Chapter 4
Commuting traces of biderivations

4.1 Introduction

A mapping $D : R \times R \rightarrow R$ is said to be symmetric if $D(x, y) = D(y, x)$ for all $x, y \in R$. A mapping $f : R \rightarrow R$ defined by $f(x) = D(x, x)$, where $D : R \times R \rightarrow R$ is a symmetric mapping is called trace of $D$. In 1980, Maksa [89] introduced the concept of a biderivation. A biadditive mapping $D : R \times R \rightarrow R$ is said to be a biderivation if for all $x, y \in R$, the mappings $y \mapsto D(x, y)$ and $x \mapsto D(x, y)$ are derivations of $R$. Later it was shown that symmetric biderivations are related to general solution of some functional equations. The notion of additive commuting mapping is closely connected with the notion of a biderivation. Every commuting additive mapping $f : R \rightarrow R$ gives rise to a biderivation. Linearizing $[f(x), x] = 0$ for all $x \in R$, we get $[f(x), y] = [x, f(y)]$ for all $x, y \in R$ and hence we note that the mapping $(x, y) \mapsto [f(x), y]$ is a biderivation on $R$ (moreover all derivations appearing are inner).

Section 4.2 deals with the study of $n$-centralizing traces of symmetric biderivations of a semiprime ring. The main result is the following: Let $R$ be a semiprime ring, $I$ a nonzero ideal of $R$ and $n$ be a fixed positive integer. Let $R$ be $n!$-torsion free for $n > 1$ and $2$-torsion free for $n = 1$. Suppose there exists a symmetric biderivation $D : R \times R \rightarrow R$ such that the mapping $f : R \rightarrow R$ is $n$-centralizing on $I$, where $f$ stands for the trace of $D$. Then $f$ is $n$-commuting on $I$. Moreover we extend the
result for a Lie ideal of \( R \).

In section 4.3, we study symmetric generalized biderivations of prime rings. The notion of generalized symmetric biderivation was introduced by Nurcan in [13]. Let \( R \) be a ring and \( D : R \times R \to R \) be a biadditive map. A biadditive mapping \( \Delta : R \times R \to R \) is said to be a generalized biderivation if for every \( x \in R \), the map \( y \mapsto \Delta(x, y) \) is a generalized derivation of \( R \) associated with function \( y \mapsto D(x, y) \) for all \( x, y \in R \) as well as for every \( y \in R \), the map \( x \mapsto \Delta(x, y) \) is a generalized derivation of \( R \) associated with function \( x \mapsto D(x, y) \) for all \( x, y \in R \). The trace \( g \) of a symmetric generalized biderivation \( \Delta \) defined by \( g(x) = \Delta(x, x) \), satisfies \( g(x + y) = g(x) + g(y) + 2\Delta(x, y) \) for all \( x, y \in R \).

Recently in [117, Theorem 2] Yenigul et.al proved a result of Vukman [108, Theorem 4] for a two sided ideal \( I \) of a prime ring \( R \) which states that if there exist symmetric biderivations \( D_1 : R \times R \to R \) and \( D_2 : R \times R \to R \) such that \( D_1(d_2(x), x) = 0 \) for all \( x \in I \), where \( d_2 \) is the trace of \( D_2 \), then either \( D_1 = 0 \) or \( D_2 = 0 \). We obtain the result for a symmetric generalized biderivation \( \Delta \) with associated biderivation \( D \) of \( R \) with trace \( f \) satisfying \( \Delta(f(x), x) = 0 \) for all \( x \in I \) and conclude that either \( \Delta = 0 \) or \( R \) is commutative.

Finally we investigate the commutativity of a semiprime ring \( R \) satisfying various identities involving the trace \( f \) of the symmetric biadditive mapping \( D \) on \( R \).

4.2 \( n \)-centralizing traces of symmetric biderivations

Definition 4.2.1 (Symmetric mapping) A mapping \( D : R \times R \to R \) is said to be symmetric if \( D(x, y) = D(y, x) \) for all \( x, y \in R \).

Definition 4.2.2 (Biadditive mapping) A mapping \( D : R \times R \to R \) is called biadditive if it is additive in both arguments.
Definition 4.2.3 (Trace) A mapping $f : R \rightarrow R$ defined by $f(x) = D(x, x)$, where $D : R \times R \rightarrow R$ is a symmetric mapping is called the trace of $D$.

Remark 4.2.1

(i) The trace $f$ of $D$ satisfies the relation $f(x+y) = f(x) + f(y) + D(x, y) + D(y, x)$ for all $x, y \in R$.

(ii) If $D$ is symmetric, then the trace $f$ of $D$ satisfies the relation $f(x+y) = f(x) + f(y) + 2D(x, y)$ for all $x, y \in R$.

Definition 4.2.4 (Biderivation) A biadditive mapping $D : R \times R \rightarrow R$ is said to be a biderivation on $R$ if $D(xy, z) = D(x, z)y + xD(y, z)$ and $D(x, yz) = D(x, y)z + yD(x, z)$ for all $x, y, z \in R$.

Example 4.2.1 Let $R$ be a ring and $\lambda \in Z(R)$, the centre of $R$. Then the mapping $(x, y) \mapsto \lambda[x, y]$ is a biderivation on $R$.

In 1987, Bell and Martindale [29] proved that if a semiprime ring $R$ admits a derivation $d$ which is nonzero on a nonzero left ideal $I$ of $R$ and centralizing on $I$, then $R$ must contain a nonzero central ideal. Deng and Bell [49] proved the result for $n$-centralizing mappings. Now we prove the following:

Theorem 4.2.1 Let $R$ be a semiprime ring and $L$ be a nonzero square closed Lie ideal of $R$. Let $n$ be a fixed positive integer. Let $R$ be $n!$-torsion free for $n > 1$ and 2-torsion free for $n = 1$. Suppose there exists a symmetric biderivation $D : R \times R \rightarrow R$ such that the mapping $f : R \rightarrow R$ is $n$-centralizing on $L$, where $f$ stands for the trace of $D$. Then $f$ is $n$-commuting on $L$.

The following lemma due to Deng and Bell [49] is essential to prove our theorem.
Lemma 4.2.1 [49, Lemma 1] Let $n$ be a positive integer and $R$ be $n!$-torsion free semiprime ring. Let $f : R \to R$ be an additive map on $R$. For $i = 1, 2, \ldots, n$, let $F_i(x, y)$ be a generalized polynomial which is homogeneous of degree $i$ in the non-commuting indeterminates $x$ and $y$. Let $a \in R$ and $(a)$ be the additive subgroup generated by $a$. If $F_n(x, f(x)) + F_{n-1}(x, f(x)) + \ldots + F_1(x, f(x)) \in Z(R)$ for all $x \in (a)$, then $F_i(a, f(a)) \in Z(R)$ for $i = 1, 2, \ldots, n$.

Proof of Theorem 4.2.1 Assume that $n = 1$. Linearizing the condition $[f(x), x] \in Z(R)$ for all $x \in L$, we have

$$[f(x), y] + [f(y), x] + [2D(x, y), x] + [2D(x, y), y] \in Z(R) \quad \text{for all } x, y \in L. \quad (4.2.1)$$

Substituting $-y$ for $y$ in (4.2.1), we have

$$-[f(x), y] + [f(y), x] - [2D(x, y), x] + [2D(x, y), y] \in Z(R), \quad \text{for all } x, y \in L. \quad (4.2.2)$$

Subtracting (4.2.1) and (4.2.2), we get $2[f(x), y] + 4[D(x, y), x] \in Z(R)$ for all $x, y \in L$. Replacing $y$ by $x^2$ in this relation we have $8[x^2, f(x)] \in Z(R)$ for all $x \in L$. Now commuting this with $f(x)$ and using 2-torsion condition, we obtain $[x^2, f(x)] = 0$ for all $x \in L$. This implies that $[f(x), x][x, f(x)] = 0$ for all $x \in L$ i.e. $[f(x), x]^2 = 0$ for all $x \in L$. Since the centre of a semiprime ring contains no nonzero nilpotent elements, $[f(x), x] = 0$ for all $x \in L$.

