Chapter 3

Commuting values of generalized derivations

3.1 Introduction

In order to generalize Posner's classical Theorem Lanski [77] established that: Let $R$ be a prime ring, $L$ a noncommutative Lie ideal of $R$ and $d$ be a nonzero derivation of $R$. If $d$ is centralizing on $L$, then either $R$ is commutative or $\text{char } R = 2$ and $R$ satisfies $s_4$, the standard identity in four variables. Many results in literature indicate how the global structure of a ring $R$ is often tightly connected to the behaviour of additive mappings defined on $R$. In [33] Bresar proved that if $d$ and $\delta$ are derivations of $R$ such that $d(x)x - x\delta(x) \in Z(R)$, for all $x \in R$, then either $d = \delta = 0$ or $R$ is commutative. Lee and Wong [83] extended Bresar's result for Lie ideals. As a partial extension of the result of Bresar, Lee and Shiue [85, Theorem 2] proved the following: Let $R$ be a prime ring, $f(x_1, \ldots, x_n)$ a multilinear polynomial over $C$, the extended centroid of $R$ in noncommuting indeterminates and $d : R \to R$ be a nonzero derivation of $R$. If $d(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) \in C$ for all $r_1, \ldots, r_n \in R$ and $f(x_1, \ldots, x_n)$ is not central valued on $RC$, then $\text{char } R = 2$ and $R$ satisfies $s_4$. Continuing the study in section 3.2, we prove that if $R$ is a prime ring of characteristic different from 2 admitting a nonzero generalized derivation $F$ such that $[F(u)u, F(v)v] = 0$ for any $u, v \in f(R) = \{ f(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in R \}$, then $F(x) = cx$ for all $x \in R$; $c \in C$, 50
the extended centroid of $R$ and one of the following holds: (i) $f(x_1, \ldots, x_n)^2$ is central valued on $R$; (ii) $R$ satisfies $s_4$, the standard identity of degree 4.

In section 3.3, we investigate the conditions (i) $[d(x), F(y)] = 0$; (ii) $d(x) \circ F(y) = 0$; (iii) $[d(x), F(y)] \times [x, y] = 0$; (iv) $d(x) \circ F(y) = x \circ y = 0$; (v) $d(x) \circ F(y) = x y = 0$; (vi) $[d(x), F(y)] = x y = 0$; (vii) $d(x) \circ F(y) = [x, y] = 0$; (viii) $d(x) \circ F(y) = [x, y] = 0$, for all $x, y \in I$, a nonzero ideal of a semiprime ring $R$ admitting a generalized derivation $F$ with associated derivation $d$ and prove that $R$ contains a nonzero central ideal.

In [26], Bell proved that if $R$ is a prime ring, $I$ a nonzero left ideal of $R$ with $I[I, I] = (0)$ and $d$ is a derivation of $R$ such that $d(I) \subseteq I$ and $d([x, y]) = 0$ for all $x, y \in I$, then $R$ is commutative. Recently Argac [13] generalized the result showing that if $R$ is a semiprime ring satisfying $d([x, y]) \in Z(R)$ for all $x, y \in I$, a nonzero ideal of $R$, then $R$ contains a nonzero central ideal. In section 3.4, we extend these results to the case when the generalized derivation $F$ acts on a one sided ideal of a semiprime ring $R$.

### 3.2 Commuting values of generalized derivations on multilinear polynomials

Throughout this section, unless specially stated, $K$ denotes a commutative ring with identity, $R$ is a prime $K$-algebra of characteristic different from 2 with centre $Z(R)$, $U$ a two sided Utumi quotient ring and extended centroid $C$. Let $f(x_1, \ldots, x_n)$ be a noncentral polynomial over $K$. We will use the following notation:

$$f(x_1, \ldots, x_n) = x_1 x_2 \ldots x_n + \sum_{\sigma \in S_n, \sigma \neq id} \alpha_{\sigma} x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)}$$

for some $\alpha_{\sigma} \in K$ and $S_n$ the symmetric group of degree $n$. Moreover we denote by $f^d(x_1, \ldots, x_n)$ the polynomial obtained from $f(x_1, \ldots, x_n)$ by replacing each coefficient $\alpha_{\sigma}$ with $d(\alpha_{\sigma})$. Thus $d(f(r_1, \ldots, r_n)) = f^d(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, d(r_i), \ldots, r_n)$,
for all \( r_1, r_2, \ldots, r_n \) in \( R \). Moreover, since \( R \) is a \( K \)-algebra we can assume that \( K \) is a subring of \( C \) and so, for any derivation \( d \), one has \( d(K) \subseteq C \).

Lee and Shiue [85, Theorem 2] proved the following: let \( R \) be a prime ring, \( f(x_1, \ldots, x_n) \) a multilinear polynomial over \( C \), the extended centroid of \( R \) in noncommuting indeterminates and \( d : R \to R \) be a nonzero derivation of \( R \). If \( d(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) \in C \) for all \( r_1, \ldots, r_n \in R \) and \( f(x_1, \ldots, x_n) \) is not central valued on \( RC \), then \( \text{char } R = 2 \) and \( R \) satisfies \( s_4 \).

More recently Argaç and Demir [15] obtained the following result to the case of generalized derivations.

**Theorem 3.2.1** [15, Lemma 3] Let \( R \) be a prime ring, \( f(x_1, \ldots, x_n) \) a noncentral multilinear polynomial over \( C \) in \( n \)-noncommuting indeterminates, and \( F : R \to R \) be a nonzero generalized derivation of \( R \). If \( F(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) \in C \), for all \( r_1, \ldots, r_n \in R \), then either \( \text{char } R = 2 \) and \( R \) satisfies \( s_4 \) or there exists \( b \in C \) such that \( F(x) = bx \) for all \( x \in R \) and \( f(x_1, \ldots, x_n)^2 \) is central valued on \( R \).

These facts in a prime \( K \)-algebra are natural tests which evidence that, if \( d \) is a derivation of \( R \) and \( F \) is a generalized derivation of \( R \), then the sets \( \{d(x)x \mid x \in S\} \) and \( \{F(x)x \mid x \in S\} \) are rather large in \( R \), where \( S \) is either a noncentral Lie ideal of \( R \), or the set of all the evaluations of a noncentral multilinear polynomial over \( K \).

Motivated by these observations we prove the following.

**Theorem 3.2.2** Let \( R \) be a prime ring of characteristic different from 2, \( Z(R) \) the centre of \( R \), \( U \) the two sided Utumi quotient ring of \( R \), \( f(x_1, \ldots, x_n) \) a noncentral multilinear polynomial over \( K \), \( F \) a nonzero generalized derivation of \( R \). Denote \( f(R) \) the set of all evaluations of the polynomial \( f(x_1, \ldots, x_n) \) in \( R \). If \( [F(u)v, F(v)u] = 0 \), for all \( u, v \in f(R) \), then there exists \( c \in U \) such that \( F(x) = cx \), for all \( x \in R \) and
one of the following holds:

(i) \( f(x_1, \ldots, x_n)^2 \) is central valued on \( R \);

(ii) \( R \) satisfies \( s_4 \), the standard identity of degree 4.

