Chapter 2

Commuting generalized derivations

2.1 Introduction

Let $S$ be a nonempty subset of a ring $R$. A mapping $f : R \rightarrow R$ is said to be centralizing (resp. commuting) on $S$ if for all $x \in S$, $[f(x), x] \in Z(R)$ (resp. $[f(x), x] = 0$). The first important result on commuting mappings is due to E. C. Posner [98, Theorem 2], which states that the existence of a nonzero commuting derivation on a prime ring $R$ forces $R$ to be commutative. The theorem has been extremely influential and it initiated the study of commuting derivations. A lot of work has been done on commuting derivations for example [29], [32], [33], [37], [94], [95], [96], where further references can be found. It is interesting to weaken the hypothesis of the results on commuting derivations in rings and obtain them on some well behaved subsets of rings. In section 2.2, we study commuting (skew commuting) generalized derivations on a nonzero left ideal of a semiprime ring.

In 1993 Q. Deng [48] defined $n$-centralizing and $n$-commuting mappings, a concept more general than centralizing and commuting mappings. Let $n$ be a fixed positive integer. In the mentioned paper Deng proved that if $R$ is a prime ring of characteristic either zero or $> n$, $I$ a nonzero left ideal of $R$ and $d$ is a nonzero derivation of $R$ which is $n$-centralizing on $I$, then $R$ is commutative. In section 2.3, we obtain above cited result for $n$-centralizing generalized derivation of a prime ring $R$.
in the setting of a left ideal of $R$. Finally we extend the result for a Lie ideal of a prime ring.

### 2.2 Ideals and commuting generalized derivations

**Definition 2.2.1 (Centralizing and commuting mappings)** Let $S$ be a nonempty subset of a ring $R$. A mapping $f : R \to R$ is said to be centralizing (resp. commuting) on $S$ if for all $x \in S$ $[f(x), x] \in Z(R)$ (resp. $[f(x), x] = 0$).

The two basic and obvious examples of commuting maps are the identity map and every map having its range in $Z(R)$.

**Example 2.2.1** Consider a ring $R = R_1 \oplus R_2$, where $R_1$ is a noncommutative ring having a nonzero derivation $d_1$ and $R_2$ is a commutative domain. Then $R$ is a noncommutative ring and $d : R \to R$ defined by $d(x_1, x_2) = (d_1(x_1), 0)$ is a nonzero commuting derivation on $R$.

**Example 2.2.2** Let $R$ be a 3-dimensional algebra over a field of characteristic 2, with basis $\{u_0, u_1, u_2\}$ and multiplication defined by

$$u_iu_j = \begin{cases} u_0 & \text{if } (i, j) = (1, 2) \\ 0 & \text{otherwise} \end{cases}$$

Let $d$ be the linear transformation on $R$ defined by $d(u_0) = 0$, $d(u_1) = u_1$, $d(u_2) = u_2$. It is easily verified that $d$ is a centralizing derivation on $R$.

**Definition 2.2.2 (Skew-centralizing and Skew-commuting mappings)** Let $S$ be a nonempty subset of a ring $R$. A mapping $f : R \to R$ is said to be skew-centralizing (resp. skew-commuting) on $S$ if for all $x \in S$ $f(x) \circ x \in Z(R)$ (resp. $f(x) \circ x = 0$).
Example 2.2.3 Let $S$ be any commutative ring and

$$R = \left\{ \begin{pmatrix} w & 0 & 0 \\ x & w & 0 \\ y & z & w \end{pmatrix} \mid w, x, y, z \in S \right\}.$$ Define $f : R \to R$ by

$$f \begin{pmatrix} w & 0 & 0 \\ x & w & 0 \\ y & z & w \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y & 0 & 0 \end{pmatrix}.$$ Then $f$ is skew-centralizing on $R$.

Several authors have proved commutativity of prime and semiprime rings $R$ admitting automorphisms or derivations which are centralizing on appropriate subsets of $R$. The culminating theorem in this series due to Mayne [93], asserts that if a prime ring $R$ admits either a nonidentity automorphism or a nonzero derivation which is centralizing (commuting) on some nonzero ideal $I$ of $R$, then $R$ is commutative. Further Bell and Martindale [29, Theorem 3] proved that a semiprime ring $R$ must have a nontrivial central ideal if it admits an appropriate endomorphism or a derivation which is centralizing on some nontrivial one sided ideal. We obtain the theorem of Bell and Martindale for a generalized derivation commuting on a left ideal of $R$.

Theorem 2.2.1 Let $R$ be a semiprime ring and $I$ be a nonzero left ideal of $R$. Let $F$ be a generalized derivation with associated derivation $d$ such that $Id(I) \neq (0)$. If $F$ is commuting on $I$, then $[I, I]d(I) = (0)$ and there exists $0 \neq \alpha \in Z(R)$ such that $\alpha I \subseteq Z(R)$.

For developing the proof we require the following lemmas:

Lemma 2.2.1 [72, Chapter 4] In a semiprime ring $R$

(i) The centre of $R$ contains no nonzero nilpotent elements.

(ii) $R$ does not contain any nonzero nilpotent left ideals.

(iii) If $P$ is a nonzero prime ideal of $R$ and $a, b \in R$ such that $aRb \subseteq P$, then either
Lemma 2.2.2 [78, Theorem] Let $R$ be a semiprime ring and $I$ be a nonzero left ideal of $R$. Let $d$ be a derivation on $R$. If for some positive integers $t_0, t_1, t_2, \ldots, t_n$ and all $x \in I$, the identity $[[[d(x^{t_0}), x^{t_1}], \ldots], x^{t_n}] = 0$ holds, then either $d(1) = 0$ or else $d(I)$ and $d(R)I$ are contained in a nonzero central ideal of $R$. In particular when $R$ is a prime ring, $R$ is commutative.

Proof of Theorem 2.2.1 We have $[F(x), x] = 0$ for all $x \in I$. Linearization yields that

\[ [F(x), y] + [F(y), x] = 0 \text{ for all } x, y \in I. \quad (2.2.1) \]

Replacing $y$ by $yx$ in (2.2.1), we have

\[ y[F(x), x] + [F(x), y]x + [F(y), x]x + y[d(x), x] + [y, x]d(x) = 0 \text{ for all } x, y \in I. \quad (2.2.2) \]

Using (2.2.1), we have

\[ y[d(x), x] + [y, x]d(x) = 0 \text{ for all } x, y \in I. \quad (2.2.3) \]

Substituting $ry$ for $y$ in (2.2.3), we have

\[ ry[d(x), x] + r[y, x]d(x) + [r, x]yd(x) = 0 \text{ for all } x, y \in I, r \in R. \quad (2.2.4) \]

Application of (2.2.3), yields that

\[ [r, x]yd(x) = 0 \text{ for all } x, y \in I, r \in R. \quad (2.2.5) \]

Replacing $y$ by $ry$ in (2.2.5), we get

\[ [r, x]ryd(x) = 0 \text{ for all } x, y \in I, r \in R. \quad (2.2.6) \]

By (2.2.5), we have $[x, R]Id(x) = 0$ for all $x \in I$. Take the family $\{P_{\alpha}\}$ of prime ideals of $R$ such that $\cap P_{\alpha} = (0)$. Let $x_1$ be a fixed element of $I$, thus for any $x_2 \in I$ and by (2.2.5), it follows

\[ (0) = [x_1 + x_2, R]Id(x_1 + x_2) = [x_1, R]Id(x_2) + [x_2, R]Id(x_1). \quad (2.2.7) \]
Moreover, for any $P_a$, by (2.2.6) and Lemma 2.2.1 (iii), either $Id(x_1) \subseteq P_a$, or $[x_1, R] \subseteq P_a$. In case $Id(x_1) \subseteq P_a$, then a fortiori $[x_2, R]Id(x_1) \subseteq P_a$. On the other hand, if $[x_1, R] \subseteq P_a$, then by (2.2.7) it follows again $[x_2, R]Id(x_1) \subseteq P_a$. Therefore in any case $[I, R]Id(I) \subseteq P_a$, for any $\alpha$. This implies

$$[I, R]Id(I) \subseteq \cap P_a = (0).$$

(2.2.8)

In particular we get $[I, I]RId(I) = (0)$ and from this we also have $[I, I]d(I)R[I, I]d(I) = (0)$. Hence, by the semiprimeness of $R$ we get

$$[I, I]d(I) = (0).$$

(2.2.9)

Moreover, for any $r, s \in R$, $x, y, z \in I$ and by (2.2.9) it follows

$$0 = [rx, s]yd(z) = [r, s]xyd(z)$$

(2.2.10)

and replacing $y$ with $ty$ in (2.2.10), for any $t \in R$, we get $[r, s]xtyd(z) = 0$, that is $[R, R]Id(I) = (0)$. Again by the semiprimeness of $R$, it follows $[R, R]Id(I) = (0)$ and a fortiori $[R, R]RId(I) = (0)$. This last implies easily that $[I, R]d(I) = (0)$, that is $Id(I) \subseteq Z(R)$.