Now Suppose that $n > 1$. Linearizing the condition $[f(x), x^n] \in Z(R)$ we get

$$[f(x), x^n] + [f(x), x^{n-1}y + \ldots + yx^{n-1}] + [f(y), x^{n-1}y + \ldots + yx^{n-1}]$$

$$+ [f(y), x^n] + [2D(x, y), x^n] + [2D(x, y), x^{n-1}y + \ldots + yx^{n-1}] \in Z(R)$$

for all $x, y \in L$. Using Lemma 4.2.1 and the fact that $[f(x), x^n] \in Z(R)$ for all $x \in L$, we obtain

$$[f(x), x^{n-1}y + \ldots + yx^{n-1}] + [2D(x, y), x^{n-1}y + \ldots + yx^{n-1}]$$

$$+ [f(y), x^n] + [2D(x, y), x^n] \in Z(R), \quad \text{for all } x, y \in L. \quad (4.2.3)$$
Replacing \( y \) by \(-y\) in (4.2.3), we have

\[
-\left[ f(x), x^{n-1}y + \cdots + yx^{n-1} \right] + \left[ 2D(x,y), x^{n-1}y + \cdots + yx^{n-1} \right] \\
+\left[ f(y), x^n \right] - \left[ 2D(x,y), x^n \right] \in Z(R), \text{ for all } x, y \in L. 
\] (4.2.4)

Now subtracting (4.2.3) and (4.2.4), we get

\[
2\left[ f(x), x^{n-1}y + \cdots + yx^{n-1} \right] + 4\left[ D(x,y), x^n \right] \in Z(R) \text{ for all } x, y \in L. 
\] (4.2.5)

Substituting \( x^2 \) for \( y \) in (4.2.5), we find that \( 2\left[ f(x), nx^{n+1} \right] + 4\left[ D(x,x^2), x^n \right] \in Z(R) \) for all \( x \in L \). This implies that \( 2(4 + n)x\left[ f(x), x^n \right] \in Z(R) \) for all \( x \in L \). i.e. \( 2(4 + n)(x\left[ f(x), x^n \right])^n \in Z(R) \) for all \( x \in L \). Commuting with \( f(x) \) and using torsion condition, we get

\[
[x^n[f(x), x^n], f(x)] = 0 \text{ for all } x \in L. 
\] (4.2.6)

This implies that

\[
[f(x), x^n]^{n+1} = 0 \text{ for all } x \in L. 
\] (4.2.7)

Since the centre of a semiprime ring contains no nonzero nilpotent elements, we have \([f(x), x^n] = 0\), for all \( x \in L \).

Using the similar techniques with slight modifications, we can prove the following:

**Theorem 4.2.2** Let \( R \) be a semiprime ring and \( I \) be a nonzero left ideal of \( R \). Let \( n \) be a fixed positive integer. Let \( R \) be \( n! \)-torsion free for \( n > 1 \) and 2-torsion free for \( n = 1 \). Suppose there exists a symmetric biderivation \( D : R \times R \rightarrow R \) such that the mapping \( f : R \rightarrow R \) is \( n \)-centralizing on \( I \), where \( f \) stands for the trace of \( D \). Then \( f \) is \( n \)-commuting on \( I \).

The following are known results:

**Lemma 4.2.2** [57, Corollary 2] If \( R \) is a semiprime ring and \( I \) is an ideal of \( R \), then \( I \cap A(I) = (0) \).
Lemma 4.2.3 [108, Theorem 4] Let $R$ be a $2$-torsion free semiprime ring. Suppose there exists a symmetric biderivation $D : R \times R \rightarrow R$ such that $D(f(x), x) = 0$ for all $x \in R$, where $f$ denotes the trace of $D$. In this case we have $D = 0$.

Lemma 4.2.4 Let $R$ be a prime ring of characteristic different from two and $I$ be a nonzero ideal of $R$. If $D$ is symmetric biderivation such that $D(x, x) = 0$ for all $x \in I$, then either $D = 0$ or $R$ is commutative.

Proof Let $D(x, x) = 0$ for all $x \in I$. Linearization yields that $2D(x, y) = 0$ for all $x, y \in I$. Since characteristic of $R$ is different from two, we have $D(x, y) = 0$ for all $x, y \in I$. Replacing $y$ by $ry$, we get $D(x, r)y = 0$ for all $x, y \in I$ and $r \in R$. Substitute $sx$ for $x$, we obtain $D(s, r)xy = 0$ for all $x, y \in I$ and $r, s \in R$. This implies that $D(s, r)R[x, y] = 0$ for all $x, y \in I$ and $r, s \in R$. Primeness of $R$ yields that either $[x, y] = 0$ or $D(r, s) = 0$ for all $x, y \in I$ and $r, s \in R$. If $[x, y] = 0$ for all $x, y \in I$, then $I$ is commutative and hence $R$ is commutative. Later gives that $D = 0$.

The following theorem extends a result due to Vukman [109, Theorem 1].

Theorem 4.2.3 Let $R$ be a prime ring of characteristic not two and three and $I$ be an ideal of $R$. If $D_1, D_2$ are the symmetric biderivations of $R$ with trace $f_1, f_2$ respectively such that $f_1(x)f_2(x) = 0$ for all $x \in I$, then either $D_1 = 0$ or $D_2 = 0$ unless $[I, I] = 0$.

Proof Suppose that $f_1(x)f_2(x) = 0$ for all $x \in I$. (4.2.8)

Linearization yields that

$$f_1(y)f_2(x) + 2D_1(x, y)f_2(x) + f_1(x)f_2(y) + 2D_1(x, y)f_2(y) + 2f_1(x)D_2(x, y) + 2f_1(y)D_2(x, y) + 4D_1(x, y)D_2(x, y) = 0 \text{ for all } x, y \in I. \ (4.2.9)$$
Substitute \(-y\) for \(y\) in (4.2.9) to get

\[
f_1(y)f_2(x) - 2D_1(x, y)f_2(x) + f_1(x)f_2(y) - 2D_1(x, y)f_2(y)
\]

\[-2f_1(x)D_2(x, y) - 2f_1(y)D_2(x, y) + 4D_1(x, y)D_2(x, y) = 0 \text{ for all } x, y \in I. \quad (4.2.10)\]

Adding (4.2.9) and (4.2.10) and using 2-torsion freeness of \(R\), we obtain

\[
f_1(y)f_2(x) + f_1(x)f_2(y) + 4D_1(x, y)D_2(x, y) = 0 \text{ for all } x, y \in I. \quad (4.2.11)\]

Replacing \(y\) by \(y + z\) in (4.2.11), we find

\[
\begin{align*}
f_1(y)f_2(x) + f_1(x)f_2(y) + 4D_1(x, y)D_2(x, z) + & f_1(x)f_2(z) + 4D_1(x, y)D_2(x, y) + 4D_1(x, z)D_2(x, z) \quad (4.2.12) \\
+ 4D_1(x, y)D_2(x, z) + 4D_1(x, z)D_2(x, y) = 0 \text{ for all } x, y, z \in I.
\end{align*}
\]

Using (4.2.11), (4.2.12) gives that

\[
4D_1(y, z)f_2(x) + 4f_1(x)D_2(y, z) + 8D_1(x, y)D_2(x, z) + 8D_1(x, z)D_2(x, y) = 0 \text{ for all } x, y, z \in I. \quad (4.2.13)
\]

Substitute \(y\) for \(x\) in (4.2.13), we get

\[
12D_1(y, z)f_2(y) + 12f_1(y)D_2(y, z) = 0 \text{ for all } y, z \in I. \quad (4.2.14)
\]

Replace \(z\) by \(zu\) in (4.2.14) and use (4.2.14) to obtain

\[
[f_1(y), z]D_2(y, u) + D_1(y, z)[u, f_2(y)] = 0 \text{ for all } y, z, u \in I. \quad (4.2.15)
\]

Again replace \(z\) by \(f_1(y)z\) in (4.2.15) to get

\[
\begin{align*}
f_1(y)[f_1(y), z]D_2(y, u) + f_1(y)D_1(y, z)[u, f_2(y)] + & D_1(y, f_1(y))z[u, f_2(y)] = 0 \text{ for all } y, z, u \in I. \quad (4.2.16)
\end{align*}
\]

Comparing (4.2.15) and (4.2.16), we arrive at

\[
D_1(y, f_1(y))z[u, f_2(y)] = 0 \text{ for all } y, z, u \in I. \quad (4.2.17)
\]
This implies that \( D_1(y, f_1(y))Rz[u, f_2(y)] = 0 \) for all \( y, z, u \in I \). Primeness of \( R \) yields that either \( D_1(y, f_1(y)) = 0 \) or \( z[u, f_2(y)] = 0 \) for all \( y, z, u \in I \). If \( D_1(y, f_1(y)) = 0 \) for all \( y \in I \), then conclusion follows from Lemma 4.2.3. Now consider the case when \( z[u, f_2(y)] = 0 \) for all \( y, z, u \in I \). Hence we get \( [u, f_2(y)] = 0 \) for all \( y, u \in I \). Linearization yields that \([u, D_2(x, y)] = 0 \) for all \( x, y, u \in I \). Replacing \( x \) by \( xz \), we have \([u, x]D_2(z, y) + D_2(x, y)[u, z] = 0 \) for all \( x, y, u, z \in I \). In particular, we get \([x, z]D_2(z, y) = 0 \) for all \( x, y, z \in I \). This implies that \([x, z]D_2(z, y) = 0 \) for all \( x, y, z \in I \). Since \([I, I] \neq 0\), primeness of \( I \) yields that \( D_2(z, y) = 0 \) for all \( z, y \in I \).