To prove the above theorem, we shall require the following lemmas:

**Lemma 3.2.1** [52, Theorem 1.3] Let \( \Gamma \) be an associative ring with identity, \( A \) a central prime algebra over \( \Gamma \) and \( F \) be the field of fractions of \( \Gamma \). Then:

(i) \( A \otimes_\Gamma F \) is a prime algebra over \( F \).

(ii) If \( A \) is finitely generated as an ideal, then \( A \otimes_\Gamma F \) is central over \( F \).

(iii) If the centre of \( A \) is nonzero, then \( 1 \in A \otimes F \) and \( a \otimes F \) is central over \( F \).

**Lemma 3.2.2** [88, Theorem 4] If in a ring \( R \) every multilinear nil polynomial vanishes, then the same holds for \( M_n \), the \( n \times n \) matrix ring over \( R \).

**Lemma 3.2.3** [115, Lemma 2] Let \( D \) be a division ring and \( V_D \) an infinite dimensional vector space over \( D \). Suppose that \( R \) is a dense subring of \( \text{End}_D V \) and that \( \Phi(X) \) is a generalized polynomial with coefficients in \( \text{End}_D V \). If \( f(x_1, \ldots, x_i) \) is a multilinear polynomial such that \( \Phi(f(x_1, \ldots, x_i)) = 0 \) for all \( x_1, \ldots, x_i \in R \), then \( \Phi(x) = 0 \) for all \( x \in \text{End}_D V \).

First we study the case when the generalized derivation \( F \) is inner, induced by the elements \( a, b \in U \), that is, for all \( x \in R \), \( F(x) = ax + xb \). Now we prove the following:

**Proposition 3.2.1** Let \( a, b \) be elements of \( U \). Suppose that

\[
\left[ (af(r) + f(r)b)f(r), (af(s) + f(s)b)f(s) \right] = 0
\]
for all \( r = (r_1, \ldots, r_n) \in R^n \) and \( s = (s_1, \ldots, s_n) \in R^n \). Then \( R \) satisfies a nontrivial generalized polynomial identity, unless when \( a = -b \in C \).

**Proof** Denote by \( T = Q *_C C \{x_1, \ldots, x_n\} \) the free product over \( C \) of the \( C \)-algebra \( U \) and the free \( C \)-algebra \( C \{x_1, \ldots, x_n\} \). Any element of \( T \) is a generalized polynomial with coefficients in \( U \) (We refer the reader to [24] and [43] for more details on generalized polynomial identities). For sake of clearness we denote \( X = (x_1, \ldots, x_n) \), \( Y = (y_1, \ldots, y_n) \) and

\[
\Phi(X, Y) = \left[ (af(X) + f(X)b)f(X), (af(Y) + f(Y)b)f(Y) \right].
\]

Suppose that \( R \) does not satisfy any nontrivial generalized polynomial identity. Thus \( \Phi(X, Y) = 0 \in T \), and by Theorem 1.3.3 it follows that both \( a \in C \) and \( b \in C \). Hence \( R \) satisfies \((a + b)^2[f(X)^2, f(Y)^2] \). Again since \( R \) is not a GPI-ring, we have \((a + b)^2[f(X)^2, f(Y)^2] = 0 \in T \). Finally, since \( a + b \) cannot be a zero divisor and \( f(x_1, \ldots, x_n)^2 \) cannot be central valued on \( R \), we get the required \( a + b = 0 \) conclusion.

**Proposition 3.2.2** Let \( R = M_p(F) \), the ring of \( p \times p \) matrices over the field \( F \) of characteristic different from 2, with \( p > 1 \), \( a, b \) elements of \( R \) such that

\[
\left[ (af(r) + f(r)b)f(r), (af(s) + f(s)b)f(s) \right] = 0
\]

for all \( r = (r_1, \ldots, r_n) \in R^n \) and \( s = (s_1, \ldots, s_n) \in R^n \). Then \( b \in Z(R) \) and one of the following holds:

(i) \( a + b = 0 \);

(ii) \( f(x_1, \ldots, x_n)^2 \) is central valued on \( R \);

(iii) \( R \) satisfies \( s_4 \), the standard identity of degree 4.

**Proof** Since \( f(x_1, \ldots, x_n) \) is not central, by Lemma 3.2.2, there exist \( u_1, \ldots, u_n \in M_p(F) \) and \( \gamma \in F - \{0\} \), such that \( f(u_1, \ldots, u_n) = \gamma e_{kl} \), with \( k \neq l \). Here \( e_{kl} \) denotes the usual matrix unit with 1 in \((k, l)\)-entry and zero elsewhere. Moreover,
since the set \( \{ f(v_1, \ldots, v_n) \mid v_1, \ldots, v_n \in M_p(F) \} \) is invariant under the action of all \( F \)-automorphisms of \( M_p(F) \), then for any \( i \neq j \) there exist \( r_1, \ldots, r_n \in M_p(F) \) such that \( f(r_1, \ldots, r_n) = e_{ij} \).

Say \( b = \sum_{rs} b_{rs} e_{rs} \), where \( b_{rs} \in F \) and \( e_{rs} \) are the usual unit matrices. Assume first that \( b \) is not a diagonal matrix, for example let \( b_{ji} \neq 0 \), for some \( j \neq i \). In (3.2.1) assume that \( f(x_1, \ldots, x_n) = e_{ij} \) and \( f(y_1, \ldots, y_n) = e_{ji} \), then

\[
\begin{bmatrix}
(ae_{ij} + e_{ij} b) e_{ij}, (ae_{ji} + e_{ji} b) e_{ji}
\end{bmatrix} = 0
\]  

(3.2.2)

and right multiplying by \( e_{jj} \) it follows that \(-b_{ij} b_{ji} e_{jj} = 0\), that is \( b_{ji} b_{ij} = 0 \). Since we assume \( b_{ij} \neq 0 \), then \( b_{ij} = 0 \).

Let \( \varphi \) and \( \chi \) be inner automorphisms on \( M_p(F) \) defined respectively as follows:

\[
\varphi(x) = (1 + e_{ij}) x (1 - e_{ij}) = x + e_{ij} x - xe_{ij} - e_{ij} xe_{ij}
\]

\[
\chi(x) = (1 - e_{ij}) x (1 + e_{ij}) = x - e_{ij} x + xe_{ij} - e_{ij} xe_{ij}.
\]

Since the set \( \{ f(r_1, \ldots, r_n) \mid r_i \in R \} \) is invariant under the action of \( \varphi \) and \( \chi \), the elements \( \varphi(b) \) and \( \chi(b) \) must satisfy the same conditions which are satisfied by \( b \). Thus, if denote \( \varphi(b) = \sum b'_{rs} e_{rs} \) and \( \chi(b) = \sum b''_{rs} e_{rs} \), with \( b'_{rs}, b''_{rs} \in F \), we have that both \( b'_{im} b''_{mi} = 0 \) and \( b'_{im} b''_{ml} = 0 \), for all \( l \neq m \).

By easy computation we notice that \( b'_{ji} = b''_{ji} = b_{ji} \neq 0 \), therefore both \( b'_{ij} = 0\) and \( b''_{ij} = 0 \), that is

\[
0 = b'_{ij} = b_{jj} - b_{ii} - b_{ji}
\]

and

\[
0 = b''_{ij} = -b_{jj} + b_{ii} - b_{ji}.
\]

Comparing these last two equalities and since \( \text{char } F \neq 2 \), we get the contradiction \( b_{ji} = 0 \). The previous argument says that \( b \) is a diagonal matrix, \( b = \sum_{rr} b_{rr} e_{rr} \).