Therefore for $x, y, z \in I$, we have $xd(z)y + xzd(y) = xd(zy) \in Z(R)$, and since $xd(zy) \in Z(R)$, it follows that also $xd(z)y \in Z(R)$, for any $x, y, z \in I$. Moreover, by $Id(I) \neq (0)$, there exist $x_0, z_0 \in I$ such that $0 \neq x_0d(z_0) = \alpha \in Z(R)$. Hence, for all $y \in I$, we get $\alpha y \in Z(R)$, that is $\alpha I \subseteq Z(R)$.

**Theorem 2.2.2** Let $R$ be a semiprime ring and $I$ be a nonzero left ideal of $R$. Let $F$ be a generalized derivation with associated derivation $d$ such that $Id(I) \neq (0)$. If $F$ is skew-commuting on $I$, then $[I, I]d(I) = (0)$ and there exists $0 \neq \alpha \in Z(R)$ such that $\alpha I \subseteq Z(R)$.

**Proof** We have $F(x)x + xF(x) = 0$ for all $x \in I$. Linearization yields that

$$F(x)y + F(y)x + yF(x) + xF(y) = 0 \text{ for all } x, y \in I.$$  

(2.2.11)
Replacing $y$ by $yx$ in (2.2.11), we have

$$(F(x)y + F(y)x + xF(y))x + yd(x)x + xyd(x) + yxF(x) = 0 \quad \text{for all } x, y \in I. \quad (2.2.12)$$

Comparing (2.2.11) and (2.2.12), we get

$$-yF(x)x + yxF(x) + yd(x)x + xyd(x) = 0 \quad \text{for all } x, y \in I. \quad (2.2.13)$$

This implies that

$$y[x, F(x)] + yd(x)x + xyd(x) = 0 \quad \text{for all } x, y \in I. \quad (2.2.14)$$

Replacing $y$ by $ry$ in (2.2.14), we obtain

$$r(y[F(x), x] + yd(x)x) + xryd(x) = 0 \quad \text{for all } x, y \in I, r \in R. \quad (2.2.15)$$

From (2.2.14) and (2.2.15), we have

$$-rxyd(x) + xryd(x) = [x, r]yd(x) = 0 \quad \text{for all } x, y \in I, r \in R. \quad (2.2.16)$$

This implies that $[x, r]Ryd(x) = (0)$ for all $x, y \in I$ and $r \in R$. Arguing in the similar manner as we have done in the proof of above theorem, we get the required result.

An immediate consequence of the above theorems is the following corollary:

**Corollary 2.2.1** Let $R$ be a prime ring and $I$ be a nonzero left ideal of $R$. Let $F$ be a generalized derivation with associated derivation $d$. If either $F$ is commuting or skew commuting on $I$, then $R$ is commutative.

In [27], Bell and Daif showed that if $R$ is a semiprime ring, $I$ is a nonzero ideal of $R$ and $d$ is a derivation of $R$ such that $d([x, y]) = [x, y] = 0$, for all $x, y \in I$, then $I \subseteq Z(R)$. Following this line of investigation, in [99] a similar situation was considered, in case the derivation $d$ is replaced by a generalized derivation $F$. More precisely in [99] following result is proved:
Theorem 2.2.3  Let $R$ be a prime ring with center $Z(R)$, $F$ be a generalized derivation on $R$, $I$ be a nonzero ideal of $R$. Then $R$ is commutative if one of the following holds:

(i) $F([x, y]) = [x, y]$, for all $x, y \in I$;
(ii) $F([x, y]) = -[x, y]$, for all $x, y \in I$;
(iii) $F(xy + yx) = xy + yx$, for all $x, y \in I$;
(iv) $F(xy + yx) = -xy - yx$, for all $x, y \in I$.

More recently in [50] Dhara generalized this result to the semiprime case and proved the following:

Theorem 2.2.4  Let $R$ be a semiprime ring and $F$ be a generalized derivation on $R$, with associated derivation $d$. If $I$ is a nonzero ideal of $R$ such that $d(I) \neq 0$, then $I \subseteq Z(R)$, in case one of the following conditions holds:

(i) $F([x, y]) = \pm [x, y]$, for all $x, y \in I$;
(ii) $F(xy + yx) = \pm (xy + yx)$, for all $x, y \in I$;

More precisely $d(Z(R))I \subseteq Z(R)$.

We extend the previous cited results to the case when the generalized derivation $F$ acts on one sided ideal of $R$ considering the conditions:

(i) $F(x)F(y) - xy = 0$, for all $x, y \in I$;
(ii) $F(x)F(y) - [x, y] = 0$, for all $x, y \in I$;
(iii) $F([x, y]) - [x, y] = 0$, for all $x, y \in I$;
(iv) $F([x, y]) - [x, y] \in Z(R)$, for all $x, y \in I$. 
Lemma 2.2.3 [57, Lemma 1.1.5] If $R$ is a semiprime ring, then the centre of a nonzero one sided ideal is contained in the centre of $R$. Moreover any commutative one sided ideal is contained in the centre of $R$.

Lemma 2.2.4 [119, Lemma 1.3] Let $R$ be a semiprime ring and $a \in R$ some fixed element. If $a[x,y] = 0$ for all $x,y \in R$, then there exists an ideal $I$ of $R$ such that $a \in I \subseteq Z(R)$.

Proposition 2.2.1 If $R$ is a semiprime ring, $I$ is a nonzero left ideal of $R$ and $0 \neq a \in R$, such that $aI = (0)$, then $Ia = (0)$.

Proof Since $a(RI) = (0)$, we have $Ia(RI)a = (0)$ and the conclusion follows from the semiprimeness of $R$.

Proposition 2.2.2 If $R$ is a semiprime ring and $0 \neq a \in R$, such that $[R,R]a = (0)$, then $a[R,R] = (0)$ and there exists a central ideal $I$ of $R$ such that $a \in I$.

Proof Since $(0) = [R,R^2]a = [R,R]Ra$, then a fortiori $a[R,R]Ra[R,R] = (0)$, that is $a[R,R] = (0)$. In this situation we conclude from Lemma 2.2.4.

Theorem 2.2.5 Let $R$ be a semiprime ring and $I$ be a nonzero left ideal of $R$. Let $F$ be a generalized derivation of $R$ with associated derivation $d$. If $Id(I) \neq (0)$ and $F(x)F(y) - xy = 0$ for all $x,y \in I$, then $[I, I]d(I) = (0)$ and there exists $0 \neq \alpha \in Z(R)$ such that $\alpha I \subseteq Z(R)$.

