Chapter 1

Introduction

1.1 Development of Finsler geometry

From a historical point of view, Finsler geometry is originated in the famous dissertation of Paul Finsler [116] in 1918. However, the fundamental concept of a Finsler geometry may be traced back to the famous lecture of Riemann at Göttingen in 1854. He discussed the various possibilities by means of which an $n$-dimensional manifold may be equipped with a metric and paid special attention to a metric defined by the positive square root of a positive definite quadratic differential form. In this way, he laid down the foundation of Riemannian geometry. Along with this he also discussed that the positive fourth root of a fourth order differential form might serve as a metric function and suggested a possibility of studying more general geometry other than the Riemannian geometry, but he was not hopeful about the geometrical interpretation of the results in such spaces. It is clear from his following comments:

*Investigation of this more general class would actually require no essential different principles, but it would be rather time*
consuming [zeitraubend, in the original German] and throw relatively little new light on the study of space especially since the results cannot be expressed geometrically.

Due to the RIEMANN’s above comment, no mathematician went in the direction of this field for more than sixty years. In 1918, a young German geometer PAUL FINSLER [116] studied this geometry systematically in his thesis written under the guidance of CARATHEÓDARY. PAUL FINSLER succeeded in developing the foundations for the theory of such geometry. This is the reason why the name Finsler space was accepted for the general class of such spaces first defined by RIEMANN. Later on several mathematicians involved themselves in the development of Finsler geometry. The history of development of Finsler geometry can be divided into four periods.

The first period of the history of Finsler geometry began in 1904, when three geometers J. H. TAYLOR, J. L. SYNGE and L. BERWALD simultaneously started the work in this field. A special parallelism was introduced by TAYLOR and SYNGE while the concept of connection in the theory of Finsler spaces was given by BERWALD. BERWALD developed a theory with particular reference to the theory of curvature in which the Ricci’s lemma does not hold good.

The beginning of second period of the development of Finsler geometry lies in 1934, when the monograph of E. CARTAN[25, 26] was published on the topic Finsler Geometry. In this monograph, he introduced a system of postulates to give uniquely a Finsler connection from the fundamental function. Also, he showed that it was possible to define connection coefficients and hence covariant derivatives in such a way that Ricci’s lemma will remain satisfied. On this basis E. CARTAN [25, 26] developed the the-
ory of curvature and torsion as well as he discussed the theory of Finsler spaces by this approach. After this, many mathematicians including E. T. Davies [27, 28], S. Golab [198], H. Hombu [42], O. Varga [115] and V. V. Wagner [239, 240] developed Finsler geometry along Cartan’s approach. A special Finsler space was introduced by G. Randers to study the theory of gravitation and electromagnetism. As a result of this many physicists paid attention to Finsler geometry.

In 1951, a young German geometer H. Rund [51] introduced a new process of parallelism from the standpoint of Minkowskian geometry. This is the beginning of the third period. It was emphasized that the local metric of a Finsler space is a Minkowskian one. Cartan introduced parallelism from the standpoint of local Euclidean geometry. E. T. Davis (Math. Rev.) and A. Deicke (Zentralblatt.) observed that both the parallelism introduced by Cartan and Rund were same. Later on several mathematicians such as W. Barthel [241], A. Deicke [2], D. Lauwitz [24] and R. Sulanke [193] worked on Finsler geometry along the Rund’s approach.

The fourth period began in 1963, when H. Akabar Zadeh [34] developed the modern theory of Finsler space by introducing a theory of connections in fibre bundles. During this period, Finsler geometry came under the influence of Topology. This modernization led to a global definition of connections in a Finsler space and re-examine the Cartan’s system of axioms. In 1970, Makoto Matsumoto organized a symposium on The Models of Finsler Spaces. Several mathematicians especially, S. S. Chern, D. Bao [21, 22], Z. Shen [244], R. L. Bryant, D. Burogo and S. Ivanow further worked in this field on the above line.
Contrary to Riemann’s opinion, Finsler geometry has abundance of applications in various branches of physics and biology such as thermodynamics, optics, evolution, ecology, information and developmental biology etc. R. S. Ingarden applied this theory to the theory of electron microscope. P. L. Antonelli used this in the study of biology.

Many Indian mathematicians also contributed significantly towards the development of Finsler geometry. Some of them are R. S. Mishra, R. N. Sen, U. P. Singh [231]-[236], B. B. Sinha [7], H. D. Pande [35], R. B. Misra [172]-[178], R. S. Sinha [191, 192], R. S. D. Dubey, P. N. Pandey [126]-[170], B. N. Prasad [9]-[11], C. S. Bagewadi [13]-[18], S. K. Narasimhamurthy [202]-[209], H. S. Shukla, T. N. Pandey [224] and M. K. Gupta [78]-[80].

Now, we shall discuss some preliminary concepts which are necessary for the discussion of the subsequent chapters of the thesis.

1.2 Differentiable Manifold

Topological Manifolds

Intuitively, a manifold is a generalization of curves and surfaces to higher dimensions. The concept of a manifold is central to many parts of geometry and modern mathematical physics because it allows more complicated structures to be described and understood in terms of the relatively well-understood properties of Euclidean space.

Let M be a topological space and φ be a homeomorphism of an open set $U \subset M$ onto an open subset of $\mathbb{R}^n$. Then $U$ is called a coordinate neighbourhood and φ is called a coordinate map. The pair $(U, \phi)$ is called coordinate chart. An $n$-dimensional topological manifold $M$ is a
second countable, Hausdorff and locally Euclidean space of dimension $n$.

**Compatiable Charts**

An atlas on a locally Euclidean space $M$ is a collection $U = (U_\alpha, \phi_\alpha)$ of pairwise $C^\infty$ - compatiable charts [73] that covers $M$. Two charts $(U, \phi : U \to R^n)$ and $(V, \psi : V \to R^n)$ of a topological manifold are $C^\infty$ - compatiable if the two transition functions $\phi \circ \psi^{-1} : \psi(U \cap V) \mapsto \phi(U \cap V)$ and $\psi \circ \phi^{-1} : \phi(U \cap V) \mapsto \psi(U \cap V)$ are $C^\infty$.

**Differentiable Manifolds**

A differentiable manifold is a topological manifold equipped with an equivalence class of atlases whose transition maps are all differentiable.