Finally, for any \( l \neq m \), consider again the inner automorphism \( \lambda(x) = (1 - e_{lm}) x (1 + \)
As above we notice that $\lambda(b)$ satisfies the same condition of $b$, so that $\lambda(b)$ must be a diagonal matrix. That is the matrix

$$(1 - e_{im})b(1 + e_{im}) = b - e_{im}a + be_{im} - e_{im}b = b + (b_{mm} - b) e_{ml}$$

is diagonal, which implies that $b_{ll} = b_{mm}$ and $b$ is central in $M_p(F)$.

Hence, if we denote $c = a + b$, then $R$ satisfies

$$\left[ cf(x_1, \ldots, x_n)^2, cf(y_1, \ldots, y_n)^2 \right]$$

and of course we may assume $p \geq 3$, if not $R = M_2(F)$ and we obtain one of the required conclusions.

Let $G$ be the additive subgroup generated by the evaluations of $f(x_1, \ldots, x_n)$ in $R$, and notice that $[cx, cy] = 0$ for all $x, y \in G$. By Theorem 1.3.2 and since $\text{char } R \neq 2$, we have that one of the following holds:

(i) $f(x_1, \ldots, x_n)^2$ is central valued on $R$;

(ii) there exists a noncentral Lie ideal $L$ of $R$ such that $L \subseteq G$.

Since in the first case we are done, we consider the only case when $L \subseteq G$. Moreover we recall that, since $\text{char } R \neq 2$, we also have $[R, R] \subseteq L \subseteq G$. Therefore we may assume that

$$\left[ c[x_1, x_2], c[y_1, y_2] \right]$$

is satisfies by $R$. In particular for $[x_1, x_2] = e_{ij}$ and $[y_1, y_2] = e_{ik}$, with $i, j, k$ different indices, and right multiplying (3.2.4) by $e_{jj}$, we obtain $-ce_{ik}ce_{ij} = 0$. By using the same above argument we may prove that $c$ is a central matrix in $M_p(F)$. Thus $c^2[[r_1, r_2], [s_1, s_2]] = 0$, for all $r_1, r_2, s_1, s_2 \in R$. We conclude by proving that if $c \neq 0$, then a contradiction follows. In fact, we remark that $0 \neq c \in Z(R)$ implies $[[r_1, r_2], [s_1, s_2]] = 0$, for all $r_1, r_2, s_1, s_2 \in R$ and in particular, for $[r_1, r_2] = e_{12}$ and $[s_1, s_2] = e_{21}$, we have $0 = e_{11} - e_{22} \neq 0$. Therefore $c$ must be zero and in any case we
obtain one of the conclusions of Proposition.

**Proposition 3.2.3** Let \( R \) be a prime ring of characteristic different from 2, \( a, b \) elements of \( R \) such that

\[
\left[ (af(r) + f(r)b)f(r), (af(s) + f(s)b)f(s) \right] = 0
\]

(3.2.5)

for all \( r = (r_1, \ldots, r_n) \in R^n \) and \( s = (s_1, \ldots, s_n) \in R^n \). Then \( b \in Z(R) \) and one of the following holds:

(i) \( a + b = 0 \);

(ii) \( f(x_1, \ldots, x_n)^2 \) is central valued on \( R \);

(iii) \( R \) satisfies \( s_4 \), the standard identity of degree 4.

**Proof** Also here, for sake of clearness we denote \( X = (x_1, \ldots, x_n) \), \( Y = (y_1, \ldots, y_n) \)

and

\[
\Phi(X,Y) = \left[ (af(X) + f(X)b)f(X), (af(Y) + f(Y)b)f(Y) \right].
\]

By Proposition 3.2.1, \( \Phi(X, Y) \) is a nontrivial generalized polynomial identity for \( R \). Moreover \( U \) and \( U \otimes_C \overline{C} \) are both centrally closed algebras by Lemma 3.2.1 and in case \( C \) is infinite, they satisfy the same generalized polynomial identities. Hence, replacing \( R \) by \( U \) or \( U \otimes_C \overline{C} \), as well as \( C \) is finite or infinite, we may assume, without loss of generality, \( C = Z(R) \) and \( R \) is a \( C \)-algebra centrally closed. By Theorem 1.3.11, \( R \) is a primitive ring which is isomorphic to a dense ring of linear transformations of a vector space \( V \) over \( C \).

Consider the case \( \text{dim}_C(V) = p \), with \( p \) finite positive integer \( \geq 2 \). In this condition \( R \) is a simple ring which satisfies a nontrivial generalized polynomial identity, moreover \( M_p(C) \) satisfies the same generalized identity of \( R \) and we get the conclusion by Proposition 3.2.2.
Let now $\text{dim}_C V = \infty$. Since the set $\{ f(r_1, \ldots, r_n) | r_1, \ldots, r_n \in R \}$ is dense in $R$, from (3.2.5) we have
\[ (ar + rb)r, (as + sb)s = 0 \] (3.2.6)
for all $r, s \in R$ by Lemma 3.2.3. We will prove that in this case $a = -b \in C$.

Suppose first that there exists $v \in V$ such that $\{v, bv\}$ are linearly $C$-independent. Then that there exists some vectors $w, u \in V$ such that $\{v, bw, w, u\}$ are linearly $C$-independent.

By the density of $R$ we have that there exist $r, s \in R$ such that
\[ rw = 0, \quad sw = v, \quad ru = v, \quad rv = 0, \quad sv = 0, \quad rbv = v, \quad sbv = u \]
and the following contradiction follows:
\[ 0 = (ar + rb)r, (as + sb)s w = v \neq 0. \]
Therefore $\{v, vb\}$ are linearly $C$-dependent for all $v \in V$ and standard arguments show that $b \in C$. Thus we have that
\[ (a + b)r^2, (a + b)s^2 = 0 \] (3.2.7)
for all $r, s \in R$ and denote $a + b = c$. As above suppose first that there exists $v \in V$ such that $\{v, cv\}$ are linearly $C$-independent. Then there exists some vectors $u \in V$ such that $\{v, cv, u\}$ are linearly $C$-independent and by the density of $R$, there exist $r, s \in R$ such that
\[ ru = 0, \quad su = v, \quad sv = v, \quad rcv = v, \quad rv = v \]
and the following contradiction follows:
\[ 0 = (a + b)r^2, (a + b)s^2 u = cv \neq 0. \]
Therefore $\{v, cv\}$ are linearly $C$-dependent for all $v \in V$, so that $c \in C$. Hence $c^2[r^2, s^2] = 0$ for all $r, s \in R$, forces $c = 0$, since $R$ cannot satisfy any polynomial
As an easy consequence we have the following:

**Corollary 3.2.1** Let $R$ be a prime ring of characteristic different from 2, $a$ be an element of $R$ such that

$$\left[(af(r) - f(r)a)f(r), (af(s) - f(s)a)f(s)\right] = 0$$

for all $r = (r_1, \ldots, r_n) \in R^n$ and $s = (s_1, \ldots, s_n) \in R^n$. Then $a \in Z(R)$.

Now we equipped enough to prove our main result.