Proof If $F = 0$, then we have $xy = 0$ for all $x,y \in I$. A simple calculation yields that $[x,y] = 0$ for all $x,y \in I$. By application of Lemma 2.2.3, we get the required result. Suppose that $F \neq 0$ and

$$F(x)F(y) - xy = 0 \text{ for all } x,y \in I. \quad (2.2.17)$$

Replacing $y$ by $yz$ in (2.2.17), we obtain

$$F(x)F(y)z + F(x)yd(z) - xyz = 0 \text{ for all } x,y,z \in I. \quad (2.2.18)$$
In view of (2.2.17), (2.2.18) yields that $F(x)yd(z) = 0$ for all $x, y, z \in I$. Substitute $xu$ for $x$ to get

$$F(x)uyd(z) + xd(u)yd(z) = xd(u)yd(z) = 0 \quad \text{for all } x, y, z, u \in I. \quad (2.2.19)$$

Which yields that $[I, I]d(I)R[I, I]d(I) = (0)$. By semiprimeness of $R$, we have $[I, I]d(I) = (0)$. This implies that $[R, R]Rd(I) = (0)$. By semiprimeness of $R$, we find that $[R, R]Id(I) = (0)$ and hence $[R, R]RId(I) = (0)$. This implies that $[Id(I), R]Rd(I, R) = (0)$, that is $Id(I) \subseteq Z(R)$. Therefore for $x, y, z \in I$, we have $xd(z)y + xzd(y) = xd(zy) \in Z(R)$ and since $zd(zy) \in Z(R)$, it follows that also $xd(z)y \in Z(R)$, for any $x, y, z \in I$. Moreover, by $Id(I) \neq (0)$, there exist $x_0, z_0 \in I$ such that $0 \neq x_0d(z_0) = \alpha \in Z(R)$. Hence, for all $y \in I$, we get $\alpha y \in Z(R)$, that is $\alpha I \subseteq Z(R)$.

**Theorem 2.2.6** Let $R$ be a semiprime ring and $I$ be a nonzero left ideal of $R$. Let $F$ be a generalized derivation of $R$ with associated derivation $d$. If $Id(I) \neq (0)$ and $F(x)F(y) - [x, y] = 0$ for all $x, y \in I$, then $[I, I]d(I) = (0)$ and there exists $0 \neq \alpha \in Z(R)$ such that $\alpha I \subseteq Z(R)$.

**Proof** If $F = 0$, then we have $[x, y] = 0$ for all $x, y \in I$. Replacing $y$ by $ry$, we obtain $[x, r]y = 0$ for all $x, y \in I$ and $r \in R$. This implies that $[x, r]R[x, r] = 0$ for all $x \in I$ and $r \in R$. By semiprimeness of $R$, we get $[x, r] = 0$ for all $x \in I$ and $r \in R$ and hence $I \subseteq Z(R)$.

If $F \neq 0$, then we have

$$F(x)F(y) - [x, y] = 0 \quad \text{for all } x, y \in I. \quad (2.2.20)$$

Substitute $yx$ for $y$ in (2.2.20) to get

$$F(x)F(y)x + F(x)yd(x) - [x, y]x = 0 \quad \text{for all } x, y \in I. \quad (2.2.21)$$

In view of (2.2.20), (2.2.21) yields that
\[ F(x)y_d(x) = 0 \text{ for all } x, y \in I. \quad (2.2.22) \]

Replacing \( y \) by \( F(z)y \) in (2.2.22) and using (2.2.20), we have
\[ [z, x]y_d(x) = 0 \text{ for all } x, y, z \in I. \quad (2.2.23) \]

Replacing \( y \) by \( r[y, x] \) in (2.2.23), we obtain
\[ [z, x]r[y, x]d(x) = 0 \text{ for all } x, y, z \in I, \quad r \in R. \quad (2.2.24) \]

If we substitute \( d(x)r \) for \( r \) and \( y \) for \( z \) in (2.2.24), then we get
\[ [y, x]d(x)R[y, x]d(x) = 0 \text{ for all } x, y \in I. \quad (2.2.25) \]

Since \( R \) is semiprime, we have
\[ [x, y]d(x) = 0 \text{ for all } x, y \in I. \quad (2.2.26) \]

Replacing in (2.2.26), \( y \) by \( ry \), for any \( r \in R \), it follows
\[ 0 = [x, ry]d(x) = [x, r]y_d(x) \]
so that, for all \( r, s \in R \)
\[ 0 = [x, rs]y_d(x) = r[x, s]y_d(x) + [x, r]s_yd(x) = [x, r]s_yd(x) \]
that is
\[ [x, R]Ry_d(x) = (0) \text{ for all } x, y \in I. \quad (2.2.27) \]

Moreover, since \( R \) is semiprime, it follows
\[ [x, R]I_d(x) = (0) \text{ for all } x \in I. \quad (2.2.28) \]

Take now the family \( \{P_\alpha\} \) of prime ideals of \( R \) such that \( \cap P_\alpha = (0) \). Let \( x_1 \) be a fixed element of \( I \), thus for any \( x_2 \in I \) and by (2.2.28), it follows
\[ (0) = [x_1 + x_2, R]I_d(x_1 + x_2) = [x_1, R]I_d(x_2) + [x_2, R]I_d(x_1). \quad (2.2.29) \]
Moreover, for any $P_\alpha$, by (2.2.27) and Lemma 2.2.1 (iii), either $\text{Id}(x_1) \subseteq P_\alpha$, or $[x_1, R] \subseteq P_\alpha$. In case $\text{Id}(x_1) \subseteq P_\alpha$, then a fortiori $[x_2, R]\text{Id}(x_1) \subseteq P_\alpha$. On the other hand, if $[x_1, R] \subseteq P_\alpha$, then by (2.2.29) it follows again $[x_2, R]\text{Id}(x_1) \subseteq P_\alpha$. Therefore in any case $[I, R]\text{Id}(I) \subseteq P_\alpha$, for any $\alpha$. This implies

$$[I, R]\text{Id}(I) \subseteq \cap P_\alpha = (0). \quad (2.2.30)$$

In particular we get $[I, I]R\text{Id}(I) = (0)$ and from this we also have $[I, I]\text{Id}(I)R[I, I]\text{Id}(I) = (0)$. Hence, by the semiprimeness of $R$ we get

$$[I, I]\text{Id}(I) = (0). \quad (2.2.31)$$

Moreover, for any $r, s \in R, x, y, z \in I$ and by (2.2.31) it follows

$$0 = [rx, s]yd(z) = [r, s]xyd(z) \quad (2.2.32)$$

and replacing $y$ with $ty$ in (2.2.32), for any $t \in R$, we get $[r, s]txyd(z) = 0$, that is $[R, R]IR\text{Id}(I) = (0)$. Again by the semiprimeness of $R$, it follows $[R, R]\text{Id}(I) = (0)$ and a fortiori $[R, R]R\text{Id}(I) = (0)$. This last implies easily that $[\text{Id}(I), R][I, I]\text{Id}(I), R] = (0)$, that is $\text{Id}(I) \subseteq Z(R)$.

Therefore for $x, y, z \in I$, we have $xd(z)y + xzd(y) = zd(xy) \in Z(R)$, and since $zd(xy) \in Z(R)$, it follows that also $xd(z)y \in Z(R)$, for any $x, y, z \in I$. Moreover, by $\text{Id}(I) \neq (0)$, there exist $x_0, z_0 \in I$ such that $0 \neq x_0d(z_0) = \alpha \in Z(R)$. Hence, for all $y \in I$, we get $\alpha y \in Z(R)$, that is $\alpha I \subseteq Z(R)$, as required.

### 2.3 Ideals and $n$-centralizing generalized derivations

In [48], Q. Deng defined $n$-centralizing and $n$-commuting mappings, concepts more general than centralizing (commuting) mappings.
Definition 2.3.1 \((n\)-Centralizing and \(n\)-commuting mappings\) Let \(n\) be a fixed positive integer. A mapping \(f : R \rightarrow R\) is said to be \(n\)-centralizing (resp. \(n\)-commuting) on a nonempty subset \(S\) of \(R\), if 
\([f(x), x^n] \in Z(R)\) (resp. \([f(x), x^n] = 0\)) for all \(x \in S\).

Example 2.3.1 Let \(R = R_1 \oplus R_2\), where \(R_1\) and \(R_2\) are nonzero rings, \(R_2\) is a commutative ring. Define a map \(d : R \rightarrow R\) such that \(d(x, y) = (0, y)\). Then it can be verified that \(d\) is an \(n\)-commuting derivation on \(R\).