1.3 **Finsler Space and Metric tensor**

Let $R$ be a region of an $n$-dimensional space $M^n$, which is completely covered by a coordinate system such that each point $P$ of $R$ is represented by $n$ tuples $(x^i : i = 1, 2, 3 \ldots n)$ of real numbers, called coordinates of $P$.

Let $C : x^i = x^i(t)$ be a curve of class $C^1$ and the components of tangent vector at the point $P(x^i)$ be denoted by $y^i$, i.e. $y^i = dx^i/dt = \dot{x}^i$. The combination $(x^i, y^i)$ is called as line-element of the curve $C$ with centre at $P$. For convenience, we shall denote the line element $(x^i, y^i)$ by $(x, y)$. In the line element $(x, y)$, $x$ and $y$ are called positional and directional coordinates respectively.

Let us consider a function $L(x, y)$ defined for all line-elements in $R$ such that:

(a) The function $L(x, y)$ is positively homogeneous of degree one in $y^i$, i.e.

$$L(x, ky) = kL(x, y) \quad \text{for} \quad k > 0.$$
(b) The function $L(x, y)$ is positive unless all $y^i$ vanish simultaneously, i.e.
\[ L(x, y) > 0 \quad \text{with} \quad \sum (y^i)^2 \neq 0. \]

(c) The quadratic form
\[ \{ \dot{\partial}_i \dot{\partial}_j L^2(x, y) \} \eta^i \eta^j \]
is positive definite for all variables $\eta^i$.

With the help of this function, we define the *distance* $ds$ between two neighbouring points $P(x^i)$ and $Q(x^i + dx^i)$ by
\[ (1.3.1) \quad ds = L(x^i, dx^i). \]

The space equipped with such metric is called a *Finsler space* of dimension $n$ and is denoted by $F^n$. The function $L(x, y)$ is called the *fundamental function* or *metric function* of the Finsler space $F^n$.

Let us define $n^2$ quantities $g_{ij}$ of line-elements by
\[ (1.3.2) \quad g_{ij}(x, y) = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L^2(x, y). \]

These quantities constitute the components of a tensor of type (0,2), called *metric tensor*. From (1.3.1) it is obvious that the tensor $g_{ij}(x, y)$ is positively homogeneous of degree zero in $y^i$ and symmetric in $i$ and $j$. In view of Euler’s theorem on homogeneous function, we have
\[ (1.3.3) \quad g_{ij}(x, y) y^i y^j = L^2(x, y). \]

By virtue of Euler’s theorem on homogeneous functions, we have
\[ (1.3.4) \quad (a) \ y^i \dot{\partial}_i L = L, \quad (b) \ y^i \dot{\partial}_i \dot{\partial}_j L = 0, \quad (c) \ g_{ij} y^i y^j = L^2. \]

The *angular metric tensor* $h_{ij}$ is defined as
\[ (1.3.5) \quad h_{ij} = L \dot{\partial}_i \dot{\partial}_j L = g_{ij} - l_i l_j, \]
where $l_i = \dot{\partial}_i L$.

It is positively homogeneous of degree 0 and satisfies $h_{ij} y^i = 0$. 

1.4 Tangent space and its dual space

Suppose $F^n = (M, L(x, y))$ be a Finsler space over an $n$-dimensional manifold $M$ and $C : x^i = x^i(t)$ be a curve in $F^n$. Let us consider a change of local coordinates represented by

$$
(1.4.1) \quad \bar{x}^i = \bar{x}^i(x^i(t)).
$$

Then the components $y^i$ of the tangent vector to the curve $C$ are transformed according as

$$
(1.4.2) \quad \bar{y}^i = \left( \partial_j \bar{x}^i \right) y^j.
$$

In terms of differentials, the equation (1.4.2) may be written as

$$
(1.4.3) \quad d\bar{x}^i = \left( \partial_j \bar{x}^i \right) dx^j.
$$

A system of $n$-quantities $X^i$ whose transformation law under (1.4.1) is analogous to that of $y^i$ constitutes a contravariant vector attached to the point $P(x^i)$ of $F^n$. Such contravariant vectors attached to $P(x^i)$ constitute the elements of a vector space. This vector space is called the tangent space at $P(x^i)$ and is denoted by $T_n(P)$ or $T_n(x^i)$. The notion of tangent space is independent of the metric imposed on the underlying manifold $M^n$. However the introduction of a metric on $M^n$ gives rise to the measurement of length in each tangent space.

Corresponding to each contravariant vector $y^i$ of the tangent space $T_n(P)$, there is a covariant vector $y_i$ such that

$$
(1.4.4) \quad y_i = g_{ij}(x, y) y^j.
$$

The set of all such covariant vectors associated with the point $P$ of $F^n$ forms a vector space called the dual tangent space at $P$ and is denoted
by $T_n'(P)$. The metric function of the dual tangent space is Hamiltonian function $H(x^i, y_i)$ satisfying the three requisite conditions for a Finslerian metric as stated in Section 1.3. Analogous to the metric tensor $g_{ij}(x, y)$, we define a tensor $g^{ij}(x^k, y_k)$ as follows:

\begin{equation}
    g^{ij}(x^k, y_k) = \frac{1}{2} \bar{\partial}_i \bar{\partial}_j H^2(x^k, y_k),
\end{equation}

where $\bar{\partial}_i$ denotes the partial differentiation with respect to the covariant vector $y_i$. The quantities $g^{ij}(x^k, y_k)$ constitute the components of a tensor of type $(2, 0)$.