**Proof of Theorem 3.2.2** We assume $F(x) = ax + d(x)$, for some $a \in U$ and $d$ derivation on $U$. Since $R$ and $U$ satisfy the same differential identities by Theorem 1.3.7, $U$ satisfies

$$\left[(af(x_1, \ldots, x_n) + d(f(x_1, \ldots, x_n)))f(x_1, \ldots, x_n),
(af(y_1, \ldots, y_n) + d(f(y_1, \ldots, y_n)))f(y_1, \ldots, y_n)\right].$$

That is

$$\left[(af(x_1, \ldots, x_n) + f^d(x_1, \ldots, x_n) + \sum f(x_1, \ldots, d(x_i), \ldots, x_n))f(x_1, \ldots, x_n),
(af(y_1, \ldots, y_n) + f^d(y_1, \ldots, y_n) + \sum f(y_1, \ldots, d(y_j), \ldots, y_n))f(y_1, \ldots, y_n)\right].$$

In case $d$ is an inner derivation of $U$, then there exists $q \in U$ such that

$$d(x) = [q, x]$$

for all $x \in R$ and $U$ satisfies

$$\left[((a + q)f(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)q)f(x_1, \ldots, x_n),
((a + q)f(y_1, \ldots, y_n) - f(y_1, \ldots, y_n)q)f(y_1, \ldots, y_n)\right].$$
In this case, by Proposition 3.2.3 we have that $q \in C$, that is $d = 0$ and $F(x) = ax$ for all $x \in R$, moreover either $f(x_1, \ldots, x_n)^2$ is central valued on $R$ or $R$ satisfies $s_4$.

Thus assume that $d$ is not an inner derivation of $U$. By Theorem 1.3.6 and (3.2.5), $U$ satisfies the generalized polynomial identity

$$\left(af(x_1, \ldots, x_n) + f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, x_i)\right)f(x_1, \ldots, x_n),$$

$$\left(af(y_1, \ldots, y_n) + f^d(y_1, \ldots, y_n) + \sum_j f(y_1, \ldots, y_j)\right)f(y_1, \ldots, y_n)$$

and in particular $U$ satisfies

$$\left(\sum_i f(x_1, \ldots, x_i)\right)f(x_1, \ldots, x_n), \left(\sum_j f(y_1, \ldots, y_j)\right)f(y_1, \ldots, y_n).$$

If replace any $x_i$ with $[r, x_i]$ and any $y_j$ with $[r, y_j]$, for a suitable $r \notin C$, then $U$ satisfies

$$\left[r, f(x_1, \ldots, x_n)\right]f(x_1, \ldots, x_n), \left[r, f(y_1, \ldots, y_n)\right]f(y_1, \ldots, y_n)$$

and by Corollary 3.2.1 a contradiction follows.

### 3.3 Ideals and commuting values of generalized derivations

We begin with the following lemmas which are essential to prove our results.

**Lemma 3.3.1** [29, Theorem 3] Let $R$ be a semiprime ring and $I$ be a nonzero left ideal of $R$. If $R$ admits a derivation which is nonzero on $I$ and centralizing on $I$, then $R$ contains a nonzero central ideal.

**Lemma 3.3.2** Let $R$ be a 2-torsion free semiprime ring and $I$ be a nonzero ideal of $R$. If $d$ is a nonzero derivation of $R$ such that $Id^2(I) = (0)$, then $I \subseteq Z(R)$. 
Proof Suppose that $xd^2(y) = 0$ for all $x, y \in I$. Replacing $y$ by $yz$ and using 2-torsion freeness of $R$, we get $xd(y)d(z) = 0$ for all $x, y, z \in I$. Substitute $yu$ for $y$ to get

$$2xd(y)ud(z) = 0 \text{ for all } x, y, u, z \in I. \quad (3.3.1)$$

Again replacing $u$ by $rx$ in (3.3.1) and using the fact that $R$ is 2-torsion free, we obtain $xd(y)rd(x) = 0$ for all $x, y, z \in I$ and $r \in R$. In particular, we have $xd(x)Rx \in I$. Using semiprimeness of $R$, we find

$$xd(x) = 0 \text{ for all } x \in I. \quad (3.3.2)$$

If we replace $u$ and $x$ by $xr$, left multiplying by $d(x)$ and right multiplying by $x$ in (3.3.1), then we get $d(x)xrd(y)xrd(z)x = 0$ for all $x, y, z \in I$ and $r \in R$. In particular, we have $d(x)xrd(x)xrd(x)x = 0$ for all $x \in I$ and $r \in R$. Since $R$ is semiprime, we obtain

$$d(x)x = 0 \text{ for all } x \in I. \quad (3.3.3)$$

Subtracting (3.3.2) and (3.3.3) to get $[d(x), x] = 0$ for all $x \in I$. Application of Lemma 3.3.1 completes the proof.

Theorem 3.3.1 Let $R$ be a 2-torsion free semiprime ring and $I$ be a nonzero ideal of $R$. If $F$ is a generalized derivation with associated derivation $d$ such that $d(x) \circ F(y) = 0$ for all $x, y \in I$, then $R$ contains a nonzero central ideal.

Proof Suppose that

$$d(x) \circ F(y) = 0 \text{ for all } x, y \in I. \quad (3.3.4)$$

Replacing $y$ by $yz$ in (3.3.4), we have

$$d(x) \circ yd(z) + F(y)[z, d(x)] = 0 \text{ for all } x, y, z \in I. \quad (3.3.5)$$

Substitute $zd(x)$ for $z$ in (3.3.5), we find

$$d(x) \circ yzd^2(x) = 0 \text{ for all } x, y, z \in I. \quad (3.3.6)$$
This implies that
\[d(x)yzd^2(x) = -yzd^2(x)d(x)\text{ for all } x, y, z \in I.\] (3.3.7)

Replacing \(y\) by \(ry\) in (3.3.7) and use (3.3.7), we obtain
\[\[d(x), r\]yzd^2(x) = 0 \text{ for all } x, y, z \in I, \ r \in R.\] (3.3.8)

This implies that \([d(x), r]Ryzd^2(x) = (0)\) for all \(x, y, z \in I\) and \(r \in R\). On simplification, we get \([d(x), y]Rd^2(x)R[d(x), y]Rd^2(x) = (0)\) for all \(x, y \in I\). Since \(R\) is semiprime, we obtain \([d(x), y]d^2(x) = 0\) for all \(x, y \in I\). Linearization yields that
\[\[d(x), y]d^2(x) + [d(z), y]d^2(x) = 0 \text{ for all } x, y, z \in I.\] (3.3.9)

Substitute \(yu\) for \(y\) in (3.3.9) to get
\[\[d(x), y]ud^2(z) + [d(z), y]ud^2(x) = 0 \text{ for all } x, y, z, u \in I.\] (3.3.10)

Replace \(u\) by \(ud^2(x)w[d(z), y]u\) in (3.3.10), we obtain
\[\[d(x), y]ud^2(x)w[d(z), y]ud^2(x) + [d(z), y]ud^2(x)w[d(z), y]ud^2(x) = 0 \text{ for all } x, y, z, w, u \in I.\] (3.3.11)

This implies that
\[\[d(z), y]ud^2(x)w[d(z), y]ud^2(x) = 0 \text{ for all } x, y, z, w, u \in I.\] (3.3.12)

Substitute \(wr\) for \(w\) in (3.3.12) and using semiprimeness of \(R\), we get \([d(z), y]ud^2(x)w = 0\) for all \(y, z, u, w \in I\). Replacing \(w\) by \(r[d(z), y]u\) and using semiprimeness of \(R\), we obtain
\[\[d(z), y]ud^2(x) = 0 \text{ for all } x, y, z, u \in I.\] (3.3.13)

Again replacing \(x\) by \(xv\) in (3.3.13) and use (3.3.13), we find
\[2[d(z), y]ud(x)d(v) = 0 \text{ for all } x, y, z, u, v \in I.\] (3.3.14)

Since \(R\) is 2-torsion free, we have \([d(z), y]ud(x)d(v) = 0\) for all \(x, y, z, u, v \in I\).