Example 2.3.2 Let \(R = M_2(GF(2))\) denote the Galois field of two elements and \(f : R \rightarrow R\) be defined by \(f \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha + \gamma & 0 \\ 0 & \beta + \delta \end{pmatrix}\) for \(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in R\). Then \(f\) is a \(GF(2)\)-linear map. A direct computation yields that 
\([f(x), x^6] = 0\) for all \(x \in R\). However,
\[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.\] Hence \(f\) is 6-commuting linear map.

In [86] T. K. Lee extended the definition of a generalized derivation as follows: An additive mapping \(F : J \rightarrow U\) such that \(F(xy) = F(x)y + xd(y)\), for all \(x, y \in J\), where \(U\) is the right Utumi quotient ring of \(R\), \(J\) is a dense right ideal of \(R\) and \(d\) is a derivation from \(J\) to \(U\). In the mentioned paper he proved that every generalized derivation of \(R\) can be extended to a generalized derivation of \(U\). In fact there exists \(a \in U\) and a derivation \(d\) of \(U\) such that \(F(x) = ax + d(x)\) for all \(x \in U\) by Theorem 1.3.10. A corresponding form for dense left ideals as follow: An additive mapping \(F : I \rightarrow U\) is called a generalized derivation if there exists a derivation \(d : I \rightarrow U\) such that \(F(xy) = xF(y) + d(x)y\) for all \(x, y \in I\), where \(U\) is a left Utumi quotient ring of \(R\), \(I\) is a dense left ideal of \(R\). Following the same method as in [86], one can extend \(F\) uniquely to a generalized derivation of \(U\). The extended generalized derivation of \(U\) can also be denoted by \(F\) and has the form \(F(x) = xa + d(x)\) for all \(x \in U\), where \(d\) is a derivation of \(U\). We prove a result in spirit of Posner's Theorem.
and the results of Deng [48, Theorem 2], Deng and Bell [49, Theorem 2]. In fact we prove the following:

**Theorem 2.3.1** Let $R$ be a prime ring, $F$ be a nonzero generalized derivation of $R$, $I$ a nonzero left ideal of $R$, $n \geq 1$ a fixed positive integer such that $F$ is $n$-centralizing on the set $[I, I]$. Then there exists $a \in U$ and $\alpha \in C$, the extended centroid of $R$ such that $F(x) = xa$, for all $x \in R$ and $I(a - \alpha) = (0)$ unless when $x_1 s_4(x_2, x_3, x_4, x_5)$ is an identity for $I$.

To prove the above theorem, we assume the conclusion $I$ satisfies $x_1 s_4(x_2, x_3, x_4, x_5)$ of Theorem 2.3.1 is false. Thus there exists $a_1, a_2, a_3, a_4, a_5 \in I$ such that $a_1 s_4(a_2, a_3, a_4, a_5) \neq 0$. Our goal is to ultimately arrive to prove that in this case there exists $a \in U$ such that $F(x) = xa$ for all $x \in R$ and $I[a, I] = (0)$.

**Fact 2.3.1** In all that follows let $T = U \ast_C C\{X\}$ be the free product over $C$ of the $C$-algebra $U$ and the free $C$-algebra $C\{X\}$, with $X$ the countable set consisting of noncommuting indeterminates $\{x_1, x_2, \ldots, x_n, \ldots\}$. The elements of $T$ are called generalized polynomials with coefficients in $U$. $I$, $IR$ and $IU$ satisfy the same generalized polynomials identities with coefficients in $U$. We refer the reader to [25] and [43] for the definitions and the related properties of these objects. Recall that if $B$ is a basis of $U$ over $C$, then any element of $T = U \ast_C C\{x_1, x_2, \ldots, x_n\}$ can be written in the form $g = \sum \alpha_i m_i$, where $\alpha_i \in C$ and $m_i$ are $B$-monomials, that is, $m_i = q_0 y_1 \ldots y_n q_n$, with $q_i \in B$ and $y_i \in \{x_1, x_2, \ldots, x_n\}$. By Theorem 1.3.4 it is shown that a generalized polynomial $g = \sum \alpha_i m_i$ is the zero element of $T$ if and only if any $\alpha_i$ is zero. As a consequence, if $a_1, a_2 \in U$ are linearly independent over $C$ and $a_1 g_1(x_1, x_2, \ldots, x_n) + a_2 g_2(x_1, x_2, \ldots, x_n) = 0 \in T$, for some $g_1, g_2 \in T$, then both $g_1(x_1, x_2, \ldots, x_n)$ and $g_2(x_1, x_2, \ldots, x_n)$ are the zero element of $T$.

**Lemma 2.3.1** [40, Theorem 1] Let $R$ be a prime ring and $I$ a nonzero right ideal
of $R$. If $I$ possesses a central differential identity, then $R$ is a PI-ring.

**Lemma 2.3.2** [41, Theorem 1] Assume that $Q$ possesses a nontrivial idempotents $e = e^2 \neq 0, 1$ and that $A$ is an additive subgroup of $R$ invariant under all special automorphisms of $R$. Then either $A \subseteq Z(R)$ or $A$ contains a proper Lie ideal of $R$, unless $\text{char } R = 2$ and $\dim_{C} RC = 4$.

**Lemma 2.3.3** [45, Main Theorem] Let $R$ be a $K$-algebra without nonzero nil right ideals and let $f(X_1, \ldots, X_t)$ be a multilinear polynomial over $K$, where $K$ is a commutative ring with unity. Suppose that $a \in R$ and $af(x_1, \ldots, x_t)^n = 0$ for all $x_1, \ldots, x_t \in I$, a right (resp. left) ideal of $R$, where $n = n(x_1, \ldots, x_t)$ depends on $x_1, \ldots, x_t$. Then $af(x_1, \ldots, x_t)I = 0$ (resp. $af(x_1, \ldots, x_t) = 0$) for all $x_1, \ldots, x_t \in I$.

**Lemma 2.3.4** [82, Theorem 2] Let $R$ be a semiprime ring and $I$ a right ideal of $R$. Then, for each positive integer $m$, $I^n$ and $I$ satisfy the same GPIs with coefficients in $U$.

**Lemma 2.3.5** [83, Proposition] Let $A$ be an algebra over an infinite field $F$ and $K$ be a field extension over $F$. If $A$ satisfies a GPI $p(X_1, X_2, \ldots, X_m)$, so does $A \otimes_F K$.

**Lemma 2.3.6** Either $R$ is a ring satisfying a nontrivial generalized polynomial identity (GPI) or there exists $a \in U$ such that $F(x) = xa$ for all $x \in R$ and $I(a - \alpha) = (0)$ for some $\alpha \in C$, where $I$ is a nonzero left ideal of $R$ and $C$ the extended centroid of $R$.