The two sets of quantities $g_{ij}$ and $g^{ij}$ defined by (1.3.2) and (1.4.5) are related by

\begin{equation}
    g_{ij} g^{jk} = \delta^k_i = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k. \end{cases}
\end{equation}

Substituting (1.3.2) in (1.4.4) and using Euler’s theorem on homogeneous functions, we get

\begin{equation}
    y_i := g_{ij} y^j = \frac{1}{2} \dot{\partial}_i L^2 = L \dot{\partial}_i L.
\end{equation}

The vector $y_i$ also satisfies the following relations:

\begin{equation}
    \begin{align*}
        a) & \quad y_i y^i = L^2, & b) & \quad g_{ij} = \dot{\partial}_i y_j = \dot{\partial}_j y_i.
    \end{align*}
\end{equation}

1.5 Magnitude of a vector and the notion of orthogonality

We may define the magnitude of a vector and the angle between two vectors with the help of the metric tensor $g_{ij}(x, y)$ in two different ways:
1. Let $\xi^i$ and $\eta^i$ be two vectors. Then the scalar $|\xi|$ given by

$$|\xi|^2 = g_{ij}(x^k, \xi^k) \xi^i \xi^j$$

is called the *magnitude* of the vector $\xi^i$. The ratio

$$\cos(\xi, \eta) = \frac{g_{ij}(x^k, \xi^k) \xi^i \eta^j}{L(x^k, \xi^k) L(x^k, \eta^k)}$$

is called the *Minkowskian cosine* corresponding to the (ordered) pair of directions $(\xi^k, \eta^k)$ [51]. It is obvious that the Minkowskian cosine is not symmetrical in $\xi^k$ and $\eta^k$. The vector $\eta^k$ is normal with respect to the vector $\xi^k$ if

$$g_{ij}(x^k, \xi^k) \xi^i \eta^j = 0.$$ 

2. The scalar

$$|\xi|^2 = g_{ij}(x^k, y^k) \xi^i \xi^j$$

is called the *square of the magnitude* of the vector $\xi^i$ for the pre-assigned direction $y^k$. The *cosine of the angle* between $\xi^k$ and $\eta^k$ is given by [59]

$$\cos(\xi, \eta) = \frac{g_{ij}(x^k, y^k) \xi^i \eta^j}{[g_{ij}(x^k, y^k) \xi^i \xi^j]^{1/2} [g_{ij}(x^k, y^k) \eta^i \eta^j]^{1/2}}.$$ 

The vector $\xi^k$ is orthogonal to $\eta^k$ if

$$g_{ij}(x^k, y^k) \xi^i \eta^j = 0.$$ 

The magnitude of the element of support $y^i$ is the fundamental function $L$. Therefore the unit vector in the direction of $y^i$ is $l^i = y^i L$. 

The length of a vector $\eta^i$ in $T_n(P)$ is $L(x^i, \eta^i)$ and in view of (1.3.2)(c), it may be expressed in terms of $g_{ij}$ as $\sqrt{g_{ij} \eta^i \eta^j}$. 


1.6 Cartan tensor and the generalized Christoffel symbols

The \( n^3 \) quantities
\[
C_{ijk} := \frac{1}{2} \dot{\partial}_k g_{ij} = \frac{1}{4} \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L^2,
\]
constitute a tensor of (0, 3)-type which is \((-1)p\)-homogeneous and symmetric in all its indices. This tensor is called the \((h)hv\)-torsion tensor or Cartan tensor. By virtue of Euler’s theorem on homogeneous functions, we have
\[
(1.6.2) \quad (a) \ C_{ijk} y^i = C_{jik} y^i = C_{kji} y^i = 0, \quad (b) \ C_{ijk}^j y^j = C_{kji}^i y^i = 0,
\]
where \( C_{jk}^i \) is the associate of Cartan tensor \( C_{ijk} \) defined by
\[
(1.6.3) \quad C_{jk}^i = g^{ih} C_{ijk}.
\]
The generalized Christoffel symbols of the first and the second kinds are defined by
\[
(1.6.4) \quad \begin{align*}
& a) \ \gamma_{ijk} = \frac{1}{2} (\partial_i g_{jk} + \partial_k g_{ij} - \partial_j g_{ik}), \\
& b) \ \gamma_{ik}^h = g^{hj} \gamma_{ijk}.
\end{align*}
\]

1.7 Finsler connection

A Finsler connection \( F\Gamma \) is determined by a triad \((F^i_{jk}, N^i_j, C^i_{jk})\), where
(i) \( N^i_j \) is nonlinear connection on \( TM \) satisfying the transformation rule
\[
(1.7.1) \quad \bar{N}^l_m = N^i_j \frac{\partial \bar{x}^l}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^m} + \frac{\partial x^i}{\partial \bar{x}^m} \frac{\partial^2 x^i}{\partial \bar{x}^m \partial \bar{x}^p} \bar{y}^p,
\]
(ii) \( F^i_{jk} \) and \( C^i_{jk} \) are collections of locally defined \((0)p\)-homogeneous functions \( F^i_{jk}, C^i_{jk} : \widetilde{TM} \to \mathbb{R} \) satisfying the transformation rules
\[
(1.7.2) \quad \bar{F}^l_{mp} = F^i_{jk} \frac{\partial x^l}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^m} \frac{\partial x^k}{\partial \bar{x}^p} + \frac{\partial x^i}{\partial \bar{x}^m} \frac{\partial^2 x^i}{\partial \bar{x}^m \partial \bar{x}^p}.
\]
and

\[ C_{mn} = C_{jk} \frac{\partial \bar{x}^l}{\partial x^i} \frac{\partial x^j}{\partial x^m} \frac{\partial x^k}{\partial \bar{x}^n}. \]

The \( h- \) and \( v- \)covariant derivatives of a tensor \( T^i_j \) with respect to \( F \Gamma \) are given by

\[ T^i_j|_k = \partial_k T^i_j - N^r_k \dot{\partial}_r T^i_j + T^r_j F^i_{rk} - T^i_r F^r_{jk}, \]

and

\[ T^i_j|_k = \dot{\partial}_k T^i_j + T^r_j C^i_{rk} - T^i_r C^r_{jk}. \]

The \( h- \) and \( v- \)deflection tensors of \( F \Gamma (F^i_{jk}, N^i_j, C^i_{jk}) \) are given by

\[ (a) \quad D^i_j := y^i|_j = y^r F^i_{rj} - N^i_j, \quad (b) \quad d^i_j := y^i|_j = \delta^i_j + y^r C^i_{rj}. \]

**Torsion and curvature tensors of a Finsler connection**

There are five types of torsion tensors and three types of curvature tensors of a Finsler connection \( F \Gamma (F^i_{jk}, N^i_j, C^i_{jk}) \). They are listed below

**Torsion tensors**

(i) \( (h)h \)-torsion tensor: \( T(T^i_{jk}) \), where

\[ T^i_{jk} := F^i_{jk} - F^i_{kj}, \]

(ii) \( (v)h \)-torsion tensor: \( R^1(R^i_{jk}) \), where

\[ R^i_{jk} := \mathfrak{M}_{(jk)} \{ \partial_k N^i_j - N^r_k \dot{\partial}_r N^i_j \}, \]

where \( \mathfrak{M}_{(jk)} \) denotes the interchange of the indices \( j \) & \( k \) and subtraction.