Application of Lemma 4.2.4 gives that \( D_2 = 0 \).

In [108, Theorem 4] Vukman proved that if \( R \) is a 2-torsion free semiprime ring and \( D : R \times R \to R \) be a symmetric biderivation with trace \( f \) such that \( D(f(x), x) = 0 \) for all \( x \in R \), then \( D = 0 \). Further Yenigul et al. [117, Theorem 2] extended the result for a two sided ideal of a prime ring \( R \). We generalize the aforementioned results for semiprime ring in case of two sided ideal and prove the following.

**Theorem 4.2.4** Let \( R \) be a 2-torsion free semiprime ring and \( I \) be an ideal of \( R \). Let \( D \) be a symmetric biderivation on \( R \) such that \( D(I, I) \subseteq I \). If \( f \) is the trace of \( D \) such that \( D(f(x), x) = 0 \) for all \( x \in I \), then \( D = 0 \) on \( I \).

**Proof** Suppose that \( D(f(x), x) = 0 \) for all \( x \in I \). (4.2.18)

Linearization yields that

\[
D(f(x), y) + D(f(x), x) + D(f(y), x) + D(f(y), y)
+ 2D(D(x, y), x) + 2D(D(x, y), y) = 0 \quad \text{for all } x, y \in I.
\] (4.2.19)

Comparing (4.2.18) and (4.2.19), we get

\[
D(f(x), y) + D(f(y), x) + 2D(D(x, y), x) + 2D(D(x, y), y) = 0 \quad \text{for all } x, y \in I.
\] (4.2.20)
Substitute $-y$ for $y$ in (4.2.20), we find

$$-D(f(x), y) + D(f(y), x) - 2D(D(x, y), x) + 2D(D(x, y), y) = 0 \text{ for all } x, y \in I. \quad (4.2.21)$$

Adding (4.2.20) and (4.2.21) and using 2-torsion freeness of $R$, we get

$$D(f(y), x) + 2D(D(x, y), y) = 0 \text{ for all } x, y \in I. \quad (4.2.22)$$

Replace $x$ by $zx$ in (4.2.22), we obtain

$$xD(f(y), z) + D(f(y), x)z + 2xD(D(z, y), y) + 4D(x, y)D(z, y) + 2D(D(x, y), y)z = 0 \text{ for all } x, y \in I. \quad (4.2.23)$$

In view of (4.2.22), (4.2.23) reduces to

$$4D(x, y)D(z, y) = 0 \text{ for all } x, y \in I. \quad (4.2.24)$$

Since $R$ is 2-torsion free, we have $D(x, y)D(z, y) = 0$ for all $x, y, z \in I$. Substituting $xz$ for $z$ to get $D(x, y)zD(x, y) = 0$ for all $x, y, z \in I$. On simplification, we get $D(x, y)I = 0$ and $ID(x, y) = 0$ for all $x, y \in I$, i.e. $D(x, y) \in A(I)$ for all $x, y \in I$. Since $D(I, I) \subseteq I$, we obtain $D(x, y) \in I \cap A(I) = (0)$ for all $x, y \in I$ by Lemma 4.2.2. Hence we get $D = 0$ on $I$.

### 4.3 Traces of symmetric generalized biderivations

In [13] Nurcan defined generalized biderivation in rings as follows:

**Definition 4.3.1 (Generalized biderivation)** Let $R$ be a ring and $D : R \times R \to R$ be a biadditive map. A biadditive mapping $\Delta : R \times R \to R$ is said to be a generalized $D$-biderivation if for every $x \in R$, the map $y \mapsto \Delta(x, y)$ is a generalized derivation of $R$ associated with the function $y \mapsto D(x, y)$ for all $x, y \in R$ as well as for every $y \in R$, the map $x \mapsto \Delta(x, y)$ is a generalized derivation of $R$ associated with the function $x \mapsto D(x, y)$ for all $x, y \in R$, i.e. $\Delta(x, yz) = \Delta(x, y)z + yD(x, z)$ and
\[ \Delta(xy, z) = \Delta(x, z)y + xD(y, z) \] for all \( x, y, z \in R \).

**Example 4.3.1** Let \( R \) be a ring. If \( D \) is any biderivation of \( R \) and \( \alpha : R \times R \to R \) is a biadditive function such that \( \alpha(x, yz) = \alpha(x, y)z \) and \( \alpha(xy, z) = \alpha(x, z)y \) for all \( x, y, z \in R \), then \( D + \alpha \) is a generalized \( D \)-biderivation of \( R \).

We further extend Theorem 4.2.4 for a symmetric generalized biderivation of a prime ring in case of two sided ideal.

**Theorem 4.3.1** Let \( R \) be a prime ring of characteristic not two and \( I \) be a nonzero ideal of \( R \). If \( \Delta \) is a symmetric generalized biderivation with associated biderivation \( D \) of \( R \) with trace \( f \) such that \( \Delta(f(x), x) = 0 \) for all \( x \in I \), then either \( \Delta = 0 \) or \( R \) is commutative.

**Proof** Suppose that

\[ \Delta(f(x), x) = 0 \text{ for all } x \in I. \]  \( (4.3.1) \)

Linearizing (4.3.1) and using (4.3.1), we get

\[ \Delta(f(x), y) + \Delta(f(y), x) + 2\Delta(D(x, y), x) + 2\Delta(D(x, y), y) = 0 \text{ for all } x, y \in I. \]  \( (4.3.2) \)

Replacing \( y \) by \(-y\) in (4.3.2), we get

\[ -\Delta(f(x), y) + \Delta(f(y), x) - 2\Delta(D(x, y), x) + 2\Delta(D(x, y), y) = 0 \text{ for all } x, y \in I. \]  \( (4.3.3) \)

Adding (4.3.2) and (4.3.3) and using characteristic of \( R \) is not two, we find

\[ \Delta(f(y), x) + 2\Delta(D(x, y), y) = 0 \text{ for all } x, y \in I. \]  \( (4.3.4) \)

Substitute \( xz \) for \( x \) in (4.3.4) to get

\[ \Delta(f(y), xz) + xD(f(y), z) + 2\Delta(xD(z, y), y) + 2\Delta(D(x, y), z, y) = 0 \text{ for all } x, y, z \in I. \]  \( (4.3.5) \)
On simplification, we get

\[ \Delta(f(y), x)z + xD(f(y), z) + 2\Delta(x, y)D(z, y) + 2x(D(z, y), y) \]

\[ + 2\Delta(D(x, y), y)z + 2D(x, y)D(z, y) = 0 \text{ for all } x, y, z \in I. \]  

(4.3.6)

In view of (4.3.4), (4.3.6) yields that

\[ xD(f(y), z) + 2\Delta(x, y)D(z, y) + 2xD(D(z, y), y) + 2D(x, y)D(z, y) = 0 \text{ for all } x, y, z \in I. \]  

(4.3.7)

Replacing \( x \) by \( ux \) in (4.3.7), we obtain

\[ uxD(f(y), z) + 2\Delta(u, y)xD(z, y) + 2u(x, y)D(z, y) + 2uxD(D(z, y), y) \]

\[ + 2D(u, y)xD(z, y) + 2uD(x, y)D(z, y) = 0 \text{ for all } x, y, z, u \in I. \]  

