frac{n}{2 \beta} S_h + \frac{\sigma_h}{2} \right\} \\
+ \delta_k^j \left\{ \frac{-n r_0 k}{\beta} + \frac{n r_{00}}{2 \beta^2} b_k + \frac{(n-1) R}{2 L^2} l_k - \frac{(n-1)}{2 L} R_k - \frac{n S}{2 \beta^2} b_k + \frac{n}{2 \beta} S_k + \frac{\sigma_k}{2} \right\} \\
+ y^j \left\{ \frac{-n r_h k}{\beta} + \frac{n b_r y^s (\dot{\delta}_h F_{rs}^r)}{\beta^2} + \frac{n}{\beta^2} (r_{0k} b_h + r_{0h} b_k) - \frac{(n-1) R}{2 L^3} (2 l_k l_h - LL_{kh}) \right\} \\
+ \frac{(n-1)}{2 L^2} (R_h l_k + R_k l_h) - \frac{n}{\beta^3} b_k b_h (r_{00} - S) + \frac{n}{2 \beta} S_{kh} - \frac{n}{2 \beta^2} (S_h b_k + S_k b_h) \\
+ n b^j \left\{ \frac{2 L}{\beta^2} (r_{0h} l_k + r_{0k} l_h) + \frac{L^2}{\beta^2} r_{kh} - \frac{L^2 b_r y^s (\dot{\delta}_h F_{rs}^r)}{\beta^2} + \frac{L}{\beta} R_{kh} - \frac{L^2}{2 \beta^2} S_{kh} \right\} \\
- \frac{2 L^2}{\beta^3} (r_{0h} b_k + r_{0k} b_h) + \frac{L}{\beta^3} b_k b_h \left[ 3 L r_{00} + \beta R - 3 L S - \frac{L \beta \sigma_0}{(n+1)} \right] \\
+ \frac{1}{2 \beta} (R_k l_h + R_h l_k) + \frac{l_k l_h}{\beta^2} \left[ r_{00} - S - \frac{\beta \sigma_0}{(n+1)} \right] - \frac{L}{2 \beta^2} (R_k b_h + R_h b_k) \\
+ \frac{L_{kh}}{\beta^2} \left[ L r_{00} - L S + \frac{\beta R}{2} - \frac{L \beta \sigma_0}{(n+1)} \right] - \frac{L}{\beta^2} (S_k l_h + S_h l_k) + \frac{L^2}{\beta^3} (S_k b_h + S_h b_k) \\
+ \frac{(l_k b_h + l_h b_k)}{\beta^3} \left[ -2 L r_{00} + 2 L S - \frac{\beta R}{2} + \frac{L \beta \sigma_0}{(n+1)} \right] - \frac{L}{\beta(n+1)} (\sigma_h l_k + \sigma_k l_h) \\
+ \frac{L^2}{2 \beta^2(n+1)} (\sigma_h b_k + \sigma_k b_h) \right\} - \frac{g_{kh}}{\beta(n+1)} \left( \beta \sigma + 2 n s_0^j \right) \\
+ \frac{L}{\beta(n+1)} (\beta \sigma + 2 n s_0^r) \left[ L \left( \dot{\delta}_h C_{rk}^j - 2 C_{rh}^i C_{ik}^j \right) + 2 \left( l_k C_{rk}^j + l_h C_{rk}^j \right) \right] \\
- \frac{2 n L^2}{\beta^3(n+1)} b_k b_h s_0^j + \frac{2 n L}{\beta^2(n+1)} s_0^j (l_k b_h + l_h b_k) - \frac{2 n L}{\beta(n+1)} \left( l_k s_0^j + l_h s_0^j \right) \\
+ \frac{2 n L^2}{\beta^2(n+1)} \left( b_k s_h^j + b_h s_k^j \right) - \frac{2 n L^2}{\beta^2(n+1)} s_0^j \left( b_k C_{rk}^j + b_h C_{rk}^j \right) \\
+ \frac{2 n L^2}{\beta(n+1)} \left( s_k^j C_{rk}^j + s_h^j C_{rk}^j \right) \delta_1 \]
where $R_{kh} = \dot{\partial}_h R_k$ and $S_{kh} = \dot{\partial}_h S_k$.