(iii) \( (h)hv \)-torsion tensor: \( C(C^i_{jk}) \).

(iv) \( (v)hv \)-torsion tensor: \( P^1(P^i_{jk}) \), where

\[ P^i_{jk} := \dot{\partial}_k N^i_j - F^i_{kj}, \]
(v) \( (v) \) \( v \)-torsion tensor: \( S^1(S^1_{jk}) \), where
\[
S^1_{jk} := \mathcal{M}_{(jk)} \{ C^i_{jk} \}.
\]

**Curvature tensors**

(i) \( h \)-curvature tensor: \( R^2(R^2_{hjk}) \), where
\[
R^2_{hjk} := \mathcal{M}_{(jk)} \{ \partial_k F^i_{hj} - N^r_k \dot{\partial}_r F^i_{hj} + F^r_{hj} F^i_{rk} \} + C^i_{hr} R^r_{jk},
\]
with \( F^i_{hj} = \dot{\partial}_k F^i_{hj} \).

(ii) \( hv \)-curvature tensor: \( P^2(P^2_{hjk}) \), where
\[
P^2_{hjk} := F^i_{hjk} - C^i_{hk|j} + C^i_{hr} P^r_{jk},
\]

(iii) \( v \)-curvature tensor: \( S^2(S^2_{hjk}) \), where
\[
S^2_{hjk} := \mathcal{M}_{(jk)} \{ \dot{\partial}_k C^i_{hj} + C^r_{hj} C^i_{rk} \}.
\]

**Ricci identities**

The Ricci identities for a Finsler connection \( F_G \) are given by
\[
X^i_{\mid j|k} - X^i_{\mid k|j} = X^r P^i_{rjk} - X^i_{\mid r} V^r_{jk} - X^i_{\mid r} P^r_{jk},
\]
\[
\mathcal{M}_{(jk)} \{ X^i_{\mid j|k} \} = X^r R^i_{rjk} - X^i_{\mid r} T^r_{jk} - X^i_{\mid r} R^r_{jk},
\]
\[
\mathcal{M}_{(jk)} \{ X^i_{\mid j|k} \} = X^r S^i_{rjk} - X^i_{\mid r} S^r_{jk}.
\]

**1.8 Cartan connection**

Cartan connection \( C_F(F^i_{jk}, G^i_j, C^i_{jk}) \) on a Finsler space \( F^n \) is a Finsler connection which is uniquely determined by the following system of axioms (cf. [? , 89, 104]):
(i) \( CT \) is \( h \)- as well as \( v \)-metrical, i.e. \( g_{ij\mid k} = 0 \) and \( g_{ij}\mid k = 0 \),

(ii) \( CT \) is \((h)h\)- as well as \((v)v\)-torsion free, i.e.

\[
T^i_{jk} := \mathcal{M}_{(jk)} \{ F^i_{jk} \} = 0 \quad \text{and} \quad S^i_{jk} := \mathcal{M}_{(jk)} \{ C^i_{jk} \} = 0,
\]

(iii) the \( h \)-deflection tensor of \( CT \) vanishes, i.e.

\[
D^i_j := y^i \mid_j = y^r F^i_{rj} - G^i_j = 0.
\]

The connection coefficients of \( CT(F^i_{jk}, G^i_j, C^i_{jk}) \) are given by

\[
F^i_{jk} = \frac{1}{2} g^{ir} \left[ \delta_k g_{jr} + \delta_j g_{kr} - \delta_r g_{jk} \right], \quad \text{where} \quad \delta_k = \partial_k - G^r_k \dot{\partial}_r.
\]

\[
G^i_j = \frac{1}{2} \dot{\partial}_j \gamma^i_{00}, \quad \text{where} \quad \gamma^i_{00} = \gamma^i_{rs} y^r y^s,
\]

\[
C^i_{jk} = \frac{1}{2} g^{ir} \left[ \dot{\partial}_k g_{jr} + \dot{\partial}_j g_{kr} - \dot{\partial}_r g_{jk} \right].
\]

The \( h \)- and \( v \)-covariant derivatives of a tensor \( T^i_j \) with respect to \( CT \) are given by

\[
T^i_{j\mid k} = \partial_k T^i_j - G^r_k \dot{\partial}_r T^i_j + T^r_j F^i_{rk} - T^i_r F^r_{jk},
\]

and

\[
T^i_j \mid_k = \dot{\partial}_k T^i_j + T^r_j C^i_{rk} - T^i_r C^r_{jk}.
\]

respectively. It can be easily shown that

\[
\begin{cases}
(a) \; l^i_{\mid j} = 0, & (b) \; g_{ij\mid k} = 0, & (c) \; h_{ij\mid k} = 0, \\
(d) \; L \mid_i = 0, & (e) \; l^i_{\mid j} = \frac{h_{ij}}{L}, & (f) \; L \mid_i = l_i, \\
(g) \; F^i_{jk} = F^i_{kj}, & (h) \; F^i_{jk} y^j = G^i_{jk} y^j = G^i_k, & (i) \; G^i_j y^j = 2G^i.
\end{cases}
\]

### 1.9 Berwald connection

A Finsler-connection \( B\Gamma(G^i_{jk}, G^i_j, 0) \) constructed from the Cartan connection \( CT \) is called the Berwald connection. It is uniquely determined
from the fundamental Finsler function $L$ by the following system of axioms (cf. [?, 225, 220]):

(i) $B\Gamma$ is $F$-metrical, i.e. $L_{|k} = 0$,

(ii) $B\Gamma$ is (h)h- as well as (v)hv-torsion free, i.e.

$$T^i_{jk} = 0 \quad P^i_{jk} \equiv \dot{\partial}_k G^i_j - F^i_{kj} = 0,$$

(iii) $C^i_{jk} = 0$,

(iv) the $h$-deflection tensor of $B\Gamma$ vanishes, i.e. $D^i_j = 0$.