(4.3.8)

Comparing (4.3.7) and (4.3.8), we get

\[ 2\Delta(u, y)xD(z, y) + 2uD(x, y)D(z, y) + 2xD(D(z, y), y) \]

\[ - 2ux\Delta(x, y)D(z, y) = 0 \text{ for all } x, y, z, u \in I. \]  

(4.3.9)

Since \( R \) is of characteristic not two and replace \( u \) by \( x \), we have

\[ \Delta(x, y)xD(z, y) + xD(x, y)D(z, y) + D(x, y)xD(z, y) \]

\[ - x\Delta(x, y)D(z, y) = 0 \text{ for all } x, y, z \in I. \]  

(4.3.10)

This implies that

\[ [\Delta(x, y), x] + (xD(x, y) + D(x, y)x)]D(z, y) = 0 \text{ for all } x, y, z \in I. \]  

(4.3.11)

i.e., we have \([\Delta(x, y), x] + D(x^2, y)D(z, y) = 0 \text{ for all } x, y, z \in I. \) Replacing \( z \) by \( zu \), we obtain \([\Delta(x, y), x] + D(x^2, y)zD(u, y) = 0 \text{ for all } x, y, z, u \in I. \) Since \( R \) is prime, we get either \([\Delta(x, y), x] + D(x^2, y) = 0 \text{ or } D(u, y) = 0 \text{ for all } x, y, u \in I. \) Later yields that either \( R \) is commutative or \( D = 0 \) by Lemma 4.2.4.

If \( D = 0 \), then by (4.3.1) we get \( \Delta = 0 \). On the other hand, if \([\Delta(x, y), x] + D(x^2, y) = 0 \text{ for all } x, y \in I, \) then replacing \( y \) by \( yz \) we find \( \Delta(x, y)[z, x] + [\Delta(x, y), x]z+ \)
$y[D(x, z), x] + [y, x]D(x, z) + yD(x^2, z) + D(x^2, y)z = 0$ for all $x, y, z \in I$. This implies that $\Delta(x, y)[z, x] + y[D(x, z), x] + [y, x]D(x, z) + yD(x^2, z) = 0$ for all $x, y, z \in I$. In particular, if we take $x = z$, then we have $y[f(x), x] + [y, x]f(x) + yD(x^2, x) = 0$ for all $x, y \in I$. Again replace $y$ by $ry$ and use the last relation to get $[r, x]yf(x) = 0$ for all $x, y \in I$ and $r \in R$. Primeness of $R$ yields that either $f(x) = 0$ or $[x, r] = 0$ for all $x \in I$ and $r \in R$. If $f(x) = 0$ for all $x \in I$, then by (4.3.1) $\Delta(0, x) = 0$ for all $x \in I$ and hence $\Delta = 0$. Later gives $R$ is commutative.

Now we prove the above theorem for the noncommutative case.

**Theorem 4.3.2** Let $R$ be a noncommutative prime ring of characteristic not two and $I$ be a nonzero ideal of $R$. If $\Delta$ is a symmetric generalized biderivation with associated biderivation $D$ of $R$ with trace $f$ such that $\Delta(f(x), y) = 0$ for all $x, y \in I$, then $D = 0$ and hence $\Delta = 0$.

**Proof** Suppose that

$$\Delta(f(x), y) = 0 \text{ for all } x, y \in I. \quad (4.3.12)$$

Replacing $y$ by $yz$ in (4.3.12), we have

$$\Delta(f(x), y)z + yD(f(x), z) = 0 \text{ for all } x, y, z \in I. \quad (4.3.13)$$

In view of (4.3.12) and primeness of $R$, (4.3.13) yields that

$$D(f(x), z) = 0 \text{ for all } x, y, z \in I. \quad (4.3.14)$$

Substitute $x + y$ for $x$ in (4.3.14) to get

$$D(f(x), z) + D(f(y), z) + 2D(D(x, y), z) = 0 \text{ for all } x, y, z \in I. \quad (4.3.15)$$

Using (4.3.14) and the fact that $R$ is not of characteristic two we obtain

$$D(D(x, y), z) = 0 \text{ for all } x, y, z \in I. \quad (4.3.16)$$
Replacing \( y \) by \( yu \) in (4.3.16), we find
\[
yD(D(x, u), z) + D(y, z)D(x, u) + D(x, y)D(u, z) + D(D(x, y), z)u = 0 \text{ for all } x, y, z, u \in I. \tag{4.3.17}
\]
Applying (4.3.16) to obtain
\[
D(y, z)D(x, u) + D(x, y)D(u, z) = 0 \text{ for all } x, y, z, u \in I. \tag{4.3.18}
\]
Substituting \( yw \) for \( y \) in (4.3.18), we get
\[
D(y, z)wD(x, u) + D(x, y)wD(u, z) = 0 \text{ for all } x, y, z, u, w \in I. \tag{4.3.19}
\]
In particular, if we replace \( x \) by \( z \) in (4.3.19), then we obtain \( D(y, z)wD(z, u) + D(z, y)wD(u, z) = 0 \) for all \( y, z, u, w \in I \). Since \( D \) is symmetric and using the fact that \( R \) is not of characteristic two, we have \( D(y, z)wD(z, u) = 0 \) for all \( y, z, u, w \in I \). Primeness of \( I \) yields that \( D(z, u) = 0 \) for all \( z, u \in I \). Using Lemma 4.2.4, we have \( D = 0 \) and hence \( \Delta = 0 \).

### 4.4 Traces of symmetric biadditive mappings

Following lemmas are essential to prove our theorems.

**Lemma 4.4.1** [56, Lemma 1] Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a nonzero Lie ideal of \( R \). If \([L, L] \subseteq Z(R)\), then \( L \subseteq Z(R) \).

**Lemma 4.4.2** Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a nonzero square closed Lie ideal of \( R \). If \( x \circ y \in Z(R) \) for all \( x, y \in L \), then \( L \subseteq Z(R) \).

**Proof** Suppose \( L \nsubseteq Z(R) \) and \( x \circ y \in Z(R) \) for all \( x, y \in L \). Replacing \( x \) by \( 2yx \), we get \( 2y(x \circ y) \in Z(R) \) for all \( x, y \in L \). This implies that \( 2[y(x \circ y), z] = 0 \) for all \( x, y, z \in L \). On simplification and using the fact that \( R \) is 2-torsion free, we have \([y, z](x \circ y) = 0 \) for all \( x, y, z \in L \). Substitute \( 2xz \) for \( x \) to get \( 2[y, z]x[y, z] = 0 \) for all \( x, z, y \in L \). Since \( R \) is 2-torsion free semiprime ring, we have \([y, z] = 0 \) for all \( y, z \in L \) by Lemma 2.4.1. Hence using Lemma 4.4.1, we get a contradiction. This completes the proof.
Lemma 4.4.3 Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero Lie ideal of $R$. If $L^2 \subseteq Z(R)$, then $L \subseteq Z(R)$.

Proof Since $xy \in Z(R)$ for all $x, y \in L$, $xy - yx = [x, y] \in Z(R)$ for all $x, y \in L$. Using Lemma 4.4.1, we get the required result.

Very recently Ashraf et.al. [20] explored the commutativity of a prime ring $R$ admitting a generalized derivation $F$ satisfying one of the following properties: (i) $F(xy) \mp xy \in Z(R)$, (ii) $F(xy) \mp xy \in Z(R)$, (iii) $F(x)F(y) \mp xy \in Z(R)$ for all $x, y \in R$.

Motivated by the above cited result, we prove the following: Let $R$ be a semiprime ring of characteristic not two admitting a symmetric biadditive map $D$ with trace $f$ and $L$ be a nonzero Lie ideal of $R$. Then $L \subseteq Z(R)$ if for all $x, y \in L$ one of the following holds: (i) $f(xy) \mp [x, y] \in Z(R)$, (ii) $f(xy) \mp xy \in Z(R)$, (iii) $f([x, y]) \mp [x, y] \in Z(R)$, (iv) $f([x, y]) \mp xy \in Z(R)$, (v) $f(xy) \mp f(x) \mp [x, y] \in Z(R)$, (vi) $f(xy) \mp f(y) \mp [x, y] \in Z(R)$, (vii) $f([x, y]) \mp f(x) \mp [x, y] \in Z(R)$, (viii) $f([x, y]) \mp f(y) \mp [x, y] \in Z(R)$, (ix) $f([x, y]) \mp f(xy) \mp [x, y] \in Z(R)$, (x) $f(xy) \mp x \circ y \in Z(R)$, (xi) $f([x, y]) \mp x \circ y \in Z(R)$, (xii) $f(x \circ y) \mp [x, y] \in Z(R)$, (xiii) $f(x \circ y) \mp x \circ y \in Z(R)$, (xiv) $f(x) \circ f(y) \mp [x, y] \in Z(R)$, (xv) $f(x \circ y) \mp f(xy) \mp x \circ y \in Z(R)$, (xvi) $f(x)f(y) \mp x \circ y \in Z(R)$.