Differentiating (4.2.9) partially with respect to $y^k$, we have

$$G^i_{jk} = G^i_{jk} + \dot{\partial}_k D^i_{0j}, \quad (4.2.40)$$

From (1.8.7), (4.2.8) and (4.2.40), we find

$$D^i_{jk} = \dot{\partial}_j D^i_{0k} - (C_{jk|0}^{si} - C_{jk|0}^i). \quad (4.2.41)$$

Thus, we have

**Theorem 4.2.2:** The difference tensor $D^i_{jk}$ of the Cartan connection coefficients $F^*_i^{jk}$ of the Finsler space $F^*n$ with the generalized Kropina conformal changed metric $L^*$ and the Cartan connection coefficients $F^i_{jk}$ of the Finsler space $F^n$ with the metric $L$ is given by (4.2.41) together with (4.2.7) and (4.2.39).

If the generalized Kropina conformal change is homothetic, (4.2.37) reduces to

$$D^i_{00} = \frac{n}{\beta} \left\{ \frac{(n+1)L\beta^2 r_{00} + 2nL^3 \beta s_{i0} b^i}{(n+1)[(1-n)\beta^2 + nL^2 b^2]} \right\} \left\{ - \frac{2y^i}{L} + \frac{Lb^i}{\beta} \right\} - \frac{2nL^2}{(n+1)\beta} s^i_{0j}. \quad (4.2.42)$$

If the vector $b_i$ appearing in the generalized Kropina homothetic change is parallel with respect to the Cartan connection of $F^n$ i.e. $b_{i|j} = 0$ then we have $r_{ij} = 0 = s_{ij}$. This implies $s^i_{0j} = 0$. Thus from (4.2.42), we have

$$D^i_{00} = 0, \quad (4.2.43)$$

which gives

$$D^i_{jk} = 0. \quad (4.2.44)$$
From (4.2.40) and (4.2.42), we have

\[(4.2.45) \quad G^{*i}_{jk} = G^i_{jk},\]

Thus we conclude:

**Theorem 4.2.3:** For the generalized Kropina homothetic change (4.1.2), the difference tensor vanishes if the vector $b_i$ is parallel with respect to the Cartan connection of $F^n$ and the Berwald connection coefficients for both the spaces $F^n$ and $F^*_n$ are the same.

### 4.3 Projectively Related Finsler Spaces

**Definition 4.3.1.** Let us consider two Finsler spaces $F^n = (M^n, L)$ and $F^*_n = (M^n, L^*)$ on the same manifold $M^n$. Then the transformation from $F^n$ to $F^*_n$ which maps every geodesic of $F^n$ to some geodesic of $F^*_n$ is known as projective change and the Finsler spaces $F^n$ and $F^*_n$ are called projectively related Finsler spaces.

It is well known that the change $L \rightarrow L^*$ is projective if

\[(4.3.1) \quad G^{*i} = G^i + P(x, y)y^i,\]

where $P(x, y)$ is a homogeneous scalar function of degree one in $y^i$, called as projective factor.

Partial differentiation of (4.3.1) with respect to $y^j$ gives

\[(4.3.2) \quad G^{*i}_j - G^i_j = P_j y^i + P \delta^i_j,\]

A geodesic of $F^n$ is given by the system of differential equations

\[(4.3.3) \quad \frac{d^2x^i}{dt^2} + 2G^i(x, y) = \tau y^i,\]
where \( \tau = \frac{1}{L} \frac{dL}{dt} \), \( y^i = \frac{dx^i}{dt} \) and \( t \) is the parameter.

The Euler-Lagrange equations for the Finsler space \( F^n \) is given

\[
\frac{\partial L^*}{\partial x^i} - \frac{d}{dt} \left( \frac{\partial L^*}{\partial y^i} \right) = 0.
\]