The coefficients of $B\Gamma(G^i_{jk}, G^i_j, 0)$ are given by

$$\begin{cases}
G^i_{jk} = \dot{\partial}_k G^i_j, \\
G^i_j = \dot{\partial}_j G^i_i, \\
G^i = \frac{1}{2} \gamma^i_{00},
\end{cases}$$

$N^i_j$ being the nonlinear connection of $C\Gamma$.

It is worth-mentioning that the nonlinear connection of $B\Gamma$ and $C\Gamma$ are the same.

The non-vanishing torsion and curvature tensors of $B\Gamma$ are as follows:

(v)h-torsion tensor $R^i = (R^i_{jk})$,

$h$-curvature tensor $H^i_{jk} = \mathcal{M}_{(hk)} \{ \delta_h G^i_{jk} + G^r_{jk} G^i_{rh} \}$, where $G^i_{hjk} = \dot{\partial}_k G^i_{hj}$ are the components of a symmetric tensor and satisfy

$$G^i_{jkh} y^j = G^i_{kjh} y^j = G^i_{khj} y^j = 0. \quad (1.9.1)$$

Obviously, the $h$-curvature tensor $H^i_{jkh}$ is skew-symmetric in its last two lower indices and it is (0)$p$-homogeneous.

The Berwald-covariant derivative of a tensor $K^i_j$ with respect to $B\Gamma$ is given by

$$K^i_{j(k)} = \partial_h K^i_j - (\dot{\partial}_r K^i_j) G^r_h + K^r_{j} G^i_{rh} - K^i_r G^r_{jh}, \quad (1.9.2)$$
where \( G^r_k = G^r_{sk} y^s \).

The Berwald covariant differential operator commutes with the partial differential operator \( \hat{\partial}_k \) as

\[
(1.9.3) \quad \hat{\partial}_j T^i_{h(k)} - \left( \hat{\partial}_j T^i_{h} \right)_{(k)} = T^r_i G^i_{jkr} - T^i_r G^r_{jkh}.
\]

While it commutes with itself as

\[
(1.9.4) \quad T^i_{h(j)(k)} - T^i_{h(k)(j)} = T^r_i H^i_{rj;k} - T^i_r H^r_{hjk} - \left( \hat{\partial}_r T^i_h \right) H^r_{jk}.
\]

The components of Berwald- Ricci tensor are given by

\[
H^i_{jk} = H^i_{rjk}.
\]

The components of Berwald curvature tensor, Berwald deviation tensor, Berwald torsion tensor and Ricci tensor satisfy the following

\[
(1.9.5) \quad \begin{cases}
(a) \quad H^i_{j;k} = H^i_{hjk} y^h, & (b) \quad \hat{\partial}_h H^i_{jk} = H^i_{hjk}, \\
(c) \quad H^i_{k} = H^i_{j;k} y^j, & (d) \quad H^i_{jk} = -H^i_{k;j}, \\
(e) \quad H^i_{i} = (n - 1)H, & (f) \quad H_{jk} y^j = H_k, \\
g(3H^i_{jk} = \hat{\partial}_j H^i_{k} - \hat{\partial}_k H^i_{j}, & (h) \quad H_k y^k = (n - 1)H, \\
i) \quad H^r_{rjk} = H_{kj} - H_{jk}, & (j) \quad H^i_k y^k = 0, \\
k) \quad \hat{\partial}_j H_k = H_{jk}, & (l) \quad y^j \hat{\partial}_j H_{kh} = 0,
\end{cases}
\]

The curvature tensor of \( B\Gamma \) and the tensors derived from it satisfy the following (cf. [167, ?, 215]):

\[
(1.9.6) \quad \begin{cases}
(a) \quad \mathcal{G}_{(jkh)} \{ H^i_{jkh} \} = 0, \\
(b) \quad \mathcal{G}_{(ljk)} \{ H^i_{hjk;l} + H^r_{jk} G^i_{rlh} \} = 0, \\
(c) \quad \mathcal{G}_{(ljk)} \{ H^i_{jkl} \} = 0, \\
(d) \quad H^i_{k(l)} - H^i_{l(k)} + H^i_{k(l)} y^r = 0, \\
(e) \quad y_i H^i_{jk} = 0, \\
(f) \quad g_{ik} H^i_{mj} + y_i H^i_{mjk} = 0.
\end{cases}
\]
where $\Theta_{(jkh)}$ denotes the interchange of the indices $j, k \& h$ and addition. The Berwald connection coefficients and the Cartan connection coefficients are related as

\begin{equation}
G^i_{jk} = F^i_{jk} + C^i_{jk|0},
\end{equation}

where $C^i_{jk|0} = C^i_{jk|h} y^h$.

The Berwald connection $B\Gamma$ is neither $h$-metrical nor $v$-metrical as $g_{ij(k)} = -2C_{ijk|0}$ and $g_{ij|k} = \dot{\partial}_k g_{ij} = 2C_{ijk}$.

### 1.10 Projective transformation

A geodesic in $F^n$ is the path of extremum distance between two points. A curve $C : x^i = x^i(t)$ is a geodesic in $F^n$ if

\[ \frac{dy^i}{dt} + 2G^i = \tau y^i, \quad \tau = \frac{d^2 s/dt^2}{ds/dt}, \]

$s$-being the arc length of $C$.