Theorem 4.4.1 Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero Lie ideal of $R$. Let $D : R \times R \rightarrow R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f(xy) \mp [x, y] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

Proof Suppose
\[ f(xy) - [x, y] \in Z(R) \text{ for all } x, y \in L. \tag{4.4.1} \]
Replacing $y$ by $y + z$ in (4.4.1) we get
\[ f(xy) + f(xz) + 2D(xy, xz) - [x, y] - [x, z] \in Z(R) \text{ for all } x, y, z \in L. \tag{4.4.2} \]
Since \( R \) is 2-torsion free, \((4.4.1)\) yields that

\[
D(xy, xz) \in Z(R) \text{ for all } x, y, z \in L. \tag{4.4.3}
\]

Substituting \( y \) for \( z \) in \((4.4.3)\), we get

\[
f(xy) = D(xy, xy) \in Z(R) \text{ for all } x, y \in L. \tag{4.4.4}
\]

In view of \((4.4.1)\), \((4.4.4)\) yields that

\[
[x, y] \in Z(R) \text{ for all } x, y \in L. \tag{4.4.5}
\]

This implies that \([L, L] \subseteq Z(R)\). Hence \( L \subseteq Z(R) \) by Lemma 4.4.1. Similarly, we can prove the result for the case \( f(xy) + [x, y] \in Z(R) \) for all \( x, y \in L \).

**Theorem 4.4.2** Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a nonzero Lie ideal of \( R \). Let \( D : R \times R \rightarrow R \) be a symmetric biadditive mapping and \( f \) be the trace of \( D \). If \( f(xy) \in Z(R) \) for all \( x, y \in L \), then \( L \subseteq Z(R) \).

**Proof** The proof runs on the same parallel lines as of Theorem 4.4.1.

**Theorem 4.4.3** Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a nonzero Lie ideal of \( R \). Let \( D : R \times R \rightarrow R \) be a symmetric biadditive mapping and \( f \) be the trace of \( D \). If \( f(xy) + xy \in Z(R) \) for all \( x, y \in L \), then \( L \subseteq Z(R) \).

**Proof** Let

\[
f(xy) - xy \in Z(R) \text{ for all } x, y \in L. \tag{4.4.6}
\]

Replacing \( y \) by \( y + z \) we get

\[
f(xy) + f(xz) + 2D(xy, xz) - xy - xz \in Z(R) \text{ for all } x, y, z \in L. \tag{4.4.7}
\]

Using \((4.4.6)\), we obtain

\[
2D(xy, xz) \in Z(R) \text{ for all } x, y, z \in L. \tag{4.4.8}
\]
Since $R$ is 2-torsion free, we have
\[D(xy, xz) \in Z(R) \text{ for all } x, y, z \in L.\] (4.4.9)

Substituting $y$ for $z$ in (4.4.9), we get
\[f(xy) = D(xy, xy) \in Z(R) \text{ for all } x, y \in L.\] (4.4.10)

Using (4.4.6), we have $xy \in Z(R)$ for all $x, y \in L$. Hence $L^2 \subseteq Z(R)$ and by Lemma 4.4.3 $L \subseteq Z(R)$. Similarly we can prove the result if $f(xy) + xy \in Z(R)$ for all $x, y \in L$.

**Theorem 4.4.4** Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero Lie ideal of $R$. Let $D : R \times R \rightarrow R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f(xy) + xy \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

**Proof** The proof runs on the same parallel lines as of Theorem 4.4.3.

**Theorem 4.4.5** Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero Lie ideal of $R$. Let $D : R \times R \rightarrow R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f([x, y]) + [x, y] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

**Proof** Let
\[f([x, y]) - [x, y] \in Z(R) \text{ for all } x, y \in L.\] (4.4.11)

Replacing $y$ by $y + z$, we have $f([x, y] + [x, z]) - [x, y] - [x, z] \in Z(R)$ i.e. $f([x, y]) + f([x, z]) + 2D([x, y], [x, z]) - [x, y] - [x, z] \in Z(R)$ for all $x, y, z \in L$. Using (4.4.11), we get
\[2D([x, y], [x, z]) \in Z(R) \text{ for all } x, y, z \in L.\] (4.4.12)

Substituting $y$ for $z$ in (4.4.12) and using the fact that $R$ is 2-torsion free, we find
\[f([x, y]) \in Z(R) \text{ for all } x, y \in L.\] (4.4.13)

In view of (4.4.11), (4.4.13) yields that $[x, y] \in Z(R)$ for all $x, y \in L$. Thus we get $[L, L] \subseteq Z(R)$ and by Lemma 4.4.1 $L \subseteq Z(R)$. Similarly one can prove the result if
Using similar arguments as we have done in the proof of Theorem 4.4.5, we can prove the following:

**Theorem 4.4.6** Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero Lie ideal of $R$. Let $D : R \times R \rightarrow R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f([x,y]) + [y,x] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

**Theorem 4.4.7** Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero Lie ideal of $R$. Let $D : R \times R \rightarrow R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f([x,y]) + xy \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

**Proof** Let

\[ f([x,y]) - xy \in Z(R) \quad \text{for all } x, y \in L. \tag{4.4.14} \]

Replacing $y$ by $y + z$ in (4.4.14), we have $f([x,y] + [x,z]) - xy - xz \in Z(R)$ for all $x, y, z \in L$. This implies that

\[ f([x,y]) + f([x,z]) + 2D([x,y],[x,z]) - xy - xz \in Z(R) \quad \text{for all } x, y, z \in L. \tag{4.4.15} \]

Using (4.4.14) we obtain

\[ 2D([x,y],[x,z]) \in Z(R) \quad \text{for all } x, y, z \in L. \tag{4.4.16} \]

Since $R$ is 2-torsion free, (4.4.16) yields that

\[ D([x,y],[x,z]) \in Z(R) \quad \text{for all } x, y, z \in L. \tag{4.4.17} \]

In particular, if we substitute $y$ for $z$ in (4.4.17), then we have $f([x,y]) \in Z(R)$ for all $x, y \in L$. Again using (4.4.14), we get $xy \in Z(R)$ for all $x, y \in L$. Thus $L^2 \subseteq Z(R)$ and application of Lemma 4.4.3 completes the proof. Similarly we can prove the result if $f([x,y]) + xy \in Z(R)$ for all $x, y \in L$. 
Theorem 4.4.8 Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero Lie ideal of $R$. Let $D : R \times R \to R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f([x, y]) = yx \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

**Proof** The proof runs on the same parallel lines as that of Theorem 4.4.7.

Theorem 4.4.9 Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero Lie ideal of $R$. Let $D : R \times R \to R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f(xy) = f(x) + [x, y] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

**Proof** Suppose

$$f(xy) - f(x) - [x, y] \in Z(R) \text{ for all } x, y \in L. \tag{4.4.18}$$

Replacing $y$ by $y + z$, we get $f(xy) + f(xz) + 2D(xy, xz) - f(x) - [x, y] - [x, z] \in Z(R)$ for all $x, y, z \in L$. Using (4.4.18), we obtain

$$f(xz) + 2D(xy, xz) - [x, z] \in Z(R) \text{ for all } x, y, z \in L. \tag{4.4.19}$$

Substituting $-z$ for $z$ in (4.4.19), we get

$$f(xz) - 2D(xy, xz) + [x, z] \in Z(R) \text{ for all } x, y, z \in L. \tag{4.4.20}$$

Adding (4.4.19) and (4.4.20) we obtain

$$2f(xz) \in Z(R) \text{ for all } x, z \in L. \tag{4.4.21}$$

Since $R$ 2-torsion free, we have $f(xz) \in Z(R)$ for all $x, z \in L$.