Using (4.1.1) in (4.3.4), we find

\[
\frac{\partial}{\partial x^i} \left( e^\sigma(x) \frac{L^{n+1}}{\beta^n} \right) - \frac{d}{dt} \left[ \frac{\partial}{\partial y^i} \left( e^\sigma(x) \frac{L^{n+1}}{\beta^n} \right) \right] = 0,
\]

which implies

\[
e^\sigma(x)(n + 1) \frac{L^n}{\beta^n} \left[ \frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial y^i} \right] + e^\sigma(x) \frac{L^{n+1}}{\beta^n} \frac{\partial \sigma(x)}{\partial x^i} - n(n + 1) e^\sigma(x) \frac{L^{n-1}}{\beta^{n-2}} \left[ \frac{\partial}{\partial y^i} \left( \frac{L}{\beta} \right) \right] \left[ \frac{d}{dt} \left( \frac{L}{\beta} \right) \right] - n e^\sigma(x) \frac{L^{n+1}}{\beta^{n+1}} \left[ \frac{\partial \beta}{\partial x^i} - \frac{d}{dt} \frac{\partial \beta}{\partial y^i} \right] = 0,
\]

which reduces to

\[
\left[ \frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial y^i} \right] + A_i = 0.
\]

where \( A_i \) is the covariant vector defined as

\[
A_i = \frac{L}{(n + 1)} \frac{\partial \sigma(x)}{\partial x^i} - n \beta^2 \frac{L}{\beta} \left[ \frac{\partial}{\partial y^i} \left( \frac{L}{\beta} \right) \right] \left[ \frac{d}{dt} \left( \frac{L}{\beta} \right) \right] - \frac{n}{(n + 1)} \beta \left[ \frac{\partial \beta}{\partial x^i} - \frac{d}{dt} \frac{\partial \beta}{\partial y^i} \right].
\]

Thus, we conclude
Theorem 4.3.2: A Finsler space $F^n = (M^n, L)$ and the Finsler space $F^*n = (M^n, L^*)$ whose metric $L^*$ is obtained from the generalized Kropina conformal change of the metric $L$ are projectively related if and only if the covariant vector $A_i$ given by (4.3.8) vanishes identically.

4.4 Hypersurfaces of a Finsler Space $F^*n$

Consider Finslerian hypersurfaces $F^{n-1} = (M^{n-1}, L(u,v))$ of $F^n$ and $F^{*(n-1)} = (M^{n-1}, L^*(u,v))$ of $F^*n$. The functions $B^i_\alpha(u)$ may be considered as the components of $n - 1$ linearly independent vectors tangent to $F^{n-1}$. Since $N^i$ is the unit normal vector at a point $u^\alpha$ of $F^{n-1}$, the unit normal vector $N^{*i}(u,v)$ of $F^{*(n-1)}$ and the inverse projection factor $B^*\alpha_i$ along $F^{*(n-1)}$ are uniquely determined by

\begin{align}
(a) & \ g^*_{ij}B^i_\alpha N^{*j} = 0, \quad (b) \ g^*_{ij}N^{*i}N^{*j} = 1. \\
\text{and} & \\
(4.4.2) & \ B^*\alpha_i = g^{*\alpha\beta}g^*_{ij}B^j_\beta.
\end{align}

where $g^{*\alpha\beta}$ is the inverse of metric tensor $g^*_{\alpha\beta}$ of $F^{*(n-1)}$.

From (4.4.1) and (4.4.2), we have

\begin{align}
(a) & \ B^i_\alpha B^*_{i\beta} = \delta^\beta_\alpha, \quad (b) \ B^i_\alpha N^{*i} = 0, \quad (c) \ N^{*i}B^*_{i\alpha} = 0, \quad (d) \ N^{*i}N^{*i} = 1.
\end{align}

From (4.4.3), we have

\begin{align}
(4.4.4) & \ B^i_\alpha B^*_{j\alpha} + N^{*i}N^{*j} = \delta^i_j.
\end{align}

Transvection of (1.14.2)(a) with $v^\alpha$ gives

\begin{align}
(4.4.5) & \ y_jN^j = 0.
\end{align}
Transvecting (4.2.4) with $N^i N^j$ and using (1.14.2)(b) and (4.4.5), we have

$$(4.4.6) \quad g^*_{ij} N^i N^j = e^{2\sigma(x)}(n + 1)\frac{L^{2n}}{\beta^{2n}} + e^{2\sigma(x)}n(2n + 1)\frac{L^{2(n+1)}}{\beta^{2(n+1)}}(b_i N^i)^2.$$ 

which implies that \[ \sqrt{e^{2\sigma(x)}(n + 1)\frac{L^{2n}}{\beta^{2n}} + e^{2\sigma(x)}n(2n + 1)\frac{L^{2(n+1)}}{\beta^{2(n+1)}}(b_i N^i)^2} \] is a unit vector.

