7. APPENDIX A

Flow of a Fluid in a Horizontal Channel

Viscosity has the dominant role in mass transport inside micro-fluidic devices. It is the ratio of shear stress to the velocity gradient in the direction perpendicular to the plane of shear. Being the central property in a micro-fluidic environment, viscosity and its variation may be directly observed and analyzed using LOC devices.

Consider a horizontal micro-channel formed by two parallel plates (made of any typical material, like glass or polymer) as shown in figure 1.1. In order to pass a liquid through the channel a pressure difference between the inlet and outlet is required. The resulting flow of the liquid in the channel is governed by the simultaneous action of the force due to pressure difference and viscous force.

The total force acting on the liquid inside the micro-channel per unit volume may be represented as

\[ F_T = F_{\text{Pressure}} + F_{\text{Viscous}} \]  \hspace{1cm} (27)

Under the action of this force, the liquid moves with an acceleration of \( \frac{dv}{dt} \)

\[ F_T = \rho \frac{dv}{dt} \]  \hspace{1cm} (28)

where \( \rho \) is the density of liquid flowing in the channel; \( v \) is the velocity of the liquid.

\[ \rho \frac{dv}{dt} = F_{\text{Pressure}} + F_{\text{Viscous}} \]  \hspace{1cm} (29)

The force due to Pressure \( (F_{\text{Pressure}}) \) may be calculated by considering a volume element inside the channel with elemental volume as \( \Delta x \Delta y \Delta z \), and assuming the pressure gradient mainly along the length of the channel (which is along \( x \)-axis) (see figure 7.1) The assumption that pressure gradient is mainly along \( x \)-axis (and
is negligible along $y$ and $z$ axes) is valid because the micro-channel has very small dimension (thickness) along the directions perpendicular to $x$.

![Figure 7.1 Volume Element inside a Micro-channel](image)

The force due to pressure per unit volume on the $yz$ surface of the element may be written as

$$F_{\text{Pressure}} = \frac{(P(x + \Delta x, y, z) - P(x, y, z))dA}{dV}$$

where $dA$ is the area of the $yz$ surface.

$$F_{\text{Pressure}} = \frac{(P(x + \Delta x, y, z) - P(x, y, z))\Delta y\Delta z}{\Delta x\Delta y\Delta z}$$

Expanding $P(x + \Delta x, y, z)$ about $x$ using Taylor Series we get:

$$P(x + \Delta x, y, z) = P(x, y, z) + \frac{\Delta P}{\Delta x}\bigg|_x \Delta x + ..$$

Therefore the force due to pressure per unit volume at any point inside the channel may be written as

$$F_{\text{Pressure}} = \frac{(P(x + \Delta x) - P(x))}{\Delta x} \sim \frac{\Delta P}{\Delta x}$$

In the similar manner, if we consider the entire channel of length $L$ and the pressure at input to be $P_{\text{in}}$ and the pressure at the output to be $P_{\text{out}}$, then the force per unit volume may be written as
For pressure, the equation is:
\[ F_{\text{Pressure}} = \frac{P_{\text{out}} - P_{\text{in}}}{L} = \frac{\Delta P}{L} \]  
(30)

For flow to happen from input to output, \(\Delta P\) should be negative.

Viscous force acting on the liquid per unit volume may be calculated by considering a volume element in either the top or bottom portion with respect to the center of symmetry of the micro-channel (see figure 7.2).

![Figure 7.2 Positioning of the Volume Element in the Channel](image)

Velocity of the liquid on the top surface of the volume element will be less than the velocity of the liquid on the bottom surface. The unequal viscous force on both the surfaces of the element will result in the shear of the liquid element.

The viscous force per unit area on the top surface may be written as:
\[
F_{\text{Viscous Top}} \frac{dA}{dA} = -\mu \frac{dv_x}{dy} |_{y+\Delta y}
\]

This force seems to move the liquid along the left.