**Proof** We know that $F$ assumes the form $F(x) = ax + d(x)$ for all $x \in U$ and some $a \in U$, where $d$ is a derivation on $U$. Suppose $R$ does not satisfy any nontrivial GPI. We divide the proof into two cases:

**Case (i)** Suppose that $d$ is an inner derivation induced by an element $q \in U$. Let
0 ≠ b ∈ I. Since R does not satisfy any nontrivial GPI, then

\[
[[a[x_1b, x_2b] + q[x_1b, x_2b] - [x_1b, x_2b]q, [x_1b, x_2b]]_n, x_3]
\]

(2.3.1)

is the zero element in the free algebra T for all \(x_1, x_2, x_3 ∈ R\) (see Fact 2.3.1), that is

\[
((a + q)[x_1b, x_2b]^{n+1})x_3
\]

\[
+(-[x_1b, x_2b]q[x_1b, x_2b] - [x_1b, x_2b]^{n+1}(a + q)[x_1b, x_2b] + [x_1b, x_2b]^{n+1}q)x_3
\]

\[
-x_3((a + q)[x_1b, x_2b]^{n+1})
\]

\[
+x_3[x_1b, x_2b]q[x_1b, x_2b]^{n}
\]

\[
-x_3(-[x_1b, x_2b]^{n}(a + q)[x_1b, x_2b] + [x_1b, x_2b]^{n+1}q) = 0 ∈ T.
\]

(2.3.2)

If \(a + q ∉ C\), then \(a + q\) and 1 are linearly C-independent and in this case from (2.3.2) we have \((a + q)[x_1b, x_2b]^{n+1}x_3 = 0 ∈ T\), This implies that \(a + q = 0\), a contradiction. Hence \(a + q ∈ C\). Thus \(F(x) = (a + q)x - xq = x(a + q - q) = xa\) for all \(x ∈ R\). Then (2.3.2) becomes

\[
(-[x_1b, x_2b]a[x_1b, x_2b] - [x_1b, x_2b]^{n+1}a)x_3
\]

\[
-x_3([x_1b, x_2b]a[x_1b, x_2b] - [x_1b, x_2b]^{n+1}a) = 0 ∈ T.
\]

If \(ba\) and \(b\) are linearly C-independent, then from above we have that \(R\) satisfies the nontrivial generalized polynomial identity \(x_3[x_1b, x_2b]a[x_1b, x_2b]^{n}, \) a contradiction. Hence we conclude that \(ba\) and \(a\) are linearly C-dependent for all \(b ∈ I\). Thus there exists \(a ∈ C\) such that \(I(a - a) = (0)\).

Case (ii) Suppose that \(d\) is not an inner derivation of \(U\). Since \(R\) is not commutative, then there exists \(0 ≠ b ∈ I\) such that \(b ∉ C\). By our main assumption, \(R\) satisfies

\[
[[a[x_1b, x_2b] + [d(x_1)b + x_1d(b), x_2b] + [x_1b, d(x_2)b + x_2d(b)], [x_1b, x_2b]]_n, x_3].
\]

(2.3.3)

Since \(d\) is not inner derivation and by Theorem 1.3.6, we have that \(R\) satisfies

\[
[[a[x_1b, x_2b] + [y_1b + x_1d(b), x_2b] + [x_1b, y_2b + x_2d(b)], [x_1b, x_2b]]_n, x_3].
\]

(2.3.4)
and in particular
\[ [(y_1b, x_2b), [x_1b, x_2b]^m], x_3 \]  \hspace{1cm} (2.3.5)\]
is a generalized identity for \( R \). Since \( b \notin C \), then \( b \) and \( 1 \) linearly \( C \)-independent, thus (2.3.5) is a nontrivial generalized polynomial identity for \( R \), a contradiction.

**Lemma 2.3.7** Without loss of generality, we may assume that \( R \) is simple and equal to its own socle, \( RI = I \).

**Proof** By Lemma 2.3.6, \( R \) is a GPI (otherwise we are done). So \( U \) has nonzero socle \( H \) with nonzero left ideal \( J = HJ \) by Theorem 1.3.11. Note that \( H \) is simple, \( J = HJ \) and \( J \) satisfies the same basic conditions as \( I \) (we refer to [80]). Just replace \( R \) by \( H \), \( I \) by \( J \) and we get the required result.

**Lemma 2.3.8** Let \( R \) be a prime ring, \( 0 \neq c \in R \), \( I \) a nonzero left ideal of \( R \), \( m \geq 1 \) a fixed integer such that \( c[r_1, r_2]^m \in Z(R) \) for all \( r_1, r_2 \in I \). Then \( x_1s_4(x_2, x_3, x_4, x_5) \) is an identity for \( I \).

**Proof** Firstly we notice that if \( c[x_1, x_2]^m \) is a generalized polynomial identity for \( I \) then by Lemma 2.3.3 and since \( c \neq 0 \), we have \( r_1[r_2, r_3] = 0 \) for all \( r_1, r_2, r_3 \in I \) and a fortiori \( x_1s_4(x_2, x_3, x_4, x_5) \) is an identity for \( I \). Therefore we may assume that there exists \( a_1, a_2 \in I \) such that \( 0 \neq c[a_1, a_2]^m \in Z(R) \). By Lemma 2.3.1, \( R \) is a PI-ring and so \( RC \) is a finite dimensional central simple \( C \)-algebra. By Wedderburn-Artin theorem \( RC \cong M_k(D) \) for some \( k \geq 1 \) and \( D \) a finite dimensional central division \( C \)-algebra. By Lemma 2.3.4, \( c[r_1, r_2]^m \in C \) for all \( r_1, r_2 \in CI \). Without loss of generality we may replace \( R \) with \( RC \) and assume that \( R = M_k(D) \). Let \( E \) be a maximal subfield of \( D \), so that \( E \otimes_C M_k(D) \cong M_t(E) \), where \( t = k, [E : C] \). Hence \( c[r_1, r_2]^m \in C \) for all \( r_1, r_2 \in Z(M_t(E)) \) for any \( r_1, r_2 \in E \otimes_C I \) (Theorem 1.3.8 and Lemma 2.3.5). Therefore we may assume that \( R \cong M_t(E) \) and replace \( I \) with \( E \otimes_C I \). Moreover \( 0 \neq c[b_1, b_2]^m \in Z(M_t(E)) \) for \( b_1 = 1_{E \otimes_C} a_1 \) and \( b_2 = 1_{E \otimes_C} a_2 \). Then
I contains an invertible element of R and so \( I = R = M_1(E) \) and \( c[r_1, r_2]^m \in Z(R) \) for all \( r_1, r_2 \in R \). Consider the following subset of R,

\[
G = \{ a \in R \mid a[r_1, r_2]^m \in Z(R), \text{ for all } r_1, r_2 \in R \}
\]

and notice that \( G \) is a subgroup of \( R \), which is invariant under the action of all automorphisms of \( R \), moreover \( c \in G \). By Lemma 2.3.2, one of the following holds:

(i) \( R \) satisfies \( s_4 \) and \( \text{char } (R) = 2 \) (in this case we are done);

(ii) \( G \subseteq Z(R) \) and since \( c \neq 0 \), it follows that \( [r_1, r_2]^m \in Z(R) \) for all \( r_1, r_2 \in R \);

(iii) \( [R, R] \subseteq G \) which implies that \( [s_1, s_2][r_1, r_2]^m \in Z(R) \) for all \( s_1, s_2, r_1, r_2 \in R \).

In order to conclude our proof, we may assume that in any case \( [r_1, r_2]^m \in Z(R) \) for all \( r_1, r_2 \in R \). This implies easily that \( R \) must satisfy \( s_4 \).

**Lemma 2.3.9** Let \( R \) be a noncommutative prime ring, \( a, b \in R \), \( I \) a two sided ideal of \( R \), \( n \geq 1 \) a fixed integer such that \( [a[r_1, r_2] + [r_1, r_2]b, [r_1, r_2]^m] \in Z(R) \) for any \( r_1, r_2 \in I \). Then either \( a, b \in Z(R) \) or \( R \) satisfies the standard identity \( s_4 \).