Transvecting (4.2.4) with $B^i_\alpha N^j$ and using (1.14.2)(a) and (4.4.5), we have

$$(4.4.7) \quad g^*_{ij} B^i_\alpha N^j = (b_j N^j)\left\{ e^{2\sigma(x)}n(2n + 1)\frac{L^{2(n+1)}}{\beta^{2(n+1)}}(b_i B^i_\alpha) - e^{2\sigma(x)}2n(2n + 1)\frac{L^{2(n+1)}}{\beta^{2n+1}}l_i B^i_\alpha \right\},$$

which shows that $N^j$ is normal to $F^*(n-1)$ iff

$$(4.4.8) \quad (b_j N^j)\left\{ e^{2\sigma(x)}n(2n + 1)\frac{L^{2(n+1)}}{\beta^{2(n+1)}}(b_i B^i_\alpha) - e^{2\sigma(x)}2n(2n + 1)\frac{L^{2(n+1)}}{\beta^{2n+1}}l_i B^i_\alpha \right\} = 0.$$ 

This implies that either $e^{2\sigma(x)}n(2n+1)\frac{L^{2(n+1)}}{\beta^{2(n+1)}}(b_i B^i_\alpha) - e^{2\sigma(x)}2n(2n + 1)\frac{L^{2(n+1)}}{\beta^{2n+1}}l_i B^i_\alpha = 0$ or $b_j N^j = 0$.

Transvecting $e^{2\sigma(x)}n(2n + 1)\frac{L^{2(n+1)}}{\beta^{2(n+1)}}(b_i B^i_\alpha) - e^{2\sigma(x)}2n(2n + 1)\frac{L^{2(n+1)}}{\beta^{2n+1}}l_i B^i_\alpha = 0$ with $v^\alpha$ and using $y^i = B^i_\alpha v^\alpha$, we have

$$(4.4.9) \quad e^{2\sigma(x)}n(2n + 1)\frac{L^{2(n+1)}}{\beta^{2(n+1)}}b_i y^i - e^{2\sigma(x)}2n(2n + 1)\frac{L^{2(n+1)}}{\beta^{2n+1}}l_i y^i = 0,$$

which gives

$$(4.4.10) \quad -n e^{2\sigma(x)}\frac{L^{2(n+1)}}{\beta^{2n+1}} = 0,$$

which is not possible. Hence we have

$$(4.4.11) \quad b_j N^j = 0.$$
Thus, the vector $N^j$ is normal to $F^*(n-1)$ if and only if $b_j$ is tangent to $F^{n-1}$. From (4.4.5), (4.4.7) and (4.4.10), we can say that
\[ N^i = \frac{N^i}{\sqrt{e^{2\sigma(x)(n+1)}L^{2n}}}, \]
is a unit normal vector of $F^*(n-1)$. Therefore, in view of (4.4.1), we have
\[
(4.4.12) \quad N^*\!^i = N^i \sqrt{e^{2\sigma(x)(n+1)}L^{2n}/\beta^{2n}}.
\]
Transvecting (4.2.4) with $N^*\!^j$ and using (4.4.5), (4.4.11) and (4.4.12), we have
\[
(4.4.13) \quad N^*_i = g^*_{ij}N^*\!^j = \sqrt{e^{2\sigma(x)(n+1)}L^{2n}/\beta^{2n}} N_i.
\]

Hence, we conclude

**Theorem 4.4.1:** Let $F^*^n$ be the Finsler space obtained from $F^n$ by a generalized Kropina conformal change. The vector $b_i$ is tangential to the hypersurface $F^{n-1}$ of the Finsler space $F^n$ if and only if every vector normal to $F^{n-1}$ is also normal to the hypersurface $F^*(n-1)$ of $F^*^n$.