Using (4.4.18), we get

$$f(x) - [x, y] \in Z(R) \text{ for all } x, y \in L. \tag{4.4.22}$$

Replacing $x$ by $x + z$ in (4.4.22), we have

$$f(x) + f(z) + 2D(x, z) - [x, y] - [z, y] \in Z(R) \text{ for all } x, y, z \in L. \tag{4.4.23}$$
Again using $(4.4.18)$ and 2-torsion freeness of $R$, we find $D(x, z) \in Z(R)$ for all $x, z \in L$. In particular $f(x) = D(x, z) \in Z(R)$ for all $x \in L$. Since $f(xz) \in Z(R)$ and $f(x) \in Z(R)$, we have $f(xz) - f(x) \in Z(R)$ for all $x, z \in L$. Using $(4.4.18)$ we get $[x, y] \in Z(R)$ for all $x, y \in L$. Hence Lemma 4.4.1 completes the proof. The proof is similar if $f(xy) + f(x) + [x, y] \in Z(R)$ for all $x, y \in L$.

**Theorem 4.4.10** Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero Lie ideal of $R$. Let $D : R \times R \rightarrow R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f(xy) \mp f(y) \mp [x, y] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

**Proof** Let

$$f(xy) - f(y) - [x, y] \in Z(R) \text{ for all } x, y \in L. \quad (4.4.24)$$

Replacing $y$ by $y + z$, we have $f(xy) + f(xz) + 2D(xy, xz) - f(y) - f(z) - 2D(y, z) - [x, y] - [x, z] \in Z(R)$ for all $x, y, z \in L$. Using $(4.4.24)$, we get

$$2(D(xy, xz) - D(y, z)) \in Z(R) \text{ for all } x, y, z \in L. \quad (4.4.25)$$

Substituting $y$ for $z$ in $(4.4.25)$ and using the fact that $R$ is 2-torsion free, we find

$$f(xy) - f(y) \in Z(R) \text{ for all } x, y \in L. \quad (4.4.26)$$

This implies that $[x, y] \in Z(R)$ for all $x, y \in L$. Thus $[L, L] \subseteq Z(R)$. Applying Lemma 4.4.1, we obtain $L \subseteq Z(R)$. The proof is similar for the case $f(xy) + f(y) + [x, y] \in Z(R)$ for all $x, y \in L$.

**Theorem 4.4.11** Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero Lie ideal of $R$. Let $D : R \times R \rightarrow R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f([x, y]) \mp f(x) \mp [x, y] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

**Proof** Suppose

$$f([x, y]) - f(x) - [x, y] \in Z(R) \text{ for all } x, y \in L. \quad (4.4.27)$$
Replacing $x$ by $x + z$ in (4.4.27), we obtain

\[ f([x, y]) + f([z, y]) + 2D([x, y], [z, y]) - f(x) - f(z) - 2D(x, z) - [x, y] - [z, y] \in Z(R) \text{ for all } x, y, z \in L. \]  

(4.4.28)

Using (4.4.27), we have

\[ 2(D([x, y], [z, y]) - D(x, z)) \in Z(R) \text{ for all } x, y, z \in L. \]  

(4.4.29)

Substituting $x$ for $z$ in (4.4.29) and using the fact that $R$ is 2-torsion free, we obtain

\[ f([x, y]) - f(x) \in Z(R) \text{ for all } x, y \in L. \]  

(4.4.30)

Again using (4.4.27), (4.4.30) yields that $[x, y] \in Z(R)$ for all $x, y \in L$. This implies that $[L, L] \subseteq Z(R)$. Application of Lemma 4.4.1 completes the proof. Similarly we can prove the theorem, if $f([x, y]) + f(x) + [x, y] \in Z(R)$ for all $x, y \in L$.

**Theorem 4.4.12** Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero Lie ideal of $R$. Let $D : R \times R \rightarrow R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f([x, y]) + f([x, y]) \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

**Proof** Let

\[ f([x, y]) - f(y) - [x, y] \in Z(R) \text{ for all } x, y \in L. \]  

(4.4.31)

Replacing $y$ by $y + z$ we get

\[ f([x, y]) + f([x, z]) + 2D([x, y], [x, z]) - f(y) - f(z) - 2D(y, z) - [x, y] - [x, z] \in Z(R) \text{ for all } x, y, z \in L. \]  

(4.4.32)

Using (4.4.31), (4.4.32) yields that

\[ 2(D([x, y], [x, z]) - D(y, z)) \in Z(R) \text{ for all } x, y, z \in L. \]  

(4.4.33)

Substituting $y$ for $z$ in (4.4.33) and using the fact that $R$ is 2-torsion free, we get

\[ f([x, y]) - f(y) = D([x, y], [x, y]) - D(y, y) \in Z(R) \text{ for all } x, y \in L. \]  

(4.4.34)
In view of (4.4.31), (4.4.34) yields that \([x, y] \in Z(R)\) for all \(x, y \in L\) i.e. \([L, L] \subseteq Z(R)\).

Using Lemma 4.4.1, we have \(L \subseteq Z(R)\). Similarly we can prove the theorem, if \(f([x, y]) + f(y) + [x, y] \in Z(R)\) for all \(x, y \in L\).

Using the similar techniques as used in proving Theorem 4.4.11 and Theorem 4.4.12, we can prove the following:

**Theorem 4.4.13** Let \(R\) be a 2-torsion free semiprime ring and \(L\) be a nonzero Lie ideal of \(R\). Let \(D : R \times R \rightarrow R\) be a symmetric biadditive mapping and \(f\) be the trace of \(D\). If \(f([x, y]) + f(x) + [y, x] \in Z(R)\) for all \(x, y \in L\), then \(L \subseteq Z(R)\).

**Theorem 4.4.14** Let \(R\) be a 2-torsion free semiprime ring and \(L\) be a nonzero Lie ideal of \(R\). Let \(D : R \times R \rightarrow R\) be a symmetric biadditive mapping and \(f\) be the trace of \(D\). If \(f([x, y]) + f(y) + [x, y] \in Z(R)\) for all \(x, y \in L\), then \(L \subseteq Z(R)\).

**Theorem 4.4.15** Let \(R\) be a 2-torsion free semiprime ring and \(L\) be a nonzero Lie ideal of \(R\). Let \(D : R \times R \rightarrow R\) be a symmetric biadditive mapping and \(f\) be the trace of \(D\). If \(f([x, y]) + f(xy) + [x, y] \in Z(R)\) for all \(x, y \in L\), then \(L \subseteq Z(R)\).

**Proof** Let

\[
f([x, y]) - f(xy) - [x, y] \in Z(R)
\]

for all \(x, y \in L\). \hspace{1cm} (4.4.35)

Replacing \(y\) by \(y + z\) in (4.4.35) we get

\[
f([x, y]) + f([x, z]) + 2D([x, y], [x, z]) - f(xy) - f(xz)
\]

\[
-2D(xy, xz) - [x, y] - [x, z] \in Z(R)
\]

for all \(x, y, z \in L\). \hspace{1cm} (4.4.36)

Using (4.4.35) and (4.4.36), we obtain

\[
2(D([x, y], [x, z]) - D(xy, xz)) \in Z(R)
\]

for all \(x, y, z \in L\). \hspace{1cm} (4.4.37)

Since \(R\) is 2-torsion free, we have

\[
D([x, y], [x, z]) - D(xy, xz) \in Z(R)
\]

for all \(x, y, z \in L\). \hspace{1cm} (4.4.38)
Substituting $y$ for $z$ in (4.4.38), we get

$$f([x, y]) - f(xy) \in Z(R) \text{ for all } x, y \in L. \quad (4.4.39)$$

Using (4.4.35), we have $[x, y] \in Z(R)$ for all $x, y \in L$ and Lemma 4.4.1 completes the proof. The proof is same for the case $f([x, y]) + f(xy) + [x, y] \in Z(R)$ for all $x, y \in L$.