Similarly, the viscous force per unit area on the bottom surface may be written as
\[
F_{\text{Viscous Bottom}} \frac{dA}{dA} = \mu \frac{dv_x}{dy} |_{y}
\]

This force seems to move the liquid towards the right.
Therefore the net viscous force per unit area on the volume element may be written as

\[
\frac{F_{\text{net}}}{dA} = \mu \left[ \frac{dv_x}{dy} \bigg|_y - \frac{dv_x}{dy} \bigg|_{y+\Delta y} \right]
\]

\[
F_{\text{net}} = \mu \, dA \left[ \frac{dv_x}{dy} \bigg|_y - \frac{dv_x}{dy} \bigg|_{y+\Delta y} \right]
\]

Viscous force per unit volume becomes

\[
F_{\text{Viscous}} = \frac{F_{\text{net}}}{dV} = \frac{F_{\text{net}}}{dA \, \Delta y} = -\mu \left[ \frac{dv_x}{dy} \bigg|_{y+\Delta y} - \frac{dv_x}{dy} \bigg|_y \right]
\]

\[
= -\mu \left[ \frac{\frac{dv_x}{dy} \bigg|_{y+\Delta y} - \frac{dv_x}{dy} \bigg|_y}{\Delta y} \right]
\]

Expanding \( \frac{dv_x}{dy} \bigg|_{y+\Delta y} \) about \( y \) using Taylor Series we get

\[
\frac{dv_x}{dy} \bigg|_{y+\Delta y} = \frac{dv_x}{dy} \bigg|_y + \frac{d^2v_x}{dy^2} \bigg|_y \Delta y + \cdots
\]

The viscous force per unit volume can be given as

\[
F_{\text{Viscous}} = -\mu \frac{d^2v_x}{dy^2} \tag{31}
\]

In steady state (when force due to pressure is balanced by the viscous force), the liquid moves with constant velocity. In such a case, we should have:

\[
\rho \frac{dv}{dt} = 0
\]

Therefore eq. 3 becomes

\[
F_{\text{Pressure}} + F_{\text{Viscous}} = 0 \tag{32}
\]

Substituting eq. 4 and 5 in 6, we get

\[
\frac{\Delta P}{L} - \mu \frac{d^2v_x}{dy^2} = 0
\]
\[ \frac{d^2 v_x}{dy^2} = \frac{\Delta P}{L} \]  

Upon double integration of eq. 7 we get

\[ v_x = \frac{\Delta P}{\mu L} \left( \frac{y^2}{2} \right) + C_1 y + C_2 \]  

Applying the boundary conditions for a liquid flowing inside a channel, we get

1. At \( y = 0 \), \( \frac{dv_x}{dy} = 0 \). Substituting this in eq. 8, we get

   \[ 0 = 0 + C_1 \]  
   \[ C_1 = 0 \]

2. At \( y = w \), \( v_x = 0 \) (No-slip condition), where \( w \) is half width of the channel.

Substituting this in eq. 8, we get

\[ 0 = \frac{\Delta P}{\mu L} \left( \frac{w^2}{2} \right) + 0 + C_2 \]  
\[ C_2 = -\frac{\Delta P}{\mu L} \left( \frac{w^2}{2} \right) \]

Finally, substituting the values of \( C_1 \) and \( C_2 \) in eq. 8, we get

\[ v_x = \frac{\Delta P}{\mu L} \left( \frac{y^2}{2} \right) - \frac{\Delta P}{\mu L} \left( \frac{w^2}{2} \right) \]  
\[ v_x = \frac{\Delta P}{2\mu L} (y^2 - w^2) \]

Therefore,

\[ \mu = \frac{w^2 \Delta P}{2v_x L} \left( \frac{y^2}{w^2} - 1 \right) \]  

where
\( \mu \) is the viscosity of the liquid flowing in the channel
\( w \) is the half width of the channel
\( \Delta P \) is the Pressure differential between the inlet and outlet ports \( v_x \) is the velocity of the liquid flowing inside the channel
\( L \) is the length of the channel