Transvecting (4.2.4) with $B^i\!^\alpha N^*\!^j \!^\alpha N^k$ and using (4.4.5), (4.4.11) and (4.4.12), we have
\[
(4.4.14) \quad M^*\!^\alpha = C_{ijk}B^i\!^\alpha N^j \!^k - \frac{n}{\beta}b_iB^i\!^\alpha + \frac{n}{L}l_iB^i\!^\alpha,
\]
which implies
\[
(4.4.15) \quad M^*\!^\alpha = M\!^\alpha - n\left(\frac{b_i}{\beta} - \frac{l_i}{L}\right)B^i\!^\alpha.
\]
Transvecting $l_i = \dot{\partial}_i L$ with $y^i$, we have
\[
(4.4.16) \quad l_iy^i = L.
\]
In view of (4.4.16) and $\beta = b_iy^i$, (4.4.15) gives
\[
(4.4.17) \quad M^*\!^\alpha = M\!^\alpha.
\]
Hence, we have
**Theorem 4.4.2:** The covariant vector $M_\alpha$ under the generalized Kropina conformal change is invariant.

**Definition 4.4.3.** Let us consider two Finsler spaces $F^n = (M^n, L)$ and $F^{*n} = (M^n, L^*)$ on the same manifold $M^n$. Then the transformation from $F^n$ to $F^{*n}$ which maps every geodesic of $F^n$ to some geodesic of $F^{*n}$ is known as projective change and the Finsler spaces $F^n$ and $F^{*n}$ are called projectively related Finsler spaces.

Suppose that the generalized Kropina conformal change of metric is projective such change of metric is called as projective generalized Kropina conformal change of metric.

From (4.2.9) and (4.3.2), we have

\[(4.4.18) \quad D_{0j}^i = P_j y^i + P\delta_j^i.\]

Transvecting (4.4.14) with $N_i B_{\alpha}^j$ and using (1.14.5)(b), (1.14.5)(e) and (4.4.5), we have

\[(4.4.19) \quad N_i D_{0j}^i B_{\alpha}^j = 0.\]

If each geodesic of the hypersurface $F^{n-1}$ with respect to the induced metric is also a geodesic of a Finsler space $F^n$ then $F^{n-1}$ is known as totally geodesic hypersurface [49]. A totally geodesic hypersurface is characterised by $H_\alpha = 0$.

The normal curvature vector $H^*_\alpha$ on $F^{*(n-1)}$ is given by

\[(4.4.20) \quad H^*_\alpha = N_i (B_{0\alpha}^i + G^{*i}_j B_{\alpha}^j),\]

Using (4.2.9), (4.4.13) in (4.4.16), we have

\[(4.4.21) \quad H^*_\alpha = H_\alpha \sqrt{e^{2\sigma(x)}(n+1)} \frac{L_{2n}^2}{\beta^{2n}} + N_i D_{0j}^i B_{\alpha}^j \sqrt{e^{2\sigma(x)}(n+1)} \frac{L_{2n}^2}{\beta^{2n}}.\]
From (4.4.15) and (4.4.17), we have

\[(4.4.22)\]
\[H^*_{\alpha} = e^{\sigma(x)} \frac{L^n}{\beta n} \sqrt{(n + 1)} H_{\alpha},\]

which in view of (4.1.1) gives

\[(4.4.23)\]
\[H^*_{\alpha} = \sqrt{(n + 1)} \frac{L_*}{L} L^* H_{\alpha}.\]

Since \(\sqrt{(n + 1)} \frac{L_*}{L} \neq 0\), the vanishing of \(H_{\alpha}\) implies and implied by the vanishing of \(H^*_{\alpha}\).

This leads to:

**Theorem 4.4.4:** Let \(F^*n\) be the Finsler space obtained from the Finsler space \(F^n(n > 3)\) by a projective generalized Kropina conformal change then the hypersurface \(F^*(n-1)\) of \(F^*n\) is totally geodesic if and only if the hypersurface \(F^{n-1}\) of \(F^n\) is totally geodesic.