On simplification we obtain \([d(z), z]u[d(x), z]w[d(v), v] = 0\) for all \(x, z, u, v, w \in I\).
Substitute $ry$ for $u$ and $w$, we have $[d(z), z]ry[d(x), x]ry[d(v), v] = 0$ for all $x, y, z, v \in I$ and $r \in R$. In particular, we can write $y[d(z), z]ry[d(z), z]ry[d(z), z] = 0$ for all $y, z \in I$. Semiprimeness of $R$ yields that

$$y[d(z), z] = 0 \text{ for all } y, z \in I. \quad (3.3.15)$$

Replace $y$ by $yr$ and simplify to get $[d(z), z]yR[d(z), z]y = 0$ for all $y, z \in I$. Again using semiprimeness of $R$, we have

$$[d(z), z]y = 0 \text{ for all } y, z \in I. \quad (3.3.16)$$

Subtracting (3.3.15) and (3.3.16), we get $[[d(z), z], y] = 0$ for all $y, z \in I$. Application of Lemma 2.2.2 completes the proof.

**Theorem 3.3.2** Let $R$ be a 2-torsion free semiprime ring and $I$ be a nonzero ideal of $R$. If $F$ is a generalized derivation with associated derivation $d$ such that $d(x) o F(y) + x o y = 0$ for all $x, y \in I$, then $R$ contains a nonzero central ideal.

**Proof** Suppose that

$$d(x) o F(y) - x o y = 0 \text{ for all } x, y \in I. \quad (3.3.17)$$

Replacing $y$ by $yz$ in (3.3.17), we have

$$(d(x) o F(y))z + d(x) o yd(z) + F(y)[z, d(x)] - y[z, x] - (x o y)z = 0 \text{ for all } x, y, z \in I. \quad (3.3.18)$$

In view of (3.3.17), (3.3.18) reduces to

$$d(x) o yd(z) + F(y)[z, d(x)] - y[z, x] = 0 \text{ for all } x, y, z \in I. \quad (3.3.19)$$

Replacing $z$ by $zd(x)$ in (3.3.19), we get

$$d(x)yzd^2(x) + yzd^2(x)d(x) - yz[d(x), x] = 0 \text{ for all } x, y, z \in I. \quad (3.3.20)$$

Substitute $ry$ for $y$ in (3.3.20) and use (3.3.20), we find

$$[d(x), r]yzd^2(x) = 0 \text{ for all } x, y, z \in I, r \in R. \quad (3.3.21)$$
This implies that [d(x), r] R yzd^2(x) = (0) for all x, y, z \in I and r \in R. Arguing in the similar manner as we have done in the proof of Theorem 3.3.1, we get the required result.

**Theorem 3.3.3** Let R be a 2-torsion free semiprime ring and I be a nonzero ideal of R. If F is a generalized derivation with associated derivation d such that 
\[ d(x) \circ F(y) + x y = 0 \] for all x, y \in I, then R contains a nonzero central ideal.

**Proof** Suppose that d(x) \circ F(y) + x y = 0 for all x, y \in I. Replacing y by yz, we find
\[ (d(x) \circ F(y))z + d(x) \circ yd(z) + F(y)[z, d(x)] - x y z = 0 \] for all x, y, z \in I. (3.3.22)
This implies that
\[ d(x) \circ yd(z) + F(y)[z, d(x)] = 0 \] for all x, y, z \in I. (3.3.23)
Substitute zd(x) for z in (3.3.23) to get d(x) \circ yz d^2(x) = 0 for all x, y, z \in I. i.e.,
\[ d(x) yz d^2(x) + yz d^2(x) d(x) = 0 \] for all x, y, z \in I. (3.3.24)
Replacing y by ry in (3.3.24) and use (3.3.24), we get [d(x), r] yzd^2(x) = 0 for all x, y, z \in I and r \in R. This implies that [d(x), r] R yzd^2(x) = (0) for all x, y, z \in I and r \in R. Arguing in the similar manner as in the proof of Theorem 3.3.1, we get the required result.

**Theorem 3.3.4** Let R be a 2-torsion free semiprime ring and I be a nonzero ideal of R. If F is a generalized derivation with associated derivation d such that 
\[ d(x) \circ F(y) + [x, y] = 0 \] for all x, y \in I, then R contains a nonzero central ideal.

**Proof** Suppose that d(x) \circ F(y) - [x, y] = 0 for all x, y \in I. Replacing y by yz, we find
\[ (d(x) \circ F(y))z + d(x) \circ yd(z) + F(y)[z, d(x)] - y[x, z] - [x, y] z = 0 \] for all x, y, z \in I. (3.3.25)
This implies that
\[ d(x) \circ yd(z) + F(y)[z, d(x)] - y[x, z] = 0 \quad \text{for all} \quad x, y, z \in I. \]  
(3.3.26)

Following the outline of the proof of Theorem 3.3.2, we get the required result.

**Theorem 3.3.5** Let \( R \) be a 2-torsion free semiprime ring and \( I \) be a nonzero ideal of \( R \). If \( F \) is a generalized derivation with associated derivation \( d \) such that \([d(x), F(y)] = 0\) for all \( x, y \in I \), then \( R \) contains a nonzero central ideal.

**Proof** Suppose that \([d(x), F(y)] = 0\) for all \( x, y \in I \). Replacing \( y \) by \( yz \), we have
\[ [d(x), F(y)]z + F(y)[d(x), z] + [d(x), y]d(z) + y[d(x), d(z)] = 0 \quad \text{for all} \quad x, y, z \in I. \]  
(3.3.27)

This implies that
\[ F(y)[d(x), z] + [d(x), y]d(z) + y[d(x), d(z)] = 0 \quad \text{for all} \quad x, y, z \in I. \]  
(3.3.28)

Again replacing \( z \) by \( zd(x) \) in (3.3.28), we have
\[ F(y)[d(x), z]d(x) + [d(x), y]d(z)d(x) + [d(x), y]zd^2(x) + y[d(x), d(z)]d(x) \]
\[ + yz[d(x), d^2(x)] + y[d(x), z]d^2(x) = 0 \quad \text{for all} \quad x, y, z \in I. \]  
(3.3.29)

Application of (3.3.28), (3.3.29) yields that
\[ [d(x), y]zd^2(x) + yz[d(x), d^2(x)] + y[d(x), z]d^2(x) = 0 \quad \text{for all} \quad x, y, z \in I. \]  
(3.3.30)

Substitute \( ry \) for \( y \) in (3.3.30) to get \([d(x), r]yzd^2(x) = 0\) for all \( x, y, z \in I \) and \( r \in R \). Repeating the same arguments as we have done in the proof of Theorem 3.3.1, we get the required result.