**Proof** Suppose that either \( a \notin Z(R) \) or \( b \notin Z(R) \). In both cases

\[
[[a[x_1, x_2] + [x_1, x_2]b, [x_1, x_2]^m], x_3]
\]  

is a nontrivial generalized polynomial identity for \( I \) and so also for \( R \) (see [43]). By Theorem 1.3.3, (2.3.6) is also an identity for \( RC \). By Theorem 1.3.11 \( RC \) is a primitive ring with nonzero socle. There exists a vectorial space \( V \) over a division ring \( D \) such that \( RC \) is dense of \( D \)-linear transformations over \( V \). Firstly we will prove that \( \dim_D V \leq 2 \). By contradiction assume that \( \dim_D V \geq 3 \). If \( \{v, va\} \) is linearly \( D \)-independent for some \( v \in V \), then by the density of \( RC \), there exists \( w \in V \) such that \( \{v, w, va\} \) is linearly \( D \)-independent and \( x_0, y_0, z_0 \in RC \) such that \( vx_0 = 0, vy_0 = 0, vz_0 = 0, (va)x_0 = w, (va)y_0 = 0, (va)z_0 = v, w_0 = va \). This leads to a contradiction

\[
0 = v[[a[x_0, y_0] + [x_0, y_0]b, [x_0, y_0]^m]], z_0] = v.
\]
Thus \( \{v, va\} \) is linearly \( D \)-dependent for all \( v \in V \), which implies that \( a \in C \). From this, \( RC \) satisfies
\[
[[[x_1, x_2]b, [x_1, x_2]^n], x_3].
\] (2.3.7)

As above suppose that there exists \( v \in V \) such that \( \{v, vb\} \) are linearly \( D \)-independent. Then there exists \( w \in V \) such that \( \{v, vb, w\} \) is linearly \( D \)-independent and there exist \( x_0, y_0, z_0 \in RC \) such that \( vx_0 = w, vy_0 = 0, vz_0 = vb, wy_0 = v, (vb)x_0 = v, (vb)y_0 = 0, (vb)z_0 = v \). This implies that
\[
0 = v[[[x_0, y_0]b, [x_0, y_0]^n], z_0] = -v 
\]
a contradiction. Also in this case we conclude that \( \{v, vb\} \) are linearly \( D \)-dependent for all \( v \in V \) and so \( b \in C \).

The previous argument shows that if either \( a \notin C \) or \( b \notin C \), then \( \dim_D V \leq 2 \). In this condition \( RC \) is a simple ring which satisfies a nontrivial generalized polynomial identity. By Theorem 1.3.13, \( RC \subseteq M_t(F) \), for a suitable field \( F \), moreover \( M_t(F) \) satisfies the same generalized identity of \( RC \), hence
\[
[a[r_1, r_2] + [r_1, r_2]b, [r_1, r_2]^n] \in Z(M_t(F))
\]
for any \( r_1, r_2 \in M_t(F) \). If \( t \leq 2 \), then \( R \) satisfies the standard identity \( s_4 \). If \( t \geq 3 \), by the above arguments, we get \( a, b \in Z(M_t(F)) \).

**Proof of Theorem 2.3.1** By the regularity of \( R \), there exists \( e^2 = e \in RI \) such that \( Re = Ra_1 + Ra_2 + Ra_3 + Ra_4 + Ra_5 \) and \( a_ie = a_e \) for \( i = 1, \ldots, 5 \). In view of Theorem 1.3.6, we divide the proof into two cases:

**Case (i)** If \( d \) is an inner derivation induced by an element \( q \in U \), then \( I \) satisfies the
\[
[[a[x_1, x_2] + q[x_1, x_2] - [x_1, x_2]q, [x_1, x_2]^n], x_3].
\] (2.3.8)

Thus for all \( r, s, t \in R \), we have
\[
[[a[re, se] + q[re, se] - [re, se]q, [re, se]^n], t] = 0.
\] (2.3.9)
In particular for \( t = 1 - e \) and left multiplying by \( e \), we obtain

\[
e . [(a + q)[re, se] - [re, sc]q, [re, se]^n], 1 - e] = 0,
\]

(2.3.10)

that is, \( e[re, se]^{n+1}q(1-e) = 0 \) for all \( r, s \in R \). This implies that \([er, es]^{n+1}eq(1-e) = 0\). By Lemma 2.3.3 either \([eR, eR]e = 0\) or \(eq(1-e) = 0\). Since \( s_4(eRe) \neq (0) \), then a fortiori \([eRe, eRe] \neq (0)\), therefore we have \( eq = ege \in Re \) and \( F(Re) \subseteq Re \). Let \( \lambda = Re, \bar{\lambda} = \frac{\lambda}{\lambda \in R} \), Where \( r_R(\lambda) \) is the right annihilator of \( \lambda \) in \( R \). Therefore the prime ring \( \bar{\lambda} \) satisfies the generalized polynomial identity (2.3.8) and by Lemma 2.3.9 it follows \( s_4(\bar{\lambda}) = \bar{\lambda} \) or both \([a, \bar{\lambda}] = \bar{\lambda} \) and \([q, \bar{\lambda}] = \bar{\lambda} \).

Since \( s_4(\bar{\lambda}) = \bar{\lambda} \) implies the contradiction \( a_1s_4(a_2, a_3, a_4, a_5) = 0 \). We may assume \( \lambda[a, \lambda] = 0 \) and \( \lambda[q, \lambda] = 0 \). In this case, standard arguments show that there exist \( \alpha, \gamma \in C \) such that \( I(a - \alpha) = (0) \) and \( I(q - \gamma) = (0) \). Denote \( a' = a - \alpha, q' = q - \gamma \) and notice that in light of (2.3.8), we also have that

\[
[[a',x_1, x_2] + q'[x_1, x_2] - [x_1, x_2]q', [x_1, x_2]^n], x_3]
\]

(2.3.11)

is satisfies by \( I \), that is, \((a' + q')[x_1, x_2]^{n+1}\) is a generalized identity for \( I \). By Lemma 2.3.8 and since \( a_1s_4(a_2, a_3, a_4, a_5) \neq 0 \), it follows that \( a' + q' = 0 \) i.e. \( a + q \in C \). Therefore \( F(x) = ax + qx - xq \) and hence \( F(x) = xa \) for all \( x \in I \).

**Case (ii)** Suppose that \( d \) is not an inner derivation. By our main assumption \( R \) satisfies

\[
[[a|x_1e, x_2e] + [d(x_1)e + x_1d(e), x_2e] + [x_1e, d(x_2)e + x_2d(e)], [x_1e, x_2e]^n], x_3].
\]

(2.3.12)

Since \( d \) is not an inner derivation and by Theorem 1.3.6, we have

\[
[[a|x_1e, x_2e] + [y_1e + x_1d(e), x_2e] + [x_1e, y_2e + x_2d(e)], [x_1e, x_2e]^n], x_3]
\]

(2.3.13)
is a generalized identity for $R$. In particular $R$ satisfies both

$$\left[[y_1e, x_2e], [x_1e, x_2e]^n\right], x_3$$

and

$$\left[[x_1e, y_2e], [x_1e, x_2e]^n\right], x_3.$$ (2.3.15)

By replacing in (2.3.14) $y_1$ with $(1 - e)y_1$ and $x_3$ with $x_3e$, it follows that $R$ satisfies $(1 - e)y_1ex_2e[x_1e, x_2e]^nx_3e$ and by the primeness of $R$, we have

$$er_2e[r_1e, r_2e]^n = 0 \text{ for all } r_1, r_2 \in R.$$ (2.3.16)

Analogously, by replacing in (2.3.15) $y_2$ with $(1 - e)y_2$ and $x_3$ with $x_3e$, it follows that $R$ satisfies $-(1 - e)y_2ex_1e[x_1e, x_2e]^nx_3e$ and by the primeness of $R$, we have

$$er_1e[r_1e, r_2e]^n = 0 \text{ for all } r_1, r_2 \in R.$$ (2.3.17)

In light of (2.3.16) and (2.3.17), we obtain $[r_1e, r_2e]^{n+1} = 0$ for all $r_1, r_2 \in R$. Again by Lemma 2.3.3, we get $e[Re, Re] = (0)$ which is a contradiction. This completes the proof.

**Theorem 2.3.2** Let $R$ be a semiprime ring and $I$ be a nonzero left ideal of $R$. Let $F$ be a generalized derivation of $R$ with associated derivation $d$. If $Id(I) \neq (0)$ and $F([x, y]) - [x, y] = 0$ for all $x, y \in I$, then $[I, I]d(I) = (0)$ and there exists $0 \neq \alpha \in Z(R)$ such that $\alpha I \subseteq Z(R)$.