The second fundamental \(h\)-tensor \(H^*_{\alpha\beta}\) for the hypersurface \(F^*(n-1)\) of the Finsler space \(F^*n\) is given by

\[(4.4.24)\]
\[H^*_{\alpha\beta} = N^*_{i} \left( B_{i\beta}^i + F^*_{jk} B_j^i B_k^i \right) + M^*_{\alpha} H^*_{\beta},\]

where \(H^*_{\beta}\) is the normal curvature for the hypersurface \(F^*(n-1)\) of the Finsler space \(F^*n\) and given as

\[(4.4.25)\]
\[H^*_{\alpha} = \sqrt{p} \left( H_{\alpha} + N_i D_j^i B_j^i \right).\]

In view of (4.2.8), (4.4.13), (4.4.17), (4.4.19) and (4.4.23), (4.4.24) reduces to

\[(4.4.26)\]
\[H^*_{\alpha\beta} = \sqrt{p} \left( N_i \left( B_{i\beta}^j + F^*_{jk} B_j^i B_k^i \right) + M_{\alpha} H_{\beta} \right),\]

which in view of (1.14.6), gives

\[(4.4.27)\]
\[H^*_{\alpha\beta} = \sqrt{p} H_{\alpha\beta},\]
Thus in view of Lemma 1.13.2, we have:

**Theorem 4.4.5:** Let $F^*n$ be the Finsler space obtained from the Finsler space by a projective generalized Kropina conformal change then the hypersurface $F^{*(n-1)}$ of $F^*n$ is a hyperplane of second kind if and only if the hypersurface $F^{n-1}$ of $F^n$ is a hyperplane of second kind.

Transvecting (4.2.4) with $B^i_\alpha B^j_\beta N^k$ and using (1.14.2)(a), (4.4.5) and (4.4.11), we have

\[ M^*_\alpha\beta = \sqrt{\rho} C_{ijk} B^i_\alpha B^j_\beta N^k, \]

which in view of (1.14.10), reduces to

\[ M^*_\alpha\beta = \sqrt{\rho} M_{\alpha\beta}, \]

Thus in view of Lemma 1.13.3, we have:

**Theorem 4.4.6:** Let $F^*n$ be the Finsler space obtained from the Finsler space by a projective generalized Kropina conformal change then the hypersurface $F^{*(n-1)}$ of $F^*n$ is a hyperplane of the third kind if and only if the hypersurface $F^{n-1}$ of $F^n$ is a hyperplane of the third kind.

### 4.5 Hypersurfaces of Some Special Finsler Spaces

Consider a projective generalized Kropina conformal change. If there exists a projective change $L \rightarrow L^*$ of a Finsler space $F^n = (M^n, L)$ such that the Finsler space $F^{*n} = (M^n, L^*)$ is a locally Minkowskian space then $F^n$ is called projectively flat space.
In 1986, Yamada[229] proved that if $F^n$ is projectively flat then the totally geodesic hypersurface $F^{n-1}$ of $F^n$ is also projectively flat.

In 1980, Matsumoto[102] showed that a Finsler space $F^n(n > 2)$ is projectively flat iff Weyl torsion tensor $W^i_{jk}$ and Douglas tensor $D^i_{jkh}$ vanish, i.e.

\[(4.5.1) \quad (a) \ W^i_{jk} = 0, \quad (b) \ D^i_{jkh} = 0.\]

Under the projective change, Weyl torsion tensor $W^i_{jk}$ and Douglas tensor $D^i_{jkh}$ are invariant, i.e.

\[(4.5.2) \quad (a) \ W^*_{jk} = W^i_{jk}, \quad (b) \ D^*_{jkh} = D^i_{jkh}.\]

From theorem 4.4.2 and equations (4.5.1) and (4.5.2), we conclude

**Theorem 4.5.1:** Let $F^{*n}$ be the Finsler space obtained from the Finsler space $F^n(n > 3)$ by a projective generalized Kropina conformal change and $F^n$ be projectively flat. If $F^{*(n-1)}$ and $F^{n-1}$ are the hypersurfaces of these spaces and $F^{n-1}$ is totally geodesic then $F^{*(n-1)}$ is projectively flat.

A Finsler space $F^n$ is a Landsberg space if $(v)hv$-torsion tensor $P^i_{jk}$ vanishes identically.