**Theorem 3.3.6** Let \( R \) be a 2-torsion free semiprime ring and \( I \) be a nonzero ideal of \( R \). If \( F \) is a generalized derivation with associated derivation \( d \) such that
\[ [d(x), F(y)] + [x, y] = 0 \text{ for all } x, y \in I, \text{ then } R \text{ contains a nonzero central ideal.} \]

**Proof** Suppose that \( [d(x), F(y)] - [x, y] = 0 \) for all \( x, y \in I \). Replacing \( y \) by \( yz \), we have

\[
[d(x), F(y)]z + F(y)[d(x), z] + [d(x), y]d(z) + y[d(x), d(z)] = 0 \text{ for all } x, y, z \in I. \tag{3.3.31}
\]

This implies that

\[
F(y)[d(x), z] + [d(x), y]d(z) + y[d(x), d(z)] - y[x, z] = 0 \text{ for all } x, y, z \in I. \tag{3.3.32}
\]

Again replacing \( z \) by \( zd(x) \) in (3.3.32), we have

\[
F(y)[d(x), z]d(x) + [d(x), y]d(z)d(x) + y[d(x), d(z)]d(x) + yz[d(x), d(z)]d(x) = 0 \text{ for all } x, y, z \in I. \tag{3.3.33}
\]

In view of (3.3.32), (3.3.33) yields that

\[
[d(x), y]zd(x) + y[d(x), d(x)] + y[d(x), z]d(x) - yz[d(x), d(x)] = 0 \text{ for all } x, y, z \in I. \tag{3.3.34}
\]

Substitute \( ry \) for \( y \) in (3.3.34) to get \( [d(x), r]yd^2(x) = 0 \) for all \( x, y, z \in I \) and \( r \in R \). Repeating the same arguments as we have done in Theorem 3.3.1, we get the required result.

**Theorem 3.3.7** Let \( R \) be a 2-torsion free semiprime ring and \( I \) be a nonzero ideal of \( R \). If \( F \) is a generalized derivation with associated derivation \( d \) such that \([d(x), F(y)] \neq x \circ y = 0 \text{ for all } x, y \in I, \text{ then } R \text{ contains a nonzero central ideal.} \)

**Proof** Let \([d(x), F(y)] - x \circ y = 0 \text{ for all } x, y \in I. \) Replacing \( y \) by \( yz \) we have

\[
[d(x), F(y)]z + F(y)[d(x), z] + [d(x), y]d(z) + y[d(x), d(z)] - y[x, z] = 0 \text{ for all } x, y, z \in I. \tag{3.3.35}
\]

In view of given hypothesis, (3.3.35), yields that

\[
F(y)[d(x), z] + [d(x), y]d(z) + [d(x), y]d(z) - y[z, x] = 0 \text{ for all } x, y, z \in I. \tag{3.3.36}
\]
Following the outline of the proof of Theorem 3.3.6, we have $R$ contains a nonzero central ideal.

**Theorem 3.3.8** Let $R$ be a 2-torsion free semiprime ring and $I$ be a nonzero ideal of $R$. If $F$ is a generalized derivation with associated derivation $d$ such that $[d(x), F(y)] = xy = 0$ for all $x, y \in I$, then $R$ contains a nonzero central ideal.

**Proof** Suppose that $[d(x), F(y)] = xy = 0$ for all $x, y \in I$. Replacing $y$ by $yz$, we have

$$[d(x), F(y)]z + F(y)[d(x), z] + [d(x), y]d(z) + y[d(x), d(z)] - xyz = 0 \quad \text{for all } x, y, z \in I. \quad (3.3.37)$$

This implies that

$$F(y)[d(x), z] + [d(x), y]d(z) + y[d(x), d(z)] = 0 \quad \text{for all } x, y, z \in I. \quad (3.3.38)$$

Following the outline of the proof of Theorem 3.3.5, we have $R$ contains a nonzero central ideal.

### 3.4 Ideals and centralizing values of generalized derivations

In [26], Bell proved the following result: Let $R$ be a prime ring and $I$ be a nonzero left ideal of $R$ such that $I[I, I] = 0$. If $R$ admits a derivation $d$ such that $d(I) \subseteq I$ and $d([x, y]) = 0$ for all $x, y \in I$, then $R$ is commutative. Recently Argac [13] generalized the result considering the condition $d([x, y]) \in Z(R)$ for all $x, y \in I$, a nonzero ideal of a semiprime ring $R$ and prove that $R$ contains a nonzero central ideal.

In the present section we prove that a semiprime ring $R$ admitting a generalized derivation $F$ with associated derivation $d$ contains a nonzero central ideal if one of the following holds:
(i) \( R \) is 2-torsion free and \( F(xy) \in Z(R) \) for all \( x, y \in I \), a nonzero ideal of \( R \) unless \( F(I)I = IF(I) = Id(I) = (0) \);

(ii) \( F(xy) \not\equiv yx \in Z(R) \), for all \( x, y \in I \);

(iii) \( F(xy) \not\equiv [x, y] \in Z(R) \), for all \( x, y \in I \);

(iv) \( F \neq 0 \) and \( F([x, y]) = 0 \), for all \( x, y \in I \);

(v) \( F \neq 0 \) and \( F([x, y]) \in Z(R) \), for all \( x, y \in I \), unless either \( d(Z(R))I = (0) \) or \( Id(I) = (0) \).

We start with the following Lemmas.

**Lemma 3.4.1** [79, Theorem 2] Let \( R \) be a 2-torsion free semiprime ring and \( I \) be a nonzero left ideal of \( R \). If \( d \) and \( g \) are nonzero derivations of \( R \) such that \( dg \) acts as a derivation on \( I \), then \( Id(I) \subseteq A(g(R)) \) and \( Ig(I) \subseteq A(d(R)) \), where \( A(d(R)) \) and \( A(g(R)) \) denotes the annihilators of \( d(R) \) and \( g(R) \) respectively.

**Lemma 3.4.2** Let \( R \) be a semiprime ring and \( I \) be a nonzero left ideal of \( R \). Let \( F \) be a generalized derivation of \( R \) with associated derivation \( d \). If \( F(xy) = 0 \), for all \( x, y \in I \), then \( F(I)I = IF(I) = Id(I) = (0) \).

**Proof** Since \( I^2 \neq (0) \) (see Lemma 2.2.1), we assume \( F \neq id(R) \), the identity map on \( R \). Let \( x, y, z \in I \). Then \( 0 = F(xy)z + xyd(z) = xyd(z) \). Replace \( y \) with \( ry \), for any \( r \in R \) and get \( xryd(z) = 0 \), that is \( IRId(I) = (0) \) and so \( I(d(I)R)Id(I) = (0) \). By the semiprimeness of \( R \) it follows \( Id(I) = (0) \) and we also easily get \( F(I)I = (0) \). Finally by Proposition 2.2.1 it follows \( IF(I) = (0) \).