Let $F^n = (M, L)$ and $\bar{F}^n = (M, \bar{L})$ be two Finsler spaces on the common underlying manifold $M$ of dimension $n$. A transformation $F^n \to \bar{F}^n$ is said to be a projective change or a projective transformation if it transforms every geodesic of $F^n$ to a geodesic of $\bar{F}^n$. The necessary and sufficient condition for the transformation $F^n \to \bar{F}^n$ to be a projective change is that

\begin{equation}
G^i = G^i + P y^i,
\end{equation}

$P$ being a $(1)p$-homogeneous scalar. The projective deviation tensor given by (cf. [72])

\begin{equation}
W^i_j = H^i_j - H \delta^i_j - \frac{1}{n+1} (\dot{\partial}_r H^r_j - \dot{\partial}_j H) y^i,
\end{equation}
remains invariant under a projective change. Some other tensors which are invariant under a projective change are listed below (cf. [102]):

**Weyl torsion tensor (Projective torsion tensor)**

\[(1.10.3) \quad W^i_{jk} = H^i_{jk} + \mathfrak{m}_{(jk)} \left\{ \frac{1}{n + 1} \left( y^i H_{jk} + \delta^i_j H_k \right) \right\}, \]

the **Weyl curvature tensor (Projective curvature tensor)**

\[(1.10.4) \quad W^h_{ijk} = H^h_{ijk} + \mathfrak{m}_{(jk)} \left\{ \frac{1}{n + 1} \left( y^h \partial_i H_{jk} + \delta^h_i H_j k + \delta^h_j \partial_i H_k \right) \right\} \]

and the **Douglas tensor**

\[(1.10.5) \quad D^h_{ijk} = G^h_{ijk} - \frac{1}{n + 1} \left[ y^h \partial_k G_{ij} + \delta^h_i G_{jk} + \delta^h_j G_{ik} + \delta^h_k G_{ij} \right]. \]

It is clear that \(W^i_{jkh}\) is \((0)p\)-homogeneous and skew-symmetric in its last two lower indices.

The projective deviation tensor \(W^i_h\), projective torsion tensor \(W^i_{kh}\) and the projective curvature tensor \(W^i_{jkh}\) satisfy the following:

\[
(1.10.6) \quad \begin{cases} 
(a) & W^i_{kh} = W^i_{jkh} y^j, \\
(b) & W^i_h = W^i_{kh} y^k, \\
(c) & W^i_h y^h = 0, \\
(d) & y^m \partial_m W^i_h = 2 W^i_h, \\
(e) & y^i W^i_h = 0, \\
(e) & W^i_{jkh} = \partial_j W^i_{kh}, \\
(e) & W^i_{kh} = \frac{1}{3} \left\{ \partial_k W^i_h - \partial_h W^i_k \right\}.
\end{cases}
\]
1.11 Special Finsler spaces

Riemannian space

A Finsler space $F^n = (M, L(x, y))$ is a Riemannian space if the fundamental Finsler function $L(x, y)$ is written as

$$L(x, y) = \sqrt{g_{ij}(x)y^iy^j}.$$  

The class of Riemannian spaces lies inside the class of Finsler spaces and is characterized by $C_{ijk} = 0$.

Locally Minkowskian space

A Finsler space with the fundamental function $L(x, y)$ is called locally Minkowskian space if there exists a coordinate system $(x^i)$ in which $L$ is a function of $y^i$ only.

A Finsler space is locally Minkowskian space if and only if

for $B\Gamma$:

$$H^h_{ijk} = 0 = G^h_{ijk},$$

for $C\Gamma$:

$$R^h_{ijk} = 0 = C^h_{hij|k}.$$

Landsberg spaces

A Finsler space $F^n = (M, L(x, y))$ is said to be a Landsberg space if Berwald connection on it is $h$-metrical, i.e. $g_{ij(k)} = 0$.

A Landsberg space is characterized by one of the following equivalent conditions:

(a) $P^h_{ijk} = 0$,
(b) $P^h_{ij} (= C^h_{ij|0}) = 0$,
(c) $C^h_{ij|k} = C^h_{ik|j}$,
(d) $y_iG^i_{jk} = 0$. 

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Berwald spaces

A Finsler space $F^n$ is said to be a Berwald space if the Berwald connection coefficients $G^i_{jk}$ are functions of position only.

A Berwald space is characterized by any one of the following equivalent conditions:

(a) $G^h_{ijk} = 0$,

(b) $C_{hij|k} = 0$.

Berwald spaces are also called affinely connected spaces.

It is obvious that the class of Landsberg spaces is broader than that of Berwald spaces.

Projectively flat Finsler space

If there exists a projective change $F^n \rightarrow \bar{F}^n$ such that the Finsler space $\bar{F}^n$ is a locally Minkowskian space, then $F^n$ is called projectively flat Finsler space.

Symmetric Finsler space

A Finsler space $F^n$ is called symmetric if

(1.11.1) $H^i_{jkh(m)} = 0, \ H^i_{jkh} \neq 0$.

Projectively symmetric Finsler space

A Finsler space $F^n$ is called projectively symmetric if

(1.11.2) $W^i_{jkh(m)} = 0, \ W^i_{jkh} \neq 0$. 
Finsler space with conservative Berwald curvature

A Finsler space \( F^n \) is called with conservative Berwald curvature if

\[ H^r_{jkh(r)} = 0, \quad H^i_{jkh} \neq 0. \]  

Finsler space with conservative projective curvature

A Finsler space \( F^n \) is called with conservative projective curvature if

\[ W^r_{jkh(r)} = 0, \quad W^i_{jkh} \neq 0. \]  

Finsler space with conservative normal projective curvature

A Finsler space \( F^n \) is called with conservative normal projective curvature if

\[ N^r_{jkh(r)} = 0, \quad N^i_{jkh} \neq 0. \]  

Finsler space with \((\alpha, \beta)\)-metric

A Finsler metric \( L(x, y) \) is called an \((\alpha, \beta)\)-metric if \( L \) is positively homogeneous function of degree one in two variables \( \alpha \) and \( \beta \), where

\[ \alpha(x, y) = \left\{ a_{ij}(x) y^i y^j \right\}^{1/2} \quad \text{and} \quad \beta(x, y) = b_i(x) y^i. \]

The examples of \((\alpha, \beta)\)-metric are

- Randers metric: \( L = \alpha + \beta \),
- Kropina metric: \( L = \alpha^2/\beta \),
- Matsumoto metric: \( L = \alpha^2/(\alpha - \beta) \).
1.12 Nonholonomic Frame for a Finsler space

Suppose $M$ be an $n$-dimensional Finsler manifold and $U$ be an open set of tangent bundle $TM$. Let the mapping

\[(1.12.1) \quad V_i : u \in U \rightarrow V_i(u) \in V_u TM, \quad i \in 1, 2, \ldots, n\]

be a vertical frame over $U$. If $V_i(u) \in V_j^i(u) \frac{\partial}{\partial y_j}|_u$, then $V_i^j(u)$ are the entries of an invertible matrix for all $u \in U$ and its inverse is denoted by $\overline{V}_j^i(u)$ such that

\[(1.12.2) \quad V_i^j\overline{V}_j^k = \delta^k_i, \quad \overline{V}_j^i V_k^j = \delta^k_i.\]

The frame $V_i^j$ is called as nonholonomic frame on $TM$.