The $(v)hv$-torsion tensor $P^i_{jk}$ for the Finsler space $F^n$ is given by

\[(4.5.3) \quad P^i_{jk} = G^i_{jk} - F^i_{jk}.\]

The $(v)hv$-torsion tensor $P^\alpha_{\beta\gamma}$ for the hypersurface $F^{n-1}$ of the Finsler space $F^n$ is given by

\[(4.5.4) \quad P^\alpha_{\beta\gamma} = 2H_{\gamma}M_{\delta\beta}\delta^\alpha + B^\alpha_k \left[ P^i_{jk} B^j_{\beta} H^{k}_{\gamma} + C^i_{jk} \left( B^j_{\beta} H_{\gamma} + B^j_{\gamma} H_{\beta} \right) N^k \right].\]

For the hyperplane of second kind, we have

\[(4.5.5) \quad H_{\alpha\beta} = 0.\]
which in view of (1.14.9)(a) gives

(4.5.6) \[ H_{\alpha} = 0. \]

Using (4.5.6) in (4.5.4), \((v)hv\)-torsion tensor \(P_{\beta\gamma}^\alpha\) for the hyperplane of second kind is given by

(4.5.7) \[ P_{\beta\gamma}^\alpha = B^i_{\beta} K_{\beta\gamma}^i, \]

where

(4.5.8) \[ K_{\beta\gamma}^i = P_{jk}^i B^j_{\beta} B^k_{\gamma}. \]

In view of (1.14.5), (4.5.8) gives

(4.5.9) \[ K_{\beta\gamma}^i = B^i_{\delta} P^\delta_{\beta\gamma} + N^i_{j} N^j_{k} K_{\beta\gamma}^i. \]

For the hyperplane \(F^{*(n-1)}\) of the second kind of the Finsler space \(F^{*n}\), the \((v)hv\)-torsion tensor \(P^*_{\beta\gamma}^\alpha\) is given by

(4.5.10) \[ P^*_{\beta\gamma}^\alpha = B^*_{i} K^*_{\beta\gamma}^i, \]

where

(4.5.11) \[ K^*_{\beta\gamma}^i = P^*_{jk}^i B^j_{\beta} B^k_{\gamma}. \]

In view of theorem 4.2.3, (4.5.3) and (4.5.8), (4.5.11) gives

(4.5.12) \[ K^*_{\beta\gamma}^i = K_{\beta\gamma}^i. \]

Using (4.4.3)(a), (4.4.3)(c), (4.4.12), (4.5.7), (4.5.9) and (4.5.12) in (4.5.10), we have

(4.5.13) \[ P^*_{\beta\gamma}^\alpha = P^\alpha_{\beta\gamma}, \]

This leads to:
Theorem 4.5.2: Let the vector $b_i$ be parallel with respect to the Cartan connection of $F^n$ and $F^{*n}$ be the Finsler space obtained from the Finsler space $F^n$ by a generalized Kropina conformal change then the hyperplane $F^{*(n-1)}$ of second kind of $F^{*n}$ is Landsberg space if and only if the hyperplane $F^{n-1}$ of second kind of $F^n$ is Landsberg space.

In 1985, Matsumoto[103] showed that if a Finsler space $F^n$ is Landsberg then the hyperplane of the first kind is also Landsberg.

Thus in view of Theorem (4.4.2), Theorem (4.5.2) and equations (4.5.5) and (4.5.6), we have

Theorem 4.5.3: Let the vector $b_i$ be parallel with respect to the Cartan connection of $F^n$ and $F^{*n}$ be the Finsler space obtained from the Finsler space $F^n$ by a projective generalized Kropina conformal change and $F^n$ be Landsberg. If $F^{*(n-1)}$ and $F^{n-1}$ are the hypersurfaces of these spaces and $F^{n-1}$ is a hyperplane of first kind then $F^{*(n-1)}$ is also Landsberg space.

A Finsler space $F^n$ is a Berwald space if the Berwald connection coefficients $G_{jk}^i$ are functions of position only.

For the hyperplane of second kind, Berwald connection coefficients $G_{jk}^i$ are given by [103]

(4.5.14) \[ G_{jk}^i = B_{ik}^j A_{jk}^i, \]

where

(4.5.15) \[ A_{jk}^i = B_{ik}^j + G_{jk}^i B_{ik}^j. \]

Using (1.14.5)(d) in (4.5.14), we have

(4.5.16) \[ A_{jk}^i = B_{ik}^j G_{jk}^i + N_{ik} A_{jk}^i. \]
For the hyperplane $F^{*(n-1)}$ of the second kind of the Finsler space $F^{*n}$, Berwald connection coefficients $G^{\alpha}_{\beta\gamma}$ are given by