The following example illustrates that \( F(I)I = IF(I) = Id(I) = (0) \) but \( d(I) \neq (0) \).
Example 3.4.1 Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}_2 \right\}$ and $I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{Z}_2 \right\}$. Define $F : R \to R$ such that 
\[ F\left( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}. \]
Then $F$ is a generalized derivation with associated derivation $d : R \to R$ such that $d\left( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. It can be easily verified that $F(xy) = 0$ for all $x, y \in I$ and $F(I)I = IF(I) = Id(I) = (0)$ but $d(I) \neq (0)$ and $R$ has no nonzero central ideal.

Theorem 3.4.2 Let $R$ be a 2-torsion free semiprime ring and $I$ be a nonzero left ideal of $R$. Let $F$ be a generalized derivation of $R$ with associated derivation $d$. If $d(I) \neq (0)$ and $F(xy) \in Z(R)$, for all $x, y \in I$, then $R$ contains some nonzero central ideal, unless $F(xy) = 0$ and $F(I)I = IF(I) = Id(I) = (0)$.

Proof If $F = id(R)$, the identity map on $R$, then $xy \in Z(R)$, for all $x, y \in I$. Commuting this with $x$, we have $x[y, x] = 0$, for all $x, y \in I$. Left multiplying by $y$, we get $yx[y, x] = 0$. On the other hand, by following the same argument we get $xy[x, y] = 0$, that is $0 = [x, y]^2 = (xy - yx)^2 \in Z(R)$. By Lemma 2.2.1 we have $[x, y] = 0$, for all $x, y \in I$ and using Lemma 2.2.3 we conclude that $I$ is a central nonzero ideal of $R$. Thus consider $F \neq id(R)$. By our hypothesis, $F(x)z + xd(z) \in Z(R)$, for all $x, z \in I$. Replacing $x$ by $xy$ in this relation, we get

\[ F(xy)z + xyd(z) \in Z(R). \quad (3.4.1) \]

If $x_0, y_0 \in I$ such that $F(x_0y_0) = 0$ then we have $x_0y_0d(z) \in Z(R)$, for all $z \in I$, that is

\[ [x_0y_0d(z), R] = (0). \quad (3.4.2) \]

Consider now $x_1, y_1 \in I$ such that $0 \neq F(x_1y_1) \in Z(R)$ and by (3.4.1)

\[ [x_1y_1d(z), r] + F(x_1y_1)[z, r] = 0 \quad (3.4.3) \]
for all \( z \in I \) and \( r \in R \). Moreover, by commuting (3.4.3) with \([z, r]\) we have
\[
[x_1y_1d(z), r][z, r] = 0.
\] (3.4.4)

Denote now \( \delta_1 \) and \( \delta_2 \) the inner derivations of \( R \) induced by \( x_1y_1d(z) \) and \( z \) respectively, thus \([\delta_1(r), \delta_2(r)] = 0 \) for all \( r \in R \). By Lemma 3.4.1, it follows that \( \delta_1(R)\delta_2(R) = \delta_2(R)\delta_1(R) = (0) \), that is
\[
[x_1y_1d(z), r][z, s] = 0 \quad \text{for all} \quad r, s \in R.
\] (3.4.5)

Therefore, in light of (3.4.2) and (3.4.5), we have that for all \( x, y, z \in I \) and \( r, s \in R \) the following holds:
\[
[xyd(z), r][z, s] = 0.
\] (3.4.6)

Replacing in (3.4.6) \( x \) with \( ux \), for any \( u \in R \), it follows \( 0 = [uxyd(z), r][z, s] = [u, r]xyd(z)[z, s] = 0 \), that is \([R, R]xyd(z)[z, R] = (0) \). Thus by Proposition 2.2.2, either \( R \) contains some nonzero central ideal or \( xyd(z)[z, R] = (0) \). In this last case we easily get \( xyd(z)R[z, R] = (0) \), for all \( x, y, z \in I \). Take now the family \( \{P_\alpha\} \) of prime ideals of \( R \) such that \( \cap P_\alpha = (0) \). Thus, by using Lemma 2.2.1, for any \( P_\alpha \) we have that either \( xyd(z) \in P_\alpha \), or \([z, R] \subseteq P_\alpha \). Notice that in this case \( xyd(z) \in P_\alpha \), then \( F(xy)[z, R] \subseteq P_\alpha \) follows from (3.4.3). Moreover if \( F(xy) = 0 \) for all \( x, y \in I \), then the conclusion of Theorem follows from Lemma 3.4.2, so that we may assume \( 0 \neq F(x_1y_1) = \beta \in Z(R) \). Therefore in any case \([\beta z, R] \subseteq P_\alpha \) for all \( \alpha \), that is \([\beta z, R] \subseteq \cap P_\alpha = (0) \), and so \( \beta I \) is a central ideal of \( R \).

**Corollary 3.4.1** Let \( R \) be a 2-torsion free semiprime ring and \( I \) be a nonzero left ideal of \( R \). Let \( F \) be a generalized derivation of \( R \) with associated derivation \( d \). If \( d(I) \neq (0) \) and \( F(xy) - xy \in Z(R) \), for all \( x, y \in I \), then \( R \) contains some nonzero central ideal, unless \((F(x) - x)I = I(F(x) - x) = (0)\), for all \( x \in I \) and \( Id(I) = (0) \).

**Proof** It is sufficient to define the generalized derivation \( G \) of \( R \) as follows: \( G(x) = F(x) - x \) for all \( x \in R \). Thus \( G(xy) \in Z(R) \) for all \( x, y \in I \) and the result follows from Theorem 3.4.2.
Theorem 3.4.3 Let $R$ be a semiprime ring and $I$ be a nonzero left ideal of $R$. Let $F$ be a generalized derivation of $R$, with associated derivation $d$. If $d(I) \neq (0)$ and $F(xy) - yx \in Z(R)$, for all $x, y \in I$, then $R$ contains some nonzero central ideal.

Proof Of course in case $F = 0$, $yx \in Z(R)$ for all $x, y \in I$ and we conclude by the same above argument (see the first part of Theorem 3.4.2). Thus we consider $F \neq 0$. Let $x, y, z$ be any elements of $I$, then by our hypothesis we have both

$$F(x(yz)) - (yz)x \in Z(R) \quad (3.4.7)$$

and

$$F((xy)z) - z(xy) \in Z(R). \quad (3.4.8)$$

Comparing (3.4.7) with (3.4.8) we get $zxy - yzx \in Z(R)$, that is $[y, zx] \in Z(R)$ for all $x, y, z \in I$. Assume first that there exist $x_0, z_0 \in I$ such that $[x_0 z_0, I] \neq (0)$. Hence, if denote by $\delta$ the inner derivation of $R$, induced by the element $x_0 z_0$, one has that $\delta(I) \neq (0)$. Moreover $\delta(I) \subseteq Z(R)$ and a fortiori $[\delta(y), y] = 0$ for all $y \in I$. By Lemma 2.2.2, we conclude that $R$ must contain some nonzero central ideal. On the other hand, in case $[xz, I] = (0)$, for all $x, z \in I$, we have that $[I^2, I] = (0)$ implying that $[I^2, I^2] = (0)$, that is $I^2$ is a commuting one sided ideal of $R$. In this case, by Lemma 2.2.3, $I^2 \subseteq Z(R)$ and we are done again.

The same proof can be used in the following:

Theorem 3.4.4 Let $R$ be a semiprime ring and $I$ be a nonzero left ideal of $R$. Let $F$ be a generalized derivation of $R$, with associated derivation $d$. If $d(I) \neq (0)$ and $F(xy) + yx \in Z(R)$, for all $x, y \in I$, then $R$ contains some nonzero central ideal.