1.13 Finslerian subspace

An $m$-dimensional subspace $M^m$ ($1 < m < n$) of the underlying manifold $M^n$ may be represented parametrically by the equations $x^i = x^i(u^\alpha)$, where $u^\alpha$ are the Gaussian coordinates on $M^m$ (Latin indices run from 1 to $n$, while Greek indices take values from 1 to $m$). We assume that the matrix of projection factors $B_{\alpha}^i = \frac{\partial x^i}{\partial u^\alpha}$ is of rank $m$. If the supporting element $y^i$ at a point $u = (u^\alpha)$ of $M^m$ is assumed to be tangent to $M^m$ then we may write $y^i = B_{\alpha}^i(u) v^\alpha$. Thus $v = (v^\alpha)$ may be supposed as the supporting element of $M^m$ at the point $u^\alpha$. Since the function $L(u,v) = L(x(u),y(u,v))$ gives rise to a Finsler metric on $M^m$, we get an $m$-dimensional Finsler space $F^m = (M^m, L(u,v))$.

A covariant vector $Y_i$ is said to be normal to $F^m$ if it satisfies

$$Y_i B_{\alpha}^i = 0.$$

These are $m$ equations for the determination of $n$ functions $Y_i$. Since the rank of the matrix of $B_{\alpha}^i = \frac{\partial x^i}{\partial u^\alpha}$ is $m$, We may thus choose
(n−m) linearly independent and mutually orthogonal unit vectors \( N_i \) \((a = m+1, \ldots, n)\) such that

\[(1.13.1) \quad N_i B^i_a = 0.\]

With respect to a given direction \( y^i \) in \( T_n(P) \), we may thus choose a set of normals satisfying

\[(1.13.2) \quad (a) \ g_{ij} B^i_a N^j_a = 0, \quad (b) \ g_{ij} N^i_a N^j_a = \delta_{ab}, \quad (c) \ N^i_a = g^{ij}_a N_j.\]

The inverse projection factors \( B^\alpha_i(u, v) \) of \( B_\alpha^i \) are defined as

\[(1.13.3) \quad B^\alpha_i = g^{\alpha\beta}_i g_{ij} B^j_\beta,\]

where \( g^{\alpha\beta} \) is the inverse of the metric tensor \( g_{\alpha\beta} \) of \( F^m \).

From (1.13.2) and (1.13.3), it follows that

\[(1.13.4) \quad (a) \ B^i_a B^\beta_j = \delta^{\beta}_a, \quad (c) \ N^i_a B^\alpha_i = 0, \quad (d) \ N^i_a N_j = \delta^i_j,\]

and further

\[(1.13.5) \quad B^i_a B^\alpha_j + N^i_a N^j_a = \delta^i_j.\]

For the induced Cartan connection \( ICT = (F^\alpha_\beta, G^\alpha_\beta, C^\alpha_\beta) \) on \( F^m \), the second fundamental h-tensor \( H_{\alpha\beta} \) and the normal curvature vector \( H_\alpha \) in a normal direction \( N^i_a \) are given by

\[(1.13.6) \quad H_{\alpha\beta} = N_a (B^i_\alpha B^j_\beta + F^i_\alpha B^j_\beta B^k_\gamma) + M^a_\alpha H_\beta,\]

and

\[(1.13.7) \quad H_\alpha = N_a (B^i_0 B^j_\alpha + G^i_\alpha B^j_\beta) ,\]

where \( M^a_\alpha = C^i_\alpha N^i_a N^j_a B^k_\alpha \), \( B^i_\alpha = \partial^2 x^i / \partial u^\alpha \partial u^\beta \) and \( B^i_0 = v^\alpha B^i_\alpha \).
Contracting $H_{\beta\alpha}$ by $v^\beta$, we get

\begin{equation}
(1.13.8) \quad H_{0\alpha} = (a) H_{\beta\alpha} v^\beta = (a) H_{\alpha}.
\end{equation}

The Gauss equation with respect to the induced Cartan connection of $F^m$ is written as

\begin{equation}
(1.13.9) \quad R_{\alpha\beta\gamma\delta} = R_{ijkl} B_i^i B_j^j B_k^k B_l^l + P_{ijkl} (B_l^k H_{\delta} - B_l^k H_{\gamma}) B_i^i B_j^j N_k^h (a) + S_{ijkl} B_i^i B_j^j N_k^h N_l^h H_{\gamma} H_{\delta} + (a) \left( H_{\alpha\gamma} H_{\beta\delta} - (a) H_{\alpha\delta} H_{\beta\gamma} \right),
\end{equation}

where $R_{ijkl}$, $P_{ijkl}$ and $S_{ijkl}$ are the $h$-curvature tensor, $hv$-curvature tensor and $v$-curvature tensor respectively.

If each geodesic of the subspace $F^m$ with respect to the induced metric is also a geodesic of the enveloping space $F^n$, then $F^m$ is called a totally geodesic subspace. MATSUMOTO [103] showed that

**Lemma 1.13.1:** A totally geodesic subspace is characterized by $(a) H_{\alpha} = 0$ or equivalently $(a) H_{0} = 0$.

### 1.14 Finslerian hypersurface

An $(n-1)$-dimensional subspace of an $n$-dimensional Finsler space $F^n$ is called the hypersurface $F^{n-1}$ of the Finsler space $F^n$. The metric tensor, angular metric tensor and Cartan tensor of $F^{n-1}$ are given by

\begin{equation}
(1.14.1) \quad g_{\alpha\beta} = g_{ij} B_i^i B_\beta^j, \quad h_{\alpha\beta} = h_{ij} B_i^i B_\beta^j, \quad C_{\alpha\beta\gamma} = C_{ijk} B_i^i B_\beta^j B_\gamma^k.
\end{equation}

At each point $u^\alpha$ of $F^{n-1}$, a unit normal vector $N^i(u,v)$ is defined by

\begin{equation}
(1.14.2) \quad (a) \ g_{ij} B_i^i N^j = 0, \quad (b) \ g_{ij} N^i N^j = 1.
\end{equation}

