COMMUTING DERIVATIONS OF RINGS AND THEIR GENERALIZATIONS

ABSTRACT
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The present thesis entitled "Commuting derivations of rings and their generalizations" includes a part of research work carried out by the author during the last three years at the Department of Mathematics, Aligarh Muslim University, Aligarh. The thesis comprises five chapters and each chapter is subdivided into various sections.

Chapter 1 of the thesis contains some preliminary notions, basic definitions and important well-known results which may be needed for the development of the subsequent text. This chapter as a matter of fact, aims at making the present thesis as self-contained as possible. However, the basic knowledge of the ring theory has been presumed and no attempt is made to include the proofs of the results presented in this chapter.

In 1957 E. C. Posner [98] established a very striking result which states that the existence of a nonzero commuting derivation on a prime ring $R$ forces $R$ to be commutative. The theorem has been extremely influential and it initiated the study of commuting derivations. A lot of work has been done on commuting derivations for references see [29], [32], [33], [37], [94], [95], where further references can be found out. It is interesting to weaken the hypothesis of the results on commuting derivations in rings and obtain them on some well behaved subsets of rings. In chapter 2, we study generalized derivations commuting (skew-commuting) on a nonzero left ideal of a semiprime ring. In [48], Deng defined $n$-centralizing ($n$-commuting) mappings, a concept more general than centralizing and commuting mappings and proved that if $R$ is a prime ring of characteristic either zero or $> n$, $I$ a nonzero left ideal of $R$ and $d$ is a nonzero derivation of $R$ which is $n$-centralizing on $I$, then $R$ is commutative. In section 2.3, we obtain the result for $n$-centralizing generalized derivation of a prime ring $R$ in the setting of a left ideal of $R$. Finally we extend the result for a Lie ideal of $R$.

Chapter 3 is devoted to the study of commuting values of generalized derivations. Motivated by the results of Lee and Shiu [85], Argac and Demir [15] in section 3.2 we prove the following result: Let $R$ be a prime ring of characteristic different from 2, $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over $R$ and $F$ be a nonzero generalized derivation of $R$. If $[F(u)v, F(v)u] = 0$, for any $u, v \in f(R)$, the set of all evaluations of the polynomials $f(x_1, \ldots, x_n)$ in $R$, then there exists $c \in U$ such that $F(x) = cx$, for all $x \in R$ and one of the following holds: (I) $f(x_1, \ldots, x_n)^2$ is central valued on $R$, (II) $R$ satisfies $s_4$, the standard identity of degree 4. In section 3.3, we investigate the conditions (i) $[d(x), F(y)] = 0$; (ii) $d(x) \circ F(y) = 0$; (iii) $[d(x), F(y)] \in [x, y] = 0$; (iv) $d(x) \circ F(y) \in [x \circ y] = 0$; (v) $d(x) \circ F(y) \in xy = 0$ (vi) $[d(x), F(y)] \in [x, y] = 0$; (vii) $[d(x), F(y)] \in xy = 0$; (viii) $d(x) \circ F(y) \in [x, y] = 0$, for all $x, y \in I$, a nonzero ideal of a semiprime ring $R$ admitting a generalized derivation $F$ with associated derivation $d$ and prove that $R$ contains a nonzero central ideal. Finally we extend the results of Bell [26] and Argac [13] to
the case when the generalized derivation $F$ acts on a one sided ideal of a semiprime ring $R$.