Similarly we can prove the following:

**Theorem 4.4.16** Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero Lie ideal of $R$. Let $D : R \times R \rightarrow R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f([x, y]) \triangleq f(xy) \triangleq [y, x] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

**Theorem 4.4.17** Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero square closed Lie ideal of $R$. Let $D : R \times R \rightarrow R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f(xy) \triangleq x \circ y \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

**Proof** Suppose that

$$f(xy) - x \circ y \in Z(R) \text{ for all } x, y \in L. \quad (4.4.40)$$

Replacing $y$ by $y+z$ in (4.4.40) we get $f(xy) + f(xz) + 2D(xy, xz) - x \circ y - x \circ z \in Z(R)$ for all $x, y, z \in L$. Since $R$ is 2-torsion free, we obtain $D(xy, xz) \in Z(R)$ for all $x, y, z \in L$ by using (4.4.40). If we substituting $y$ for $z$, then we get $f(xy) = D(xy, xy) \in Z(R)$ for all $x, y \in L$. In view of (4.4.40), we find that $x \circ y \in Z(R)$ for all $x, y \in L$. Hence $L \subseteq Z(R)$ by Lemma 4.4.2. Similarly, we can prove the result for the case $f(xy) + x \circ y \in Z(R)$ for all $x, y \in L$.

**Theorem 4.4.18** Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero square closed Lie ideal of $R$. Let $D : R \times R \rightarrow R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f([x, y]) \triangleq x \circ y \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$. 


Proof Let
\[ f([x, y]) + x \circ y \in Z(R) \text{ for all } x, y \in L. \] (4.4.41)
Replacing \( y \) by \( y + z \) we get
\[ f([x, y]) + f([x, z]) + 2D([x, y], [x, z]) - x \circ y - x \circ z \in Z(R) \text{ for all } x, y, z \in L. \] (4.4.42)
Using (4.4.41), we obtain \( 2D([x, y], [x, z]) \in Z(R) \) for all \( x, y, z \in L \). This implies that \( 2[D([x, y], [x, z]), r] = 0 \) for all \( x, y, z \in L \) and \( r \in R \). Since \( R \) is 2-torsion free, we have \( D([x, y], [x, z]) \in Z(R) \) for all \( x, y, z \in L \). Substituting \( y \) for \( z \), we get \( f([x, y]) = D([x, y]) \in Z(R) \) for all \( x, y \in L \). Using (4.4.41), we have \( x \circ y \in Z(R) \) for all \( x, y \in L \). Hence by Lemma 4.4.2 \( L \subseteq Z(R) \). Similarly we can prove the result if \( f([x, y]) + x \circ y \in Z(R) \) for all \( x, y \in L \).

Theorem 4.4.19 Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a nonzero Lie ideal of \( R \). Let \( D : R \times R \to R \) be a symmetric biadditive mapping and \( f \) be the trace of \( D \). If \( f(x \circ y) + [x, y] \in Z(R) \) for all \( x, y \in L \), then \( L \subseteq Z(R) \).

Proof Let
\[ f(x \circ y) - [x, y] \in Z(R) \text{ for all } x, y \in L. \] (4.4.43)
Replacing \( y \) by \( y + z \) in (4.4.43), we have \( f(x \circ y + x \circ z) - [x, y] - [x, z] \in Z(R) \) i.e. \( f(x \circ y) + f(x \circ z) + 2D(x \circ y, x \circ z) - [x, y] - [x, z] \in Z(R) \) for all \( x, y, z \in L \). Using (4.4.43), we get
\[ 2D(x \circ y, x \circ z) \in Z(R) \text{ for all } x, y, z \in L. \] (4.4.44)
Substituting \( y \) for \( z \) in (4.4.44) and using the fact that \( R \) is 2-torsion free, we find
\[ f(x \circ y) \in Z(R) \text{ for all } x, y \in L. \] (4.4.45)
In view of (4.4.45), (4.4.43) yields that \([x, y] \in Z(R) \) for all \( x, y \in L \). Thus we get \([L, L] \subseteq Z(R) \) and by Lemma 4.4.1 \( L \subseteq Z(R) \). Similarly one can prove the result if \( f([x, y]) + [x, y] \in Z(R) \) for all \( x, y \in L \).
Theorem 4.4.20 Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero square closed Lie ideal of $R$. Let $D : R \times R \to R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f(x \circ y) = x \circ y \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

Proof Let
\[
  f(x \circ y) - x \circ y \in Z(R) \quad \text{for all } x, y \in L. \tag{4.4.46}
\]
Replacing $y$ by $y + z$ in (4.4.46), we have $f(x \circ y + x \circ z) - x \circ y - x \circ z \in Z(R)$ for all $x, y, z \in L$. This implies that
\[
  f(x \circ y) + f(x \circ z) + 2D(x \circ y, x \circ z) - x \circ y - x \circ z \in Z(R) \quad \text{for all } x, y, z \in L. \tag{4.4.47}
\]
Using (4.4.46), we obtain
\[
  2D(x \circ y, x \circ z) \in Z(R) \quad \text{for all } x, y, z \in L. \tag{4.4.48}
\]
Since $R$ is 2-torsion free, (4.4.48) yields that
\[
  D(x \circ y, x \circ z) \in Z(R) \quad \text{for all } x, y, z \in L. \tag{4.4.49}
\]
In particular, if we substitute $y$ for $z$ in (4.4.49), then we have $f(x \circ y) \in Z(R)$ for all $x, y \in L$. Again using (4.4.46), we get $x \circ y \in Z(R)$ for all $x \in L$. This application of Lemma 4.4.2 completes the proof. Similarly we can prove the result if $f(x \circ y) + x \circ y \in Z(R)$ for all $x, y \in L$.

Theorem 4.4.21 Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero Lie ideal of $R$. Let $D : R \times R \to R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f(x) \circ f(y) \neq [x, y] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

Proof Suppose
\[
  f(x) \circ f(y) - [x, y] \in Z(R) \quad \text{for all } x, y \in L. \tag{4.4.50}
\]
Replacing $y$ by $y + z$ in (4.4.50), we get $f(x) \circ f(y) + f(x) \circ f(z) + 2f(x) \circ D(y, z) - [x, y] - [x, z] \in Z(R)$ for all $x, y, z \in L$. Using (4.4.50) we obtain
\[
  2(f(x) \circ D(y, z)) \in Z(R) \quad \text{for all } x, y, z \in L. \tag{4.4.51}
\]
Substituting $y$ for $z$ in (4.4.51), we get $2(f(x) \circ f(y)) \in Z(R)$ for all $x, y \in L$. Since $R$ is 2-torsion free, we have

$$f(x) \circ f(y) \in Z(R) \text{ for all } x, y \in L. \quad (4.4.52)$$

In view of (4.4.50), (4.4.52) yields that $[x, y] \in Z(R)$ for all $x, y \in L$. Hence Lemma 4.4.1 completes the proof. The proof is similar if $f(x) \circ f(y) + [x, y] \in Z(R)$ for all $x, y \in L$.

**Theorem 4.4.22** Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero square closed Lie ideal of $R$. Let $D : R \times R \to R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f(x) \circ f(y) + xy \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

**Proof** Suppose

$$f(x) \circ f(y) - [x, y] \in Z(R) \text{ for all } x, y \in L. \quad (4.4.53)$$

Replacing $y$ by $y + z$ in (4.4.53), we get

$$f(x) \circ f(y) + f(x) \circ f(z) + 2f(x) \circ D(y, z) - [x, y] - [x, z] \in Z(R) \text{ for all } x, y, z \in L. \quad (4.4.54)$$

In view of (4.4.53), (4.4.54) yields that

$$2(f(x) \circ D(y, z)) \in Z(R) \text{ for all } x, y, z \in L. \quad (4.4.55)$$

In particular, we have $2(f(x) \circ D(y, y)) = 2(f(x) \circ f(y)) \in Z(R)$ for all $x, y \in L$. Since $R$ is 2-torsion free and using (4.4.53), we obtain $[x, y] \in Z(R)$ for all $x, y \in L$. Application of Lemma 4.4.1, we get the required result.

**Theorem 4.4.23** Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero Lie ideal of $R$. Let $D : R \times R \to R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f(x \circ y) \neq f(y) \neq [x, y] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$. 