Theorem 3.4.5 Let $R$ be a semiprime ring and $I$ be a nonzero left ideal of $R$. Let $F$ be a generalized derivation of $R$, with associated derivation $d$. If $d(I) \neq (0)$
and $F(xy) - [x, y] \in Z(R)$, for all $x, y \in I$, then $R$ contains some nonzero central ideal.

**Proof** By the hypothesis we have that $F(xy) - xy + yx \in Z(R)$, for all $x, y \in I$. Denote by $G : R \to R$, the following generalized derivation of $R$: $G(r) = F(r) - r$, for all $r \in R$. Thus we have that $G(xy) + yx \in Z(R)$ for all $x, y \in I$ and the conclusion follows from Theorem 3.4.4.

**Theorem 3.4.6** Let $R$ be a semiprime ring and $I$ be a nonzero left ideal of $R$. Let $F$ be a generalized derivation of $R$, with associated derivation $d$. If $d(I) \neq (0)$ and $F(xy) + [x, y] \in Z(R)$, for all $x, y \in I$, then $R$ contains some nonzero central ideal.

**Proof** By the hypothesis we have that $F(xy) + xy - yx \in Z(R)$, for all $x, y \in I$. Denote by $C : R \to R$, the following generalized derivation of $R$: $G(r) = F(r) + r$, for all $r \in R$. Thus we have that $G(xy) - yx \in Z(R)$ for all $x, y \in I$ and the conclusion follows from Theorem 3.4.3.

Theorem 3.4.2 - Theorem 3.4.6 can not hold for arbitrary rings as can be easily seen by Example 3.4.1. We finally conclude our section with the following:

**Theorem 3.4.7** Let $R$ be a semiprime ring and $I$ be a nonzero left ideal of $R$. Let $F$ be a generalized derivation of $R$, with associated derivation $d$. If $Id(I) \neq (0)$ and $F([x, y]) = 0$, for all $x, y \in I$, then $[I, I]d(I) = (0)$ and there exists $0 \neq \alpha \in Z(R)$ such that $\alpha I \subseteq Z(R)$.

**Proof** By the hypothesis, for all $x, y \in I$, we have

$$0 = F([x, y]) = F([x, y]x) = [x, y]d(x) \quad (3.4.9)$$

Replacing in (3.4.9), $y$ by $ry$, for any $r \in R$, it follows

$$0 = [x, ry]d(x) = [x, r]yd(x) \quad (3.4.10)$$
so that, for all \( r, s \in R \)

\[
0 = [x, rs]yd(x) = r[x, s]yd(x) + [x, r]syd(x) = [x, r]syd(x) \quad (3.4.11)
\]

that is

\[
[x, R]Ryd(x) = (0), \quad \text{for all } x, y \in I. \quad (3.4.12)
\]

Moreover, since \( R \) is semiprime, it follows

\[
[x, R]Id(x) = (0), \quad \text{for all } x \in I. \quad (3.4.13)
\]

Take now the family \( \{P_a\} \) of prime ideals of \( R \) such that \( \cap P_a = (0) \). Let \( x_1 \) be a fixed element of \( I \), thus for any \( x_2 \in I \) and by (3.4.13), it follows

\[
0 = [x_1 + x_2, R]Id(x_1 + x_2) = [x_1, R]Id(x_2) + [x_2, R]Id(x_1). \quad (3.4.14)
\]

Moreover, for any \( P_a \), by (3.4.12) and Lemma 2.2.1, either \( Id(x_1) \subseteq P_a \), or \( [x_1, R] \subseteq P_a \). In case \( Id(x_1) \subseteq P_a \), then a fortiori \( [x_2, R]Id(x_1) \subseteq P_a \). On the other hand, if \( [x_1, R] \subseteq P_a \), then by (3.4.14) it follows again \( [x_2, R]Id(x_1) \subseteq P_a \). Therefore in any case \( [I, R]Id(I) \subseteq P_a \), for any \( a \). This implies

\[
[I, R]Id(I) \subseteq \cap P_a = (0). \quad (3.4.15)
\]

In particular we get \( [I, I]RId(I) = (0) \) and from this we also have \( [I, I]d(I)R[I, I]d(I) = (0) \). Hence, by the semiprimeness of \( R \) one has

\[
[I, I]d(I) = (0). \quad (3.4.16)
\]

Moreover, for any \( r, s \in R, x, y, z \in I \) and by (3.4.9) it follows:

\[
0 = [rx, s]yd(z) = [r, s]xyd(z) \quad (3.4.17)
\]

and replacing \( y \) with \( ty \) in (3.4.17), for any \( t \in R \), we get \( [r, s]xtyd(z) = 0 \), that is \( [R, R]IRId(I) = (0) \). Again by the semiprimeness of \( R \), it follows \( [R, R]Id(I) = (0) \) and a fortiori \( [R, R]RId(I) = (0) \). This last implies easily that \( Id(I), R]RId(I), R] = (0) \), that is \( Id(I) \subseteq Z(R) \). Therefore for all \( x, y, z \in I \), we have \( xd(z)y + xzd(y) = \)
xd(zy) ∈ Z(R), and since xd(zy) ∈ Z(R), it follows that also xd(x)y ∈ Z(R), for any x, y, z ∈ I. Moreover, by \( I d(I) \neq (0) \), there exist \( x_0, z_0 \in I \) such that \( 0 \neq x_0 d(z_0) = \alpha \in Z(R) \). Hence, for all \( y \in I \), we get \( \alpha y \in Z(R) \), that is \( \alpha I \subseteq Z(R) \), as required.

**Theorem 3.4.8** Let \( R \) be a semiprime ring and \( I \) be a nonzero left ideal of \( R \). Let \( F \) be a generalized derivation of \( R \), with associated derivation \( d \). If \( I d(I) \neq (0) \) and \( F([x, y]) \in Z(R) \), for all \( x, y \in I \), then either \( d(Z(R))I = (0) \) or there exists \( 0 \neq \beta \in Z(R) \) such that \( \beta I \subseteq Z(R) \).

**Proof** Of course, in case \( Z(R) = (0) \), we end up by Theorem 3.4.7. Thus we may assume \( Z(R) \neq (0) \), moreover let \( 0 \neq \alpha \in Z(R) \) such that \( d(\alpha)I \neq (0) \). By the main assumption, for all \( x, y \in I \), we have

\[
F\left( [x, \alpha y] \right) = \alpha F\left( [x, y] \right) + [x, y]d(\alpha) \in Z(R)
\]

that is \( [x, y]d(\alpha) \in Z(R) \) and a fortiori \( [d(\alpha)I, d(\alpha)I] \subseteq Z(R) \). Let \( x_0 \in I \) and denote \( J = d(\alpha)I \neq (0) \) and \( q_0 = d(\alpha)x_0 \in J \). Thus we have that for all \( y \in J \), \( [q_0, y] \in Z(R) \) and so \( [q_0, y]_2 = 0 \). As a reduction of Lemma 2.2.2, it follows that \( R \) contains some nonzero central ideal, unless \( [x, y] = 0 \) for all \( x, y \in J \). This last case implies that \( J = d(\alpha)I \subseteq Z(R) \) (see Lemma 2.2.3).