In chapter 4 we study commuting traces of biderivations. In 1980, Maksa [89] introduced the concept of a biderivation. A biadditive mapping $D : R \times R \to R$ is said to be a biderivation on a ring $R$ if for all $x, y \in R$, the mappings $y \to D(x, y)$ and $x \to D(x, y)$ are derivations of $R$. In section 4.2, we study $n$-centralizing traces of symmetric biderivations of semiprime rings. The main result is the following: Let $R$ be a semiprime ring, $I$ a nonzero ideal of $R$ and $n$ be a fixed positive integer. Let $R$ be $n$-torsion free for $n > 1$ and 2-torsion free for $n = 1$. Suppose there exists a symmetric biderivation $D : R \times R \to R$ such that the mapping $f : R \to R$ is $n$-centralizing on $I$, where $f$ stands for the trace of $D$. Then $f$ is $n$-commuting on $I$. Moreover we extend the result for a Lie ideal of $R$. In section 4.3, we study generalized biderivations of prime rings. The notion of generalized biderivation was introduced by Nurcan in [13]. Let $R$ be a ring and $D : R \times R \to R$ be a biadditive map. A biadditive mapping $\Delta : R \times R \to R$ is said to be a generalized biderivation if for every $x \in R$, the map $y \to \Delta(x, y)$ is a generalized derivation of $R$ associated with function $y \to D(x, y)$ for all $x, y \in R$ as well as for every $y \in R$, the map $x \to \Delta(x, y)$ is a generalized derivation of $R$ associated with function $x \to D(x, y)$ for all $x, y \in R$. Recently in [117, Theorem 2] Yenigul et al. proved a result of Vukman [108, Theorem 4] for a two sided ideal $I$ of a prime ring $R$ which states that if there exist symmetric biderivations $D_1 : R \times R \to R$ and $D_2 : R \times R \to R$ such that $D_1(d_2(x), x) = 0$ for all $x \in I$, where $d_2$ is the trace of $D_2$, then either $D_1 = 0$ or $D_2 = 0$. We obtain the result for a symmetric generalized biderivation $\Delta$ with associated biderivation $D$ of $R$ with trace $f$ satisfying $\Delta(f(x), x) = 0$ for all $x \in I$ and conclude that either $\Delta = 0$ or $R$ is commutative.

Finally we investigate the commutativity of a semiprime ring $R$ satisfying various identities involving the trace $f$ of the symmetric biadditive mapping $D$ on $R$.

Chapter 5 deals with the characterizations of commuting derivations. In [1], Albas and Nurcan showed that if $R$ is a noncommutative prime ring and $F$ is a generalized derivation with associated derivation $d$ of $R$ and for all $x \in R$, $[F(x), a] = 0$, then either $a \in C$ or there exists $\lambda, \eta \in C$ such that $F(x) = \eta x + \lambda(\eta x + xa)$ for all $x \in R$. Further Aydin [22] proved the result in case of a nonzero ideal of $R$. In Section 5.2 we establish the result in the setting of a one sided ideal of $R$, which states that if $I$ is a nonzero left ideal of a prime ring $R$ admitting a generalized derivation $F$ with associated derivation $d$ such that for $a \in I, a \notin Z(R)$, $[F(x), a] = 0$ (or $F([x, a]) = 0$) for all $x \in I$, then either $R$ is commutative or $d(a) \in Z(R)$ provided that the right annihilator of $I$ that is $A_r(I) = (0)$. In Section 5.3, we consider a pair of generalized derivations $(F, d)$, $(G, g)$ satisfying $F(x)G(y) = G(x)F(y)$ for all $x, y \in I$, a left ideal of a prime ring $R$ and show that $g$ can be expressed as $g(x) = \lambda d(x)$, $\lambda \in C$, the extended centroid of $R$.

Finally we investigate that a symmetric biderivation $D$ of a prime ring $R$ of characteristic not two with trace $f$ which is commuting on a left ideal $I$ of $R$ is of the form $D(x, y) = \lambda [x, y]$ for all $x, y \in I, \lambda \in C$. 

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The list of some papers based on the text which have already been published or accepted for publication in standard refereed Mathematical Journals/Research Volumes are given below.


In the end, an exhaustive bibliography of the existing material related to the subject matter of the thesis is included which may serve as source material for those, interested in the domain of the research.

References


[9] Ali, A., Bell, H. E. and Rani, R., $(\theta, \phi)$- derivations as homomorphisms or anti-