(4.5.17) \[ G^{\alpha}_{\beta\gamma} = B^{\alpha}_{i} A^{i}_{\beta\gamma}, \]

where

(4.5.18) \[ A^{i}_{\beta\gamma} = B^{i}_{\beta\gamma} + G^{i}_{jk} B^{j}_{\gamma} B^{k}_{\beta}. \]

In view of theorem 4.2.1 and (4.5.15), (4.5.18) reduces to

(4.5.19) \[ A^{i}_{\beta\gamma} = A^{i}_{\beta\gamma}. \]

Using (4.5.19) in (4.5.17), we have

(4.5.20) \[ G^{\alpha}_{\beta\gamma} = B^{\alpha}_{i} A^{i}_{\beta\gamma}. \]

Using (4.4.3)(a), (4.4.3)(c), (4.4.12) and (4.5.16) in (5.4.20), we have

(4.5.21) \[ G^{\alpha}_{\beta\gamma} = G^{\alpha}_{\beta\gamma}. \]

Thus, we have:

**Theorem 4.5.4:** Let the vector $b_{i}$ be parallel with respect to the Cartan connection of $F^{n}$ and $F^{*n}$ be the Finsler space obtained from the Finsler space $F^{n}$ by a generalized Kropina conformal change then the hyperplane $F^{*(n-1)}$ of the second kind of $F^{*n}$ is a Berwald space if and only if the hyperplane $F^{n-1}$ of the second kind of $F^{n}$ is a Berwald space.

If a Finsler space $F^{n}$ is Berwald then the hyperplane of the first kind is also Berwald space[229].

Thus in view of Theorem (4.4.2), Theorem (4.5.4) and equations (4.5.5) and (4.5.6), we have

**Theorem 4.5.5:** Let the vector $b_{i}$ be parallel with respect to the Cartan connection of $F^{n}$ and $F^{*n}$ be the Finsler space obtained from the Finsler
space $F^n$ by a projective generalized Kropina conformal change and $F^n$ be Berwald. If $F^{* (n-1)}$ and $F^{n-1}$ are the hypersurfaces of these spaces and $F^{n-1}$ is a hypersurface of first kind then $F^{* (n-1)}$ is also Berwald space.

Since the $(v)h$-torsion tensor $R^i_{jk}$ is given by

\begin{equation}
R^i_{jk} = \frac{\delta G^i_j}{\partial x^k} - \frac{\delta G^i_k}{\partial x^j}.
\end{equation}

where

\begin{equation}
\delta \frac{\partial}{\partial x^k} = \frac{\partial}{\partial x^k} - G^r_k \frac{\partial}{\partial x^r}.
\end{equation}

In view of theorem (4.2.3), we have

\begin{equation}
R^s^i_{jk} = R^i_{jk}.
\end{equation}

Thus the $(v)h$-torsion tensor for hyperplane of second kind is given by

\begin{equation}
R^\alpha_{\beta \gamma} = R^i_{jk} B^\alpha_i B^\beta_j B^k_{\gamma}.
\end{equation}

In 1986, Matsumoto[104] showed that a Finsler space $F^n$ is locally Minkowskian if and only if it is a Berwald space satisfying

\begin{equation}
R^i_{jk} = 0.
\end{equation}

Thus from (4.5.24), (4.5.25), (4.5.26) and theorem 4.5.4, we have

**Theorem 4.5.6:** Let the vector $b_i$ be parallel with respect to the Cartan connection of $F^n$ and $F^{*n}$ be the Finsler space obtained from the Finsler space $F^n$ by a generalized Kropina conformal change then the hyperplane $F^{* (n-1)}$ of second kind of $F^{*n}$ is locally Minkowskian space if and only if the hyperplane $F^{n-1}$ of second kind of $F^n$ is locally Minkowskian space.

From (4.5.5), (4.5.6) and theorem 4.5.6, we have:
Theorem 4.5.7: Let the vector $b_i$ be parallel with respect to the Cartan connection of $F^n$ and $F^*n$ be the Finsler space obtained from the Finsler space $F^n$ by a projective generalized Kropina conformal change and $F^n$ be locally Minkowskian. If $F^{*(n-1)}$ and $F^{n-1}$ are the hypersurfaces of these spaces and $F^{n-1}$ is a hypersurface of first kind then $F^{*(n-1)}$ is also locally Minkowskian space.