**Proof** If $F = 0$, then we have $[x, y] = 0$ for all $x, y \in I$ i.e. $I$ is commutative and hence $I \subseteq Z(R)$ by Lemma 2.2.3. If $F \neq 0$ and $F([x, y]) - [x, y] = 0$ for all $x, y \in I$, then we have

$$F([x, y]) = [x, y] \text{ for all } x, y \in I.$$ (2.3.18)

Replacing $y$ by $yx$ in (2.3.18), we obtain

$$F([x, y])x + [x, y]d(x) = [x, y]x \text{ for all } x, y \in I.$$ (2.3.19)
Comparing (2.3.18) and (2.3.19) to get

\[[x, y]d(x) = 0 \text{ for all } x, y \in I.\] (2.3.20)

Substitute \(ry\) for \(r\) in (2.3.20) yields that

\[[x, r]yd(x) = 0 \text{ for all } x, y \in I, \ r \in \mathbb{R}.\] (2.3.21)

It follows that

\[[x, R]Id(x) = (0) \text{ for all } x \in I.\] (2.3.22)

Take now the family \(\{P_{\alpha}\}\) of prime ideals of \(R\) such that \(\cap P_{\alpha} = (0)\). Let \(x_1\) be a fixed element of \(I\), thus for any \(x_2 \in I\) and by (2.3.22), it follows

\((0) = [x_1 + x_2, R]Id(x_1 + x_2) = [x_1, R]Id(x_2) + [x_2, R]Id(x_1).\) (2.3.23)

Moreover, for any \(P_{\alpha}\), by (2.3.22) and Lemma 2.2.1, either \(Id(x_1) \subseteq P_{\alpha}\), or \([x_1, R] \subseteq P_{\alpha}\). In case \(Id(x_1) \subseteq P_{\alpha}\), then a fortiori \([x_2, R]Id(x_1) \subseteq P_{\alpha}\). On the other hand, if \([x_1, R] \subseteq P_{\alpha}\), then by (2.3.23) it follows again \([x_2, R]Id(x_1) \subseteq P_{\alpha}\). Therefore in any case \([I, R]Id(I) \subseteq P_{\alpha}\), for any \(\alpha\). This implies

\([I, R]Id(I) \subseteq \cap P_{\alpha} = (0).\) (2.3.24)

In particular we get \([I, I]RId(I) = (0)\) and from this we also have \([I, I]d(I)R[I, I]d(I) = (0)\). Hence, by the semiprimeness of \(R\) one has

\([I, I]d(I) = (0).\) (2.3.25)

Moreover, for any \(r, s \in R, x, y, z \in I\) and by (2.3.24) it follows:

\(0 = [rx, s]yd(z) = [r, s]xyd(z)\) (2.3.26)

and replacing \(y\) with \(ty\) in (2.3.26), for any \(t \in R\), we get \([r, s]txyd(z) = 0\), that is \([R, R]IRId(I) = (0)\). Again by the semiprimeness of \(R\), it follows \([R, R]Id(I) = (0)\) and a fortiori \([R, R]RId(I) = (0)\). This last implies easily that \([Id(I), R]R[Id(I), R] = (0)\), that is \(Id(I) \subseteq Z(R)\).
Therefore for \( x, y, z \in I \), we have \( xd(z)y + xzd(y) = xd(zy) \in Z(R) \), and since \( xd(zy) \in Z(R) \), it follows that also \( xd(z)y \in Z(R) \), for any \( x, y, z \in I \). Moreover, by \( I d(I) \neq (0) \), there exist \( x_0, z_0 \in I \) such that \( 0 \neq x_0d(z_0) = \alpha \in Z(R) \). Hence, for all \( y \in I \), we get \( \alpha y \in Z(R) \), that is \( \alpha I \subseteq Z(R) \), as required.

**Theorem 2.3.3** Let \( R \) be a semiprime ring and \( I \) be a nonzero left ideal of \( R \). Let \( F \) be a generalized derivation of \( R \) with associated derivation \( d \). If \( I d(I) \neq (0) \) and \( F([x, y]) - [x, y] \in Z(R) \) for all \( x, y \in I \), then either \( d(Z(R))I = (0) \) or there exists \( 0 \neq \alpha \in Z(R) \) such that \( \alpha I \subseteq Z(R) \).

**Proof** If \( Z(R) = (0) \), then we get the required by Theorem 2.3.2. Thus we may assume \( Z(R) \neq (0) \). Let \( 0 \neq \alpha \in Z(R) \) such that \( d(\alpha)I \neq 0 \). We have

\[
F([x, y]) - [x, y] \in Z(R) \text{ for all } x, y \in I. \tag{2.3.27}
\]

Replace \( y \) by \( \alpha y \) in (2.3.27) to get

\[
F([x, y])\alpha + [x, y]d(\alpha) - [x, y]\alpha \in Z(R) \text{ for all } x, y \in I. \tag{2.3.28}
\]

Comparing (2.3.27) and (2.3.28) we find \( [x, y]d(\alpha) \in Z(R) \) for all \( x, y \in I \) and a fortiori \( [d(\alpha)I, d(\alpha)I] \subseteq Z(R) \). Let \( x_0 \in I \) and denote \( I' = d(\alpha)I \neq (0) \) and \( q_0 = d(\alpha)x_0 \in I' \).

Thus we have that for all \( y \in I' \), \( [q_0, y] \in Z(R) \) and so \( [q_0, y]^2 = 0 \). As a reduction of Lemma 2.2.2, it follows that \( R \) contains some nonzero central ideal, unless \( [x, y] = 0 \) for all \( x, y \in I' \). This last case implies that \( I' = d(\alpha)I \subseteq Z(R) \) (see Lemma 2.2.3).

### 2.4 Lie ideals and \( n \)-centralizing generalized derivations

This section is devoted to study generalized derivations \( n \)-centralizing on Lie ideals of prime and semiprime rings. We establish Theorem 2.3.1 for a Lie ideal of a prime ring. Moreover we shall require the following lemma.
Lemma 2.4.1 Let $R$ be a prime ring and $L$ a noncentral Lie ideal of $R$. Then either there exists a nonzero ideal $I$ of $R$ such that $0 \neq [I, R] \subseteq L$ or $\text{char } R = 2$ and $R$ satisfies $s_4$.

Proof By Theorem 1.3.5 and Theorem 1.3.14, we get the required result.

Theorem 2.4.1 Let $R$ be a prime ring, $F$ a nonzero generalized derivation of $R$, $L$ a noncentral Lie ideal of $R$, $n \geq 1$ a fixed integer such that $F$ is $n$-centralizing on $L$. Then either $F(x) = \lambda x$ for all $x \in R$ and for some $\lambda \in C$ or $R$ satisfies $s_4$, the standard identity of degree 4.

Proof Assume that $R$ does not satisfy $s_4$. By Lemma 2.4.1 we have there exists a two sided ideal $I$ of $R$ such that $[I, I] \subseteq L$. In this last case we get that $[F([r_1, r_2]), [r_1, r_2]^m] \in Z(R)$ for all $r_1, r_2 \in I$.

By Theorem 1.3.10, $F$ has the form $F(x) = ax + d(x)$, for $a \in U$ and $d$ a derivation of $U$.

If $d$ is an inner derivation induced by an element $c \in U$, it follows that

$[(a + c)[r_1, r_2] - [r_1, r_2]c, [r_1, r_2]^m] \in Z(R)$

for all $r_1, r_2 \in I$ and by Lemma 2.3.9 we have that $a, c \in C$, that is $d = 0$ and $F(x) = ax$ for all $x \in R$.