Proof Suppose that
\[ f(x \circ y) - f(y) - [x, y] \in Z(R) \text{ for all } x, y \in L. \] (4.4.56)
Replacing \( y \) by \( y + z \) in (4.4.56), we get
\[ f(x \circ y) + f(x \circ z) + 2D(x \circ y, x \circ z) - f(y) - f(z) - 2D(y, z) - [x, y] - [x, z] \in Z(R) \text{ for all } x, y, z \in L. \] Using (4.4.56), we have
\[ 2(D(x \circ y, x \circ z) - D(y, z)) \in Z(R) \text{ for all } x, y, z \in L. \] Substituting \( y \) for \( z \) and using the fact that \( R \) is 2-torsion free, we get
\[ f(x \circ y) - f(y) = D(x \circ y, x \circ y) - D(y, y) \in Z(R) \text{ for all } x, y \in L. \] (4.4.57)
In view of (4.4.57), (4.4.56) yields that \([x, y] \in Z(R)\) for all \( x, y \in L \) i.e. \([L, L] \subseteq Z(R)\).
Using Lemma 4.4.1, we have \( L \subseteq Z(R) \). Similarly we can prove the theorem if
\[ f(x \circ y) + f(y) + [x, y] \in Z(R) \text{ for all } x, y \in L. \]

Theorem 4.4.24 Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a nonzero square closed Lie ideal of \( R \). Let \( D : R \times R \rightarrow R \) be a symmetric biadditive mapping and \( f \) be the trace of \( D \). If \( f(x \circ y) + f(y) + x \circ y \in Z(R) \) for all \( x, y \in L \), then \( L \subseteq Z(R) \).

Proof Suppose that
\[ f(x \circ y) - f(y) - x \circ y \in Z(R) \text{ for all } x, y \in L. \] (4.4.58)
Replacing \( y \) by \( y + z \) in (4.4.58) we have
\[ f(x \circ y) + f(x \circ z) + 2D(x \circ y, x \circ z) - f(y) - f(z) - 2D(y, z) - x \circ y - x \circ z \in Z(R) \text{ for all } x, y, z \in L. \] Using (4.4.58), we obtain
\[ 2(D(x \circ y, x \circ z) - D(y, z)) \in Z(R) \text{ for all } x, y, z \in L. \] (4.4.59)
Substitute \( y \) for \( z \) in (4.4.59) and using 2-torsion freeness of \( R \), we have
\[ D(x \circ y, x \circ y) - D(y, y) = f(x \circ y) - f(y) \in Z(R) \text{ for all } x, y \in L. \] (4.4.60)
In view of (4.4.60), (4.4.58) yields that \( x \circ y \in Z(R) \) for all \( x, y \in L \). Hence application of Lemma 4.4.2 completes the proof.
Theorem 4.4.25 Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero Lie ideal of $R$. Let $D: R \times R \rightarrow R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f(x \circ y) = f(xy) = [x, y] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

Proof Suppose that
\[
f(x \circ y) - f(xy) - [x, y] \in Z(R) \text{ for all } x, y \in L. \quad (4.4.61)
\]
Replace $y$ by $y + z$ in (4.4.61) to get $f(x \circ y) + f(x \circ z) + 2D(x \circ y, x \circ z) - f(xy) - f(xz) - 2D(xy, xz) - [x, y] - [x, z] \in Z(R)$ for all $x, y, z \in L$. In view of (4.4.61) last relation yields that $2(D(x \circ y, x \circ z) - D(xy, xz)) \in Z(R)$ for all $x, y, z \in L$. Since $R$ is 2-torsion free, we have $D(x \circ y, x \circ z) - D(xy, xz) \in Z(R)$ for all $x, y, z \in L$. Substituting $y$ for $z$, we obtain $f(x \circ y) - f(xy) \in Z(R)$ for all $x, y \in L$. Using (4.4.61), we have $[x, y] \in Z(R)$ for all $x, y \in L$ i.e. $[L, L] \subseteq Z(R)$. An application of Lemma 4.4.1 completes the proof. The proof is same for the case $f(x \circ y) + f(xy) + [x, y] \in Z(R)$ for all $x, y \in L$.

Theorem 4.4.26 Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero square closed Lie ideal of $R$. Let $D: R \times R \rightarrow R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f(x \circ y) = f(xy) = x \circ y \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

Proof Suppose that
\[
f(x \circ y) - f(xy) - x \circ y \in Z(R) \text{ for all } x, y \in L. \quad (4.4.62)
\]
Replacing $y$ by $y + z$ in (4.4.62), we have $f(x \circ y) + f(x \circ z) + 2D(x \circ y, x \circ z) - f(xy) - f(xz) - 2D(xy, xz) - x \circ y - x \circ z \in Z(R)$ for all $x, y, z \in L$. Application of (4.4.62) yields that $2(D(x \circ y, x \circ z) - D(xy, xz)) \in Z(R)$ for all $x, y, z \in L$. Since $R$ is 2-torsion free, we have $D(x \circ y, x \circ z) - D(xy, xz) \in Z(R)$ for all $x, y, z \in L$. If we substitute $y$ for $z$, then we find $f(x \circ y) - f(xy)$ for all $x, y \in L$. In view of (4.4.62), we get $x \circ y \in Z(R)$ for all $x, y \in L$. This implies that $L \subseteq Z(R)$ by Lemma 4.4.2.
Theorem 4.4.27  Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero Lie ideal of $R$. Let $D : R \times R \to R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f(x)f(y) + [x,y] \in Z(R)$ for all $x,y \in L$, then $L \subseteq Z(R)$.

Proof  Suppose

$$f(x)f(y) - [x,y] \in Z(R) \text{ for all } x,y \in L. \quad (4.4.63)$$

Substituting $y + z$ for $y$ in (4.4.63), we have

$$f(x)f(y) + f(x)f(z) + 2f(x)D(y, z) - [x, y] - [x, z] \in Z(R) \text{ for all } x,y,z \in L. \quad (4.4.64)$$

Using (4.4.63), we find

$$2f(x)D(y, z) \in Z(R) \text{ for all } x,y,z \in L. \quad (4.4.65)$$

Since $R$ is of 2-torsion free, we have

$$f(x)D(y, z) \in Z(R) \text{ for all } x,y,z \in L. \quad (4.4.66)$$

In particular if we replace $z$ by $y$ in (4.4.66), then

$$f(x)f(y) \in Z(R) \text{ for all } x,y \in L. \quad (4.4.67)$$

Comparing (4.4.67) and (4.4.63), we obtain $[x,y] \in Z(R)$ for all $x,y \in L$ i.e. $[L,L] \subseteq Z(R)$. Application of Lemma 4.4.1 completes the proof. The proof is same for the case $f(x)f(y) + [y, x] \in Z(R)$ for all $x, y \in L$.

Theorem 4.4.28  Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero Lie ideal of $R$. Let $D : R \times R \to R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f(x)f(y) + [y, x] \in Z(R)$ for all $x,y \in L$, then $L \subseteq Z(R)$.

Proof  The proof runs on the parallel lines as those of Theorem 4.4.27.

Theorem 4.4.29  Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero Lie ideal of $R$. Let $D : R \times R \to R$ be a symmetric biadditive mapping and $f$ be the
trace of $D$. If $f(x)f(y) + xy \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

**Proof** Let

\[ f(x)f(y) - xy \in Z(R) \text{ for all } x, y \in L. \quad (4.4.68) \]

Substituting $y + z$ for $y$ in (4.4.68), we have

\[ f(x)f(y) + f(x)f(z) + 2f(x)D(y, z) - xy - xz \in Z(R) \text{ for all } x, y, z \in L. \quad (4.4.69) \]

Applying (4.4.68), we obtain

\[ 2f(x)D(y, z) \in Z(R) \text{ for all } x, y, z \in L. \quad (4.4.70) \]

Since $R$ is 2-torsion free, we have

\[ f(x)D(y, z) \in Z(R) \text{ for all } x, y, z \in L. \quad (4.4.71) \]

In particular if we replace $z$ by $y$ in (4.4.71) and using (4.4.68), we find

\[ f(x)f(y) \in Z(R) \text{ for all } x, y \in L. \quad (4.4.72) \]

This implies that $xy \in Z(R)$ and hence $[x, y] = xy - yx \in Z(R)$ for all $x, y \in L$. An application of Lemma 4.4.1 completes the proof. Similarly we can prove the theorem if $f(x)f(y) + xy \in Z(R)$ for all $x, y \in L$.

**Theorem 4.4.30** Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero Lie ideal of $R$. Let $D : R \times R \rightarrow R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f(x)f(y) + xy \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

**Proof** The proof runs on the parallel lines as those of Theorem 4.4.29.