As $h_{ij} = g_{ij} - l_i l_j$, the equation (1.14.2) yields

\begin{equation}
(1.14.3) \quad (a) \ h_{ij} B_i^i N^j = 0, \quad (b) \ h_{ij} N^i N^j = 1.
\end{equation}
The inverse projection factor $B^\alpha_i(u,v)$ of $B^i_\alpha$ satisfies

(1.14.4) 

\[ B^\alpha_i B^\beta_i = \delta^\beta_\alpha, \]

From (1.14.2), (1.14.3) and (1.14.4), we have

(1.14.5) 

\[
\begin{cases}
(a) & B^\alpha_i N_i = 0, \\
(b) & N^i B^\alpha_i = 0, \\
(c) & N_i N_i = 1, \\
(d) & B^\alpha_i B^\beta_j + N^i N_j = \delta^\beta_j, \\
(e) & N_i = g_{ij} N^j.
\end{cases}
\]

The second fundamental $h$-tensor $H_{\alpha\beta}$ and the normal curvature vector $H_\alpha$ for the induced Cartan connection $IC\Gamma = (F^\alpha_\beta, G^\alpha_\beta, C^\alpha_\beta_\gamma)$ on $F^{n-1}$ are given by

(1.14.6) 

\[ H_{\alpha\beta} = N_i (B^i_{\alpha\beta} + F^i_{jk} B^j_\alpha B^k_\beta) + M_\alpha H_\beta \]

and

(1.14.7) 

\[ H_\alpha = N_i (B^i_{0\alpha} + G^i_\beta B^\beta_\alpha), \]

where

(1.14.8) 

\[ M_\alpha = C_{ijk} B^i_\alpha N^j N^k, \]

$B^i_{\alpha\beta} = \partial^2 x^i / \partial u^\alpha \partial u^\beta$ and $B^i_{0\alpha} = B^i_\beta_\alpha v^\beta$.

The equations (1.14.6) and (1.14.7) yield

(1.14.9) 

\[ H_{0\alpha} = H_{\beta\alpha} v^\beta = H_\alpha, \quad H_{\alpha0} = H_{\alpha\beta} v^\beta = H_\alpha + M_\alpha H_0. \]

The second fundamental $v$-tensor $M_{\alpha\beta}$ is defined as:

(1.14.10) 

\[ M_{\alpha\beta} = C_{ijk} B^i_\alpha B^j_\beta N^k. \]

The relative $h$- and $v$-covariant derivatives of $B^i_\alpha$ and $N^i$ are given by

(1.14.11) 

\[
\begin{align*}
B^i_{\alpha|\beta} &= H_{\alpha\beta} N^i, \\
B^i_{\alpha|\beta} &= M_{\alpha\beta} N^i, \\
N^i_{|\beta} &= -H_{\alpha\beta} B^\alpha_j g^{ij}, \\
N^i_{|\beta} &= -M_{\alpha\beta} B^\alpha_j g^{ij}.
\end{align*}
\]
Let \( X_i(x, y) \) be a vector field of \( F^n \). The relative \( h \)- and \( v \)-covariant derivatives of \( X_i \) are given by

\[
(1.14.12) \quad X_i|\beta = X_{ij} B^j_\beta + X_i|_j N^j H_\beta, \quad X_i|\beta = X_i|_j B^j_\beta.
\]

The \((v)h\)-torsion tensor for the induced Cartan connection of \( F^{n-1} \) is given by

\[
(1.14.13) \quad R_{\alpha\beta\gamma} = R_{ijk} B^i_\alpha B^j_\beta B^k_\gamma + \{ H_\beta (H_{\alpha\gamma} - P_{ijk} B^i_\alpha B^k_\gamma N^j) - \beta/\gamma \},
\]

where \( P_{ijk} \) is the \((v)hv\)-torsion tensor defined by \( P_{ijk} = C_{ijk\vert0} \).

If each path of a hypersurface \( F^{n-1} \) with respect to the induced connection is also a path of the enveloping space \( F^n \) then \( F^{n-1} \) is called a \textit{hyperplane of the first kind}.

If each \( h \)-path of a hypersurface \( F^{n-1} \) with respect to the induced connection is also an \( h \)-path of the enveloping space \( F^n \) then \( F^{n-1} \) is called a \textit{hyperplane of the second kind}.

If the unit normal vector of \( F^{n-1} \) is parallel along each curve of \( F^{n-1} \) then \( F^{n-1} \) is called a \textit{hyperplane of the third kind}.

Matsumoto [103] obtained the characteristic conditions for different kinds of hyperplanes, which are given in the following lemmas:

**Lemma 1.14.1:** A hypersurface \( F^{n-1} \) is a hyperplane of the first kind if and only if \( H_\alpha = 0 \) or equivalently \( H_0 = 0 \).

**Lemma 1.14.2:** A hypersurface \( F^{n-1} \) is a hyperplane of the second kind if and only if \( H_{\alpha\beta} = 0 \).

**Lemma 1.14.3:** A hypersurface \( F^{n-1} \) is a hyperplane of the third kind if and only if \( H_{\alpha\beta} = 0 = M_{\alpha\beta} \).
Chapter 2

A Finsler Space with Conservative Normal Projective Curvature Tensor

2.1 Introduction

A vector field in a differentiable manifold is said to be conservative if its divergence is zero. If the divergence of curvature is zero in a neighbourhood of any point \( p \) of a differentiable manifold \( M^n \) then the bending is conservative, i.e. rigid around the point \( p \). In 1980, P. N. Pandey [143] established the relation between the normal projective curvature tensor and Berwald curvature tensor. In 1984, P. N. Pandey [150] discussed projective motion in a symmetric and projectively symmetric Finsler manifold. The condition characterizing a Finsler space with conservative Berwald curvature tensor is weaker than the condition characterizing a symmetric Finsler Space. The conservativeness of curvature tensor and other tensors on K-Contact manifold, Sasakian manifold, Einstein manifold, trans- Sasakian manifold and Kaehler manifold have been discussed by several authors including U. C. De and A. A. Shaikh [230], C. S. Bagewadi and N. B. Gatti [14], C. S. Bagewadi and D. G. Prakasha [17], D.