Now assume that $d$ is not an inner derivation of $U$. If $d = 0$, then $I$ satisfies

$[[a[x_1, x_2], [x_1, x_2]^m], x_3]$\

for all $x_1, x_2 \in R$. By Lemma 2.3.9 we get the conclusion $a \in C$ and $F(x) = ax$ for all $x \in U$ and so for all $x \in R$. Assume finally $d \neq 0$. Since

$[[a[x_1, x_2] + [d(x_1), x_2] + [x_1, d(x_2)], [x_1, x_2]^m], x_3]$
for all $x_1, x_2, x_3 \in I$ is a differential identity for $I$. By Theorem 1.3.6, it follows that $I$ satisfies

$$[[a[x_1, x_2] + [y_1, x_2] + [x_1, y_2], [x_1, x_2]^n], x_3]$$

for all $x_1, x_2, x_3 \in I$. In particular

$$[[[x_1, y_2], [x_1, x_2]^n], x_3]$$

is a polynomial identity for $I$. This implies that $R$ is a PI-ring satisfying (2.4.1). Thus there exists a field $F$ such that $R$ and $M_t(F)$, the ring of $t \times t$ matrices over $F$, satisfy the same polynomial identities. Since $L$ is noncentral, $R$ must be noncommutative. Hence $t \geq 2$. In case $t = 2$, $R$ satisfies $s_4$, a contradiction. Thus $t \geq 3$. Denote $e_{ij}$ the usual matrix unit with 1 in the $(i,j)$-entry and zero elsewhere. In (2.4.1) choose $x_1 = e_{12}, x_2 = e_{21}, x_3 = e_{33}, y_2 = e_{23}$, then it follows the contradiction

$$0 = [[e_{13}, (e_{11} - e_{22})^n], e_{33}] = -e_{13}.$$

This completes the proof.

**Lemma 2.4.2** [62, Corollary 2.1] Let $R$ be a 2-torsion free semiprime ring, $L$ a noncentral Lie ideal of $R$ and $a, b \in L$.

(i) If $aLa = (0)$, then $a = 0$.

(ii) If $aL = (0)$ (or $La = (0)$), then $a = 0$.

(iii) If $L$ is square closed and $aLb = (0)$, then $ab = 0$ and $ba = 0$.

**Lemma 2.4.3** Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero square closed Lie ideal of $R$. Then $2R[L, L] \subseteq L$ and $2[L, L]R \subseteq L$.

**Proof** Since $x^2 \in L$ for all $x \in L$, we have $xy + yx = (x + y)^2 - x^2 - y^2 \in L$ for all $x, y \in L$. This implies that $2xy \in L$ for all $x, y \in L$. Since $2[x, r] \in L$ for all $x \in L$ and $r \in R$, replacing $r$ by $ry$ to get $2[x, r]y + 2r[x, y] \in L$ for all $x, y \in L$ and
$r \in R$. But $2[x, r]y \in L$ for all $x, y \in L$ and $r \in R$. Hence we have $2r[x, y] \in L$ for all $x, y \in L$ and $r \in R$, i.e., $2R[L, L] \subseteq L$. Similarly we can get $2[L, L]R \subseteq L$.

**Theorem 2.4.2** Let $R$ be a 2-torsion free semiprime ring and $L$ a nonzero square closed Lie ideal of $R$ such that $L \not\subseteq Z(R)$. If $F$ is a generalized derivation of $R$ with associated derivation $d$ such that $d(L) \subseteq L$ and $[F(x), x] = 0$ for all $x \in L$, then $[d(x), x] = 0$ for all $x \in L$.

**Proof** Suppose that $[F(x), x] = 0$ for all $x \in L$. Linearization yields that

$$[F(x), y] + [F(y), x] = 0 \text{ for all } x, y \in L. \quad (2.4.2)$$

Replacing $y$ by $2yz$ in (2.4.2) and using 2-torsion freeness of $R$, we obtain

$$y[F(x), z] + [F(x), y]z + F(y)[z, x] + [F(y), x]z + y[d(z), x] + [y, x]d(z) = 0 \text{ for all } x, y, z \in L. \quad (2.4.3)$$

In view of (2.4.2), (2.4.3) yields that

$$y[F(x), z] + F(y)[z, x] + y[d(z), x] + [y, x]d(z) = 0 \text{ for all } x, y, z \in L. \quad (2.4.4)$$

Substitute $z$ for $x$ in (2.4.4) to get

$$y[d(z), z] + [y, z]d(z) = 0 \text{ for all } y, z \in L. \quad (2.4.5)$$

Replacing $y$ by $2xy$ in (2.4.5), we have

$$2xy[d(z), x] + 2x[y, z]d(z) + 2[z, x]yd(z) = 0 \text{ for all } x, y, z \in L. \quad (2.4.6)$$

Application of (2.4.5) and using the fact that $R$ is 2-torsion free, we have

$$[x, z]yd(z) = 0 \text{ for all } x, y, z \in L. \quad (2.4.7)$$

Using Lemma 2.4.2 we obtain

$$[x, z]d(z) = 0 \text{ for all } x, z \in L. \quad (2.4.8)$$
Since \( d(L) \subseteq L \), we can replace \( x \) by \( 2d(z)x \) in (2.4.8), we obtain

\[
2[d(z), z]xd(z) = 0 \quad \text{for all } x, y, z \in L. \tag{2.4.9}
\]

This implies that \( 2[d(z), z]x[d(z), z] = 0 \) for all \( x, z \in L \). Since \( R \) is 2-torsion free, we have \( [d(z), z]L[d(z), z] = 0 \) for all \( z \in L \). Using Lemma 2.4.2 we have \( [d(z), z] = 0 \) for all \( z \in L \).

**Theorem 2.4.3** Let \( R \) be a 2-torsion free semiprime ring and \( L \) a nonzero square closed Lie ideal of \( R \) such that \( L \not\subseteq Z(R) \). If \( F \) is a generalized derivation of \( R \) with associated derivation \( d \) such that \( d(L) \subseteq L \) and \( F(x) \circ x = 0 \) for all \( x \in L \), then

\[
[d(x), x] = 0 \quad \text{for all } x \in L. \tag{2.4.10}
\]

**Proof** We have \( F(x) \circ x = F(x)x + xF(x) = 0 \) for all \( x \in L \). Linearization yields

\[
F(x)y + F(y)x + yF(x) + xF(y) = 0 \quad \text{for all } x, y \in L. \tag{2.4.11}
\]

Replacing \( y \) by \( 2yx \) in (2.4.10) and using the fact that \( R \) is 2-torsion free, we obtain

\[
(F(x)y + F(y)x + xF(y))x + yd(x)x + xyd(x) + yxF(x) = 0 \quad \text{for all } x, y \in L. \tag{2.4.12}
\]

Comparing (2.4.10) and (2.4.11), we get

\[
-yF(x)x + yxF(x) + yd(x)x + xyd(x) = 0 \quad \text{for all } x, y \in L. \tag{2.4.13}
\]

This implies that

\[
y[F(x), x] + yd(x)x + xyd(x) = 0 \quad \text{for all } x, y \in L. \tag{2.4.14}
\]

Replacing \( y \) by \( 2r[y, z] \) in (2.4.13) by Lemma 2.4.3 and use (2.4.13) to get by using 2-torsion freeness of \( R \)

\[
-px[y, z][d(x) + x[d(y, z)]d(x)] = 0 \quad \text{for all } x, y \in L. \tag{2.4.15}
\]
This implies that \([x, r][y, z]d(x) = 0\) for all \(x, y, z \in L\) and \(r \in R\). Substitute \(zr\) for \(r\), we obtain \([x, z]d(x)R[y, z]d(x) = 0\) for all \(x, y, z \in L\). Since \(R\) is semiprime, we have \([z, x]d(x) = 0\) for all \(x, z \in L\). Since \(d(L) \subseteq L\), we can replace \(z\) by \(2d(x)u\) in the last relation, we obtain
\[
2[x, d(x)]ud(x) = 0 \quad \text{for all } x, u \in L. \quad (2.4.15)
\]

A simple calculation yields that \(2[d(x), x]u[d(x), x] = 0\) for all \(x, u \in L\). Since \(R\) is 2-torsion free, we have \([d(x), x]L[d(x), x] = 0\) for all \(x \in L\). Using Lemma 2.4.2 we have \([d(x), x] = 0\) for all \(x \in L\). Hence \(d\) is commuting on \(L\).

As a consequence of above theorem we can derive following corollary.

**Corollary 2.4.1** Let \(R\) be a prime ring of characteristic not two and \(L\) a nonzero square closed Lie ideal of \(R\). Let \(F\) be a generalized derivation with associated derivation \(d\). If either \(F\) is commuting or skew commuting on \(L\), then \(L \subseteq Z